

THE OPTIMAL TAXATION THEORY REVISITED: THE CASE OF INVERSE DEMANDS*

HOANJAE PARK · KUN-HA HWANG**

This paper reformulates the optimal commodity taxation theory which is based on the distance function. It enables the change in the marginal rate of substitution between goods to be broken into scale effects and substitution effects in quantity space. An inverse demand system with these effects is constructed as a first attempt in the literature, extending Deaton's idea to the directly estimable inverse demand system. In addition, the dependence of theoretical results of the optimal taxes on the specification of market behaviors is examined following Fullerton. It suggests that reform of optimal taxation be influenced by the choice of market condition, which specifies either traditional demand systems or inverse demand systems.

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I. INTRODUCTION

In analyzing a policy problem, analysts and authors may regard either prices or quantities as choice variables. Especially in the optimum taxation, Diamond and Mirrlees (1971) use prices as controls while Atkinson and Stiglitz (1972), and Deaton (1979) use quantities. Stern (1986) abstracts this problem as follows:¹

“The choice of variables for optimization and the representations of preferences depend on the job at hand, the kind of results or information one has in mind, and the predilection of the practitioner. It is important, however, to establish

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¹ For thorough surveys of the theory of optimal taxation, see Auerbach (1985) and Stiglitz (1988) among others.

routes for passing from one approach and set of results to the other so that one can use the various techniques to bring out different aspects of the solution.”

The usual textbook treatment is to formulate the problem in price space leading to a solution in quantity space. If we reformulate the original problem in quantity space, we can derive a solution in price space as in Deaton (1979). Now we may ask ourselves what is the advantage of its reformulation in quantity space. It is to give an extremely elegant form of a solution and to obtain fresh insights quickly about the structure of an optimal taxation general equilibrium. It further relates taxes to prices and price responses without quantity responses appearing.

As the use of duality concepts has become widespread, Deaton considers the distance function dual to expenditure function in formulating the optimal taxation. We reformulate the optimal taxation theory following Deaton but in a slightly different form. We also consider the optimal taxation in inverse demand systems, which is thought to be useful to modern econometricians. As Hicks (1956) pointed out, it would be natural to estimate demand responses in the behavior of the individual consumers but price responses become at least as important in the behavior of the markets. In a way, this insight has been lost in the optimal taxation literature. Although the existing literature in the optimal taxation theory ignores completely the inverse demand case, it should be included in the literature since this case perfectly makes sense in some markets like food, rents, housing, and others.² From an economic viewpoint, the meaningful theorems characterizing demand functions are well known while the corresponding meaningful theorems for marginal valuation functions still look like a curiosum. In this, Hicks' suggestion does not necessarily lead us to such an extension. But it seems that it is not much more restrictive to extend the inverse demand case to every market than to systematically adopt the traditional demand case. For one case is not more perfect than the other and thus inverse demand case deserves some attention even in the optimal taxation literature. In addition, we will infer the dependence of theoretical results of the optimal taxation on the specification of market behaviors used to estimate demand or price responses in markets. It also should be included into the existing literature in the optimal taxation theory.

This is the only ambition of this paper. Its contribution is then obviously adding some new results into the existing literature in the optimal taxation theory, which will give more applicability and flexibility to the optimal taxation and tax reform. We follow the framework proposed by Deaton (1979, 1981) for the preference structure and Fullerton (1991) for the marginal welfare cost of taxation in comparison of market behaviors. In section 2, the distance function

² As one of other cases, proposed tax programs may remove some goods from individual opportunity sets, while introducing new goods. Other goods may also include recreational and environmental goods.

approach is applied to the Ramsey Rule. In section 3, we present an inverse demand system that is able to easily interpret the meaning of the Ramsey rule. In section 4, we infer the dependence of theoretical results of the optimal taxes on the specification of market behaviors used to estimate demand or price responses in markets. Section 5 concludes. As the main objective of this paper is methodological, we will not pay much attention to the empirical issues in practice.

II. OPTIMAL TAXATION THEORY: DISTANCE FUNCTION APPROACH

Following Deaton (1979, 1981), we consider one consumer whose name is Robinson Crusoe, who faces with prices proportional to $v = p/m$ (p denotes a price vector and m income) and purchases a quantity bundle (q) given convex preferences. Thus the distance function, $D(q, u) = 1$, describes his preference.³ A government in this Robinson Crusoe economy wishes to raise a predetermined revenue by ad valorem rate of taxation. Since there is one representative taxpayer, the government need not be concerned with questions of vertical equity or horizontal equity.⁴ The problem then becomes to choose t , or, equivalently, q , to maximize the utility function of Crusoe, subject to his initial utility level ($D(q, u) = 1$) and the government's tax revenue (R) constraint :

$$R = \sum t_i v_i q_i = \sum_i v_i (1 + t_i) q_i - 1 \quad (1)$$

where t_i is the ad valorem rate of taxation on good i .

Formally, the government maximizes utility by choosing quantities of goods subject to constraints above:

$$\begin{aligned} & \text{Max } U(q) \\ & \text{s.t. } D(q, u) = 1 \\ & \quad R = \sum_i v_i (1 + t_i) q_i - 1 \end{aligned} \quad (2)$$

The Lagrangean is

$$L = U(q) + \lambda [1 - D(q, u)] + \phi [R + 1 - \sum_i v_i (1 + t_i) q_i] \quad (3)$$

Differentiating with respect to q_i holding utility constant yields

³ See Deaton (1979), especially his property 1. The distance function can be considered a normalized money metric utility function. It is continuous in its arguments, is decreasing in u , and increasing, linearly homogeneous, and concave, first and second differentiable almost everywhere in q . In this economy, everything is normalized by income or expenditure. A similar dual approach has been surveyed in Auerbach (1985).

⁴ Vertical equity is how the tax burden varies across taxpayers of different means while horizontal equity implies how the tax burden varies across taxpayers of identical means.

$$t_i = \frac{\lambda + \phi}{\phi} \tag{4}$$

It gives immediately the Ramsey rule, i.e., the distortion by imposing taxes should be equiproportionate at the optimum, ignoring income effects, since the taxes on all goods are constant. Otherwise said, the Ramsey rule is that the proportional reduction in producers' price as a result of the imposition of the set of taxes should be the same for all goods.

Diagrammatically, figure 1 illustrates the tax problem in quantity space. With a revenue requirement, equilibrium at E on u_0 is reached. As seen in the figure, $D(u, q)$ is a scalar measure of the magnitude of q and is an measure of utility for fixed q .

If however leisure (q_1) is assumed to be untaxed, the Ramsey rule is modified as follows. Following Deaton again, we determine the demand of leisure as a function of u and q , i.e., $q_1 = \theta(q, u)$, where $q = (q_2, \dots, q_n)$. The Lagrangean is now

$$L = U(q, q_1) + \lambda [1 - D(q_1, q, u)] + \phi [R + 1 - \sum_{i=2}^n v_i (1 + t_i) q_i - v_1 q_1] \tag{5}$$

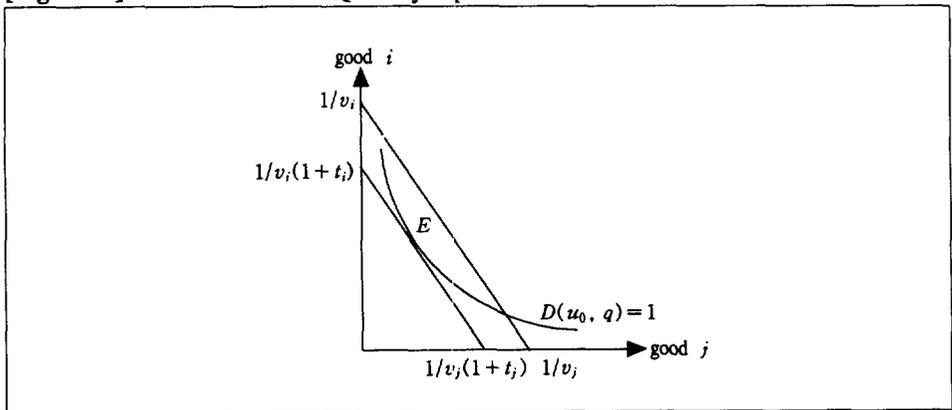
Differentiating with respect to q_i holding utility constant yields

$$t_i = - \left(\frac{\phi + \lambda}{\phi} \right) \left[1 + \frac{v_1}{v_i} \frac{\partial q_1}{\partial q_i} \right]$$

In a flexibility form, it can be rewritten as

$$t_i = - \left(\frac{\phi + \lambda}{\phi} \right) \left[1 + \frac{\partial \ln v_i / \partial \ln q_1}{\partial \ln v_1 / \partial \ln q_1} \right] \tag{6}$$

[Figure 1] Tax Problem in Quantity Space



This is the modified Ramsey rule with the untaxed good 1, leisure. It implies that goods relatively q-complementary with leisure should bear the higher tax rate since the form can be rewritten

$$t_i - t_j \propto \frac{\partial \ln v_i / \partial \ln q_1}{\partial \ln v_1 / \partial \ln q_1} - \frac{\partial \ln v_j / \partial \ln q_1}{\partial \ln v_1 / \partial \ln q_1} \propto \frac{\partial \ln (v_i / v_j)}{\partial \ln g_1} \tag{7}$$

The right-hand side is the effect of a change in leisure on the marginal rate of substitution between goods i and j along an indifference surface (MRS_{ij}). Thus it shows the intensity of substitutable interaction in their satisfaction of wants, which may be called a generalized substitutability.⁵

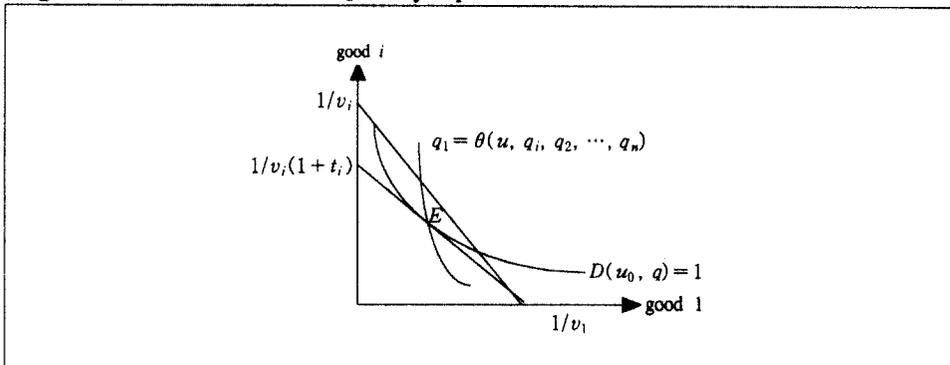
Diagrammatically, figure 2 illustrates the tax problem in quantity space with the untaxed good 1. With a revenue requirement, equilibrium at E on u_0 is reached as in figure 1. However, the government is restricted to a kind of the offer curve for good 1, leisure. It makes differences from figure 1, thus leading to the different solution.

As suggested by Deaton, we can write the marginal rate of substitution between goods i and j .

$$MRS_{ij} = v_i / v_j = u_i / u_j$$

for marginal utilities $u_i = \partial U / \partial q_i$. By using the ratio of marginal utilities between good i and j , we can decompose changes in MRS into those along an indifference curve and those out along a ray. This is analogous to the usual decomposition into income and substitution effects in the dual space. In this case, compensation is ensured by a proportional change in quantities. In particular, the effect of changes in leisure is

[Figure 2] Tax Problem in Quantity Space with the Untaxed Good



⁵ See Park (1997a) for more discussion.

$$\frac{\partial \ln MRS_{ij}}{\partial q_1} = \frac{\partial \ln(v_i/v_j)}{\partial q_1} + \frac{\partial \ln(v_i/v_j)}{\partial U} \frac{\partial U}{\partial q_1} \quad (8)$$

The first term in the right-hand side represents substitution effects, which are called generalized substitution effects in this paper. For it involves effects among three goods, not two goods. The second term represents (scale-) compensation effects, which are a proportional change in quantities. As a first attempt in this literature, we develop an interpretable inverse demand system that directly estimates q-substitution and compensation effects.⁶

III. THE OPTIMAL TAXATION IN AN INVERSE DEMAND SYSTEM

This section develops an inverse demand system having the generalized substitution effects. Its purpose is to estimate the q-substitution and scale-compensation effects directly without the inversion of the p-substitution and income effects. In addition, though it is natural to estimate demand responses in the behavior of individual consumers, price responses become at least as important in the behavior of markets as Hicks (1956) pointed out. The existing literature of the optimal taxation theory ignores completely this inverse demand case. It should be included in the literature since this case perfectly makes sense in some markets. To develop an inverse demand system with effects related to eq. (7), marginal rates of substitution are considered instead of normalized prices by changing coordinate systems to describe a consumption bundle (q^1) relative to reference bundle (q^0) using a scale expansion (s) from the reference bundle and a rotated shift in the relative proportion (r). In this coordinate system, s is defined as

$$s = D[U(q^0), q^1] \quad (9)$$

and r describes the new consumption proportion by a rotated shift:

$$r = \frac{q_i^1/q_j^1}{q_i^0/q_j^0} \quad (10)$$

In the two-good economy ($i=2$ and $j=1$), $s=r=1$ implies that $q^1=q^0$.⁷

Since any point in this coordinate can be described by an appropriate choice of r and s and there are $n-1$ independent marginal rates of substitution and $n-1$ independent quantity ratios, the marginal rate of substitution on the

⁶ See Deaton (1981). He describes how these effects can be estimated in direct demand systems by the inversion of the Slutsky substitution matrix. This paper extends it to inverse demand systems.

⁷ See Park (1997a) for diagrammatic exposition.

consumption bundle can be written as

$$MRS_{nj} = MRS_{nj}(q^0, s, r_{n1}, \dots, r_{n(n-1)}) \quad (11)$$

Taking the logarithm and totally differentiating the above, we have the changes in the $n-1$ marginal rates of substitution written as

$$d \ln MRS_{nj} = \mu_j d \ln s + \sum_{k=1}^{n-1} \beta_{jk} d \ln r_{nk} \quad (12)$$

where

$$\left. \frac{\partial \ln MRS_{nj}(q^0, s, r)}{\partial \ln s} \right|_{s=r=1} = \mu_j$$

$$\left. \frac{\partial \ln MRS_{nj}(q^0, s, r)}{\partial \ln r_{nk}} \right|_{s=r=1} = \beta_{jk}$$

$$r = [r_{n1}, \dots, r_{n(n-1)}]$$

The μ 's are the elasticities of the $n-1$ marginal rates of substitution with respect to scale and β 's are the elasticities reflecting pure substitution effects.

Following Park (1997a), β_{jk} can be written as⁸

$$\beta_{jk} = f_{kn}^* - f_{kj}^* = m_{nk}^f - m_{jk}^f \quad (13)$$

where f_{ij}^* denotes the price flexibility of good i with respect to good j and m_{ij}^f is the dual measure to the Morishima elasticity of substitution, i.e., the Morishima flexibility of substitution.⁹ The matrix, $[\beta_{jk}]$, is called as the generalized substitution matrix and interpreted that if β_{jk} is positive, then good n is more complementary to good k than good j is. If it is negative, then good n is more substitutable to good k than good j is. Thus it shows the intensity of substitutable/complementary interaction in their satisfaction of wants.

Suppose that we have three goods in the system. It then follows for $n=3$ that

$$\begin{bmatrix} d \ln(v_3/v_1) \\ d \ln(v_3/v_2) \end{bmatrix} = \begin{bmatrix} f_{13}^* - f_{11}^* & f_{23}^* - f_{21}^* \\ f_{13}^* - f_{12}^* & f_{23}^* - f_{22}^* \end{bmatrix} \begin{bmatrix} d \ln q_1 \\ d \ln q_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} d \ln s \quad (14)$$

From the above, it can be written in scalar form

$$d \ln(v_3/v_i) = \sum_{k=1}^2 \beta_{ik} d \ln q_k + \mu_i d \ln s \quad (15)$$

⁸ See Park (1997a) for details. His equations 37-43 yield the above expression.

⁹ See Blackorby and Russell (1989) for the Morishima elasticity of substitution.

If good 1 is assumed to be leisure, β_{21} in eq. (15) measures the intensity of complementarity to leisure among goods. Thus, in this system, the test on this parameter (β_{21}) directly gives which good should bear higher tax rate. It has some advantage over other demand systems. For example, the coefficient matrix is nonsingular since one equation is dropped. Another is that our estimate β_{21} directly tests $f_{13}^* - f_{12}^* + \text{error}$ whereas the separate estimates f_{13}^* and f_{12}^* in the other systems test $(f_{13}^* + \text{error}_1) - (f_{12}^* + \text{error}_2)$. Since the latter is associated with the covariance of $(\text{error}_1 - \text{error}_2)$ and the former with the variance of an error, the estimates in our system yield simpler and more accurate test results for complementarity to leisure.

The system can also recover all the own and cross price flexibilities using homogeneity conditions as in other demand systems. If we let

$$f_{11}^* = a, f_{12}^* = b, f_{13}^* = c, f_{23}^* = d, f_{22}^* = e, \quad (16)$$

we have by homogeneity that $a + b + c = 0$ and $b + e + d = 0$ so that

$$\beta_{11} = c - a, \beta_{12} = d - b, \beta_{21} = c - b, \beta_{22} = d - e. \quad (17)$$

It follows from the above that

$$\begin{aligned} f_{11}^* &= -1/3[2\beta_{11} - \beta_{21}], f_{12}^* = 1/3[\beta_{11} - 2\beta_{21}], f_{13}^* = 1/3[\beta_{11} + \beta_{21}], \\ f_{23}^* &= 1/3[\beta_{12} + \beta_{22}], f_{22}^* = -1/3[\beta_{11} - 2\beta_{21} + \beta_{12} + \beta_{22}] \end{aligned} \quad (18)$$

Thus we can recover all the price flexibilities in this system to compute optimal taxes.

IV. OPTIMAL TAXATION AND ITS MARGINAL WELFARE COSTS

In the previous section, our demand system gives direct estimates of the intensity of complementarity to leisure and thus can apply the optimal taxation rule to the commodity groups by calculating optimal tax rates. Another application of the optimal taxation theory is the area of studying the reform of a system of indirect taxes where the policy instrument is the tax on a particular good as specified in the tax laws. Stern (1987) and others have once suggested that the main use of optimal taxation theory may be for tax reform, and that policy should focus on tax reform. It is because predicting the effects of small changes from a given tax system requires only knowledge of the current position and derivatives of demand functions.

We will investigate the expected social cost of increasing revenue by changing the taxes appropriately, which is tax-induced inefficiencies. If, for example, financing of the public expenditure programs by taxes involves a welfare cost,

then comparison of welfare costs among various taxes gives the following tax reform rule: if the welfare cost of tax i is greater than that of tax j , welfare will be increased by lowering the tax rate t_i and by an offsetting increase in the tax t_j , so as to keep global tax revenue constant.¹⁰

We consider tax reforms that meet the two requirements:

$$\begin{aligned}\frac{\partial R}{\partial t_i} dt_i + \sum_k \frac{\partial R}{\partial t_k} dt_k &= 0, \\ \frac{\partial U}{\partial t_i} dt_i + \sum_k \frac{\partial U}{\partial t_k} dt_k &> 0\end{aligned}$$

In our balance-budget situation, the level of separable government expenditure is increased, and the level of taxation is increased in order to maintain government budget balance. The marginal welfare cost by the change in tax i (call it as MWC_i) may be defined as the difference between the change in consumer welfare ($\partial U/\partial t_i$) and the change in government revenue ($\partial R/\partial t_i$) where the change in consumer welfare is given by the equivalent variation (EV). Formally,

$$MWC_i = EV - dR \quad (19)$$

by the tax i .¹¹ When we consider inverse demands as market demands, quantity-constrained equivalent variation (QEV) would be relevant to the concept of the change in consumer welfare. We will see this in more details later.

If the practical application of the optimal taxation theory is dependent upon the specification of market condition, i.e., traditional demand or inverse demand systems used to estimate the reactions of the markets in question, the problem may be severe in the tax reform.¹² Thus, in this section, we examine part of answers to this interesting question theoretically. What matters for this question is whether the rankings of the different marginal welfare costs in the commodity groups by taxes will depend upon the market condition, ordinary demand systems or inverse demand systems.

In the traditional (ordinary) demand case, within our simple model of the economy with an indirect utility function, $V(t)$, and the government revenue, R ,

$$R = \sum_k t_k \cdot v_k \cdot q_k[v(t)]$$

¹⁰ See, for simple example, Vennemo (1994).

¹¹ Mayshar (1988) introduces this as a new measure of marginal welfare cost of the tax change.

¹² This problem is not a new one in other areas, especially in welfare measurement of consumer markets. See for example park (1997b). Even within a chosen market behavior, a choice of demand systems may affect the parameter values to evaluate proposed formula. One example is to choose the Rotterdam demand system or AIDS demand system in the traditional demand case. The empirical work by this choice may affect the parameter values and thus lead to different suggestion.

where $v(t) = [v_1(1+t_1), \dots, v_n(1+t_n)]$, the marginal welfare cost of the tax i can be specified as

$$MWC_i = \frac{\partial V(\cdot)}{\partial v_i} \frac{\partial v_i(t_i)}{\partial t_i} - \left[v_i q_i + \sum_k t_k v_k \frac{\partial q_k}{\partial v_i} \frac{\partial v_i(t_i)}{\partial t_i} \right] \tag{20}$$

The first term in eq. (20) indicates equivalent variation by the change in tax i . Since $\partial v_j / \partial t_i = 0$ for $j \neq i$, and $\partial v_i(t_i) / \partial t_i = v_i$, and $\partial V / \partial v_i = \lambda q_i$ by the first order conditions (λ denotes the marginal valuation of income), the equation (20) can then be rewritten as

$$MWC_i = \lambda v_i q_i - \left[v_i q_i + \sum_k t_k v_k \frac{\partial q_k}{\partial v_i} \frac{\partial v_i}{\partial t_i} \right] \tag{21}$$

where the change in the level of the indirect utility is assumed to measure the change in consumer welfare appropriately.¹³ A little numerical operation leads to an expression:

$$MWC_i = \left\{ \lambda \alpha_i - \left[\alpha_i + \sum_k t_k \alpha_k \epsilon_{ki} \right] \right\} \tag{22}$$

where α_i denotes the budget share of commodity i and ϵ_{ki} refers to the uncompensated price elasticity. If the MWC_i 's are ranked with $MWC_1 > \dots > MWC_n$, then a revenue-neutral change which increased t_n and reduced t_1 would generate a social gain of $MWC_1 - MWC_n$ per \$ of revenue switched. This would be the largest gain per \$ switched and it is the most desirable tax reform. Thus, the lower ranked alternatives may be preferable if only two taxes could be changed.¹⁴

If we assume that λ is equal to one, the marginal welfare costs of the taxes have only the third term:¹⁵

$$- \sum_k t_k \alpha_k \epsilon_{ki} \tag{23}$$

Since $\sum_k \alpha_k \epsilon_{ki} = -\alpha_i$ by Cournot aggregation¹⁶ and it is the tax-weighted sum of

¹³ In this case, the change in consumer welfare may be thought of as

$$EV = \lambda \Delta v_i(t_i) \cdot q_i = \lambda \Delta t_i \cdot v_i q_i$$

ignoring the first order approximation error which is called the Harberger triangle.

¹⁴ See Ahmad and Stern (1984) in this point.

¹⁵ One interesting point is that goods which have large budget shares and thus are relative necessities should bear the lower tax rates in order to reduce the marginal welfare. Those goods are usually relatively substitutable with leisure. This fact is implied by the Ramsey rule and would produce a progressive tax structure.

¹⁶ See Frisch (1959) for more details.

$\alpha_k \in \epsilon_{ki}$, all budget-weighted demand effects are summed with tax multipliers.¹⁷ An extreme case is that if there is only one good in this economy assuming λ equal to one, the marginal welfare cost is reduced to

$$MWC = \epsilon t \quad (24)$$

where ϵ refers to the uncompensated price elasticity. Eq. (24) shows the result provided by Browning's pioneering work.¹⁸ Another extreme case is that all tax rates are equal and thus the marginal welfare costs are simply budget shares times their tax rates:

$$MWC_i = \alpha_i t \quad (25)$$

Thus the 'rule of thumb' to evaluate marginal welfare costs may be

- 1) If the tax rates are very different from one another, then the marginal welfare costs may be affected strongly by the tax rates.
- 2) If the tax rates are similar, however, then the marginal welfare costs are influenced greatly by budget shares.
- 3) If the budget shares are close to one, the marginal welfare costs may be affected strongly by their own-price elasticities and tax rates.

We will see this point next by a simple example taken from Decoster and Schokkaert (1990). Decoster and Schokkaert estimate a differential form of the Rotterdam demand system using twelve aggregate commodities from the Belgian consumer expenditure survey 1978/1979 and the Belgian National Accounts in other context. Some commodity classification and the tax rates among those are

[Table 1] Commodity Classification, Expenditures, and Indirect Tax Rates

Commodity	Expenditures average consumer	Indirect tax rates (%)
Food	111,531	6.1741
Beverage	16,519	34.3665
Housing	12,340	7.1718
Transportation	72,001	32.0426

Note: Adapted from Decoster and Schokkaert (1990).

¹⁷ As mentioned in footnote 12, within a chosen market behavior, a choice of demand systems may affect the parameter values to evaluate proposed formula. Eq. (20) however may not be influenced heavily by the choice of systems since all the demand effects are summed as Ahmad and Stern (1984) suggest. Decoster and Schokkaert (1990) further suggest that the sensitivity to the specification of the demand system is less severe for a tax reform than it is for the computation of optimal tax rates and it however does not always lead to reliable results.

¹⁸ See Browning (1976)'s equation (6).

[Table 2] Ranking of Welfare Costs by the Rule of Thumb

Ranking	Commodities by the rule of thumb	Decoster and Schokkaert's ranking
1	Transportation	Transportation
2	Beverage	Beverage
3	Food	Food
4	Housing	Housing

Note: The third column is adapted from table 5 in Decoster and Schokkaert (1990).

shown in table 1.¹⁹

Without seeing their estimating demand responses, we applies the rule of thumb to rank the marginal welfare costs of taxation. Beverage and transportation have much higher tax rates than others and transportation has a much greater budget share, thus ranking transportation as first and beverage as second (applying rule of thumb 1 and then 2). Looking at food and housing, we can see that food has a dominant budget share and thus food may be ranked before housing (applying rule of thumb 2). Thus the ranking of welfare costs can be summarized as in table 2. Our ranking is the same as the Decoster and Schokkaert's, which possibly confirms our rule of thumb.

For the inverse demand case, we change the above setup as follows: the government revenue, R , is

$$\begin{aligned}
 R &= \sum_k v_k q_k (1 + t_k) - 1 & (26) \\
 &= \sum_k v_k(q) \cdot q_k(t_k) - 1
 \end{aligned}$$

where $q_k(t_k) = q_k(1 + t_k)$ and QEV is defined by²⁰

$$QEV = \sum_i \frac{\partial U(\cdot)}{\partial q_i} dq_i(t_i) = \lambda \int_{\Delta q_i} \sum_i v_i(q, u^1) dq_i = \lambda \sum_i v_i \Delta q_i \quad (27)$$

where the first order approximation error of the Harberger triangle, $1/2 \cdot \Delta v \cdot \Delta q$, is ignored for the simple description. QEV is the amount of additional (normalized) expenditure that would enable the consumer to maintain the new utility level u^1 while facing the initial quantities q^0 . When $q^1 < q^0$, QEV measures willingness to accept.

¹⁹ Among those twelve aggregate commodities, four commodities are chosen, which have characteristics of similar tax rates in pairs and much different ones between pairs, and of much different budget shares in pairs.

²⁰ See Park (1997b) for more details. The relationship between EV and QEV can be shown as

$$EV = -QEV + \sum_i q_i^1 \Delta v_i + \sum_i v_i^1 \Delta q_i$$

where $\Delta q_i = q_i^1 - q_i^0$. This relationship shows that EV of price changes may be obtained from QEV of quantity changes by allowing for some changes in expenditure.

In this economy, the government chooses the v 's so that the consequent quantities are such as to yield the necessary revenue. This line of reasoning will not make sense to the individual consumers. The market, however, may make either prices or quantities endogenous, thus making it sensible. In the infinitesimal change of the tax rate of commodity i , its marginal welfare cost may be expressed as

$$MWC_i = \lambda v_i \frac{\partial q_i(t_i)}{\partial t_i} - \left[v_i q_i + \sum_k \frac{\partial v_k}{\partial q_i} \frac{\partial q_i(t_i)}{\partial t_i} q_k (1 + t_k) \right] \tag{28}$$

A numerical operation multiplying v_i leads to an expression:

$$\begin{aligned} MWC_i &= \left[\lambda \alpha_i - \left[\alpha_i + \sum_k \alpha_k (1 + t_k) \int_{ki} \right] \right] \\ &= \left[\lambda \alpha_i - \sum_k \alpha_k t_k f_{ki} \right] \end{aligned} \tag{29}$$

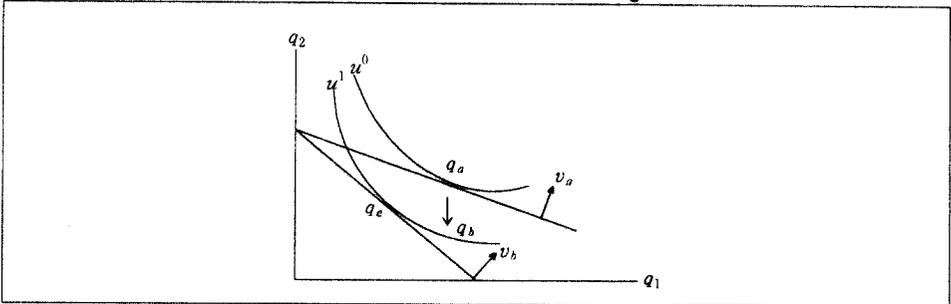
In this computation, we have used that $\sum_k \alpha_k f_{ki} = -\alpha_i$. If we assume that λ is equal to one, the marginal welfare costs of the taxes are

$$\alpha_i - \sum_k t_k \alpha_k f_{ki}. \tag{30}$$

One interesting point here is that budget shares have stronger effects on the marginal welfare costs of the taxes than those in the traditional demand case.²¹

We may then ask how large is the difference between the price elasticities and price flexibilities. This difference will lead to different marginal welfare costs of taxes on alternative market conditions, i.e., traditional demands and inverse demands. The connection between price elasticities and price flexibilities is not so close as it first appears. For the help of reader's understanding of price flexibilities, the economic meaning of the slope of uncompensated inverse

[Figure 3] Diminishing marginal valuation in normal goods



²¹ Relative necessities usually have large budget shares and thus their marginal welfare costs may be large without knowledge of others.

demand curves is illustrated in Figure 3. At q_a , a point of consumer equilibrium, the relative prices are represented by a vector v_a , normal to the indifference curve. An increase in the price of q_1 gives the price line v_b , and equilibrium moves to q_e . The price effect on the uncompensated ordinary demand for q_1 is given by the horizontal distance between q_a and q_e . Thus, the rise in the price of q_1 results in a decrease in the quantity (q_1), which shows the normality of q_1 . Now we might be interested in the effect on v_1 of an exogenous change in q_2 available for consumption. A decrease in q_2 reflected in the vertical move from q_a to q_b would lead to a decrease in the relative price of q_1 . This implies an increase in the relative price of q_2 . Moreover, the vertical intercept of the price line moves downward. Hence, for given nominal income, the fall in the quantity of q_2 will result in an absolute increase in the price of q_2 . This reveals a negatively sloped uncompensated inverse demand curve, i.e., $f_{22} < 0$. The same exercise for q_1 will show $f_{11} < 0$. The following will show the link between elasticities and flexibilities in the demand system.

Let $\epsilon = [\epsilon_{ij}]$ be the matrix of price elasticities of uncompensated ordinary demands, i.e., $q = h(p, m)$

$$\epsilon_{ij} = \frac{\partial \ln h_i(p, m)}{\partial \ln p_j}.$$

Then, for the two good case

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix}.$$

We define $F = [f_{ij}]$ be the matrix of uncompensated price flexibilities of inverse demand. Since the ordinary demand system can be written that $d \log(q) = \epsilon d \log(p) + \eta d \log(m)$, the inverse demand system could be that $d \log(p) = \epsilon^{-1} d \log(q) - \epsilon^{-1} \eta d \log(m)$, the matrix F can be obtained by inversion of ϵ :

$$\begin{aligned} F &= \epsilon^{-1} \\ &= \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\epsilon_{22}}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21}} & \frac{-\epsilon_{12}}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21}} \\ \frac{-\epsilon_{21}}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21}} & \frac{\epsilon_{11}}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21}} \end{bmatrix} \end{aligned} \tag{31}$$

It follows that the price flexibilities get involved with all the cross effects of gross substitution and complementarity. As in the study of Houck (1965), if all

cross effects are zero, then f_{ii} will equal $1/\varepsilon_{ii}$. In this case, the off-diagonal terms become zero so that the matrix is diagonal, which is not interesting case.

What is more, the empirically estimated flexibilities are much different from the inverted matrix of directly estimating price elasticities. The empirical example in Huang (1994) pursues a suggestion that inverting a matrix of elasticities to obtain measures of flexibilities or vice versa does not lead to the same figures as those estimated directly. Huang estimates a differential form ordinary demand and inverse demand systems using U.S. quarterly meat data from 1970-1990. The meat data consist of high-quality beef (beefh), manufacturing-grade beef (beefm), pork, and broilers. His empirical result is summarized in table 3. By inverting the matrix of directly estimating price elasticities in table 3, we present inverted flexibilities in table 4.

For manufacturing-grade beef, the inverted flexibility (-2.57) suggests price-inelastic demand while the estimated flexibility (-0.22) implies price-elastic demand. A more dramatic example is broiler, for which the inverted flexibility (-4.71) can be greatly different from the estimated flexibilities (-0.77). This example illustrates what can be striking numerical differences between estimated flexibilities and inverted flexibilities.

[Table 3] Directly Estimated Elasticities and Flexibilities

	Beefh	Beefm	Pork	Broiler
Estimated Elasticities				
Beefh	-1.03	-0.14	-0.19	-0.13
Beefm	0.11	-0.40	0.48	0.07
Pork	-0.01	0.04	-0.83	-0.07
Broiler	0.11	0.04	0.04	-0.19
Estimated Flexibilities				
Beefh	-0.63	-0.10	-0.03	-0.01
Beefm	-0.56	-0.22	-0.01	-0.01
Pork	-0.21	-0.02	-0.80	-0.04
Broiler	-0.61	-0.11	-0.36	-0.77

Note: Adapted from Huang (1994).

[Table 4] Inverted Flexibilities

	Beefh	Beefm	Pork	Broiler
Inverted Flexibilities				
Beefh	-0.86	0.43	0.48	0.56
Beefm	-0.28	-2.57	-1.42	-0.21
Pork	0.04	-0.12	-1.25	0.40
Broiler	-0.53	-0.33	-0.30	-4.71

As noted in Huang, elasticity and flexibility matrices obtained from any well-known procedure are not the reciprocal of one another because the two sets of regression lines in the demand system estimation differ from one another. In a traditional demand system, the sum of residuals is minimized along the quantity axis, while an inverse demand system minimizes the sum of residuals along the price axis. Therefore, it is not proper to use the inverted flexibility measurements in the case of inverse demand economy but the flexibilities from a directly estimated inverse demand should be used to assess the price effects of quantity changes by taxes in a market sense.

As a simple illustration of tax reform, using the results in table 3 and 4, we compare marginal welfare costs by elasticities in the traditional demand case with those by flexibilities in the inverse demand case to see how the ranking of commodity can be affected in the tax reform. In addition, the result using inverted flexibilities from elasticities is presented as well. It would not be very realistic because only meat expenditure is considered among total consumer expenditure. It will however give some suggestion for guiding tax reform as simple as possible. In order to do this, we assume that all indirect tax rates among these commodities are equal. One reason may be that they are in the same commodity group, i.e., meat, from the overall budget group.

[Table 5] Commodity, Expenditure Shares, and Indirect Tax Rates

Commodity	Shares in Meat Expenditure (%)	Indirect tax rates (%)
Beefh	22.6	6.1741
Beefm	22.6	6.1741
Pork	29.6	6.1741
Broiler	25.2	6.1741

Note: Figures in column 2 are adapted from Heien (1982), in which the shares of beef, pork, and broiler are 0.452, 0.296, and 0.252, respectively and the share of beef is divided into beefh and beefm equally. The figures in column 3 stem from Decoster and Schokkaert (1990) for food classification.

[Table 6] Ranking of Marginal Welfare Costs by the Estimating Elasticities, Flexibilities, and Inverted Flexibilities

Ranking of MWC	Commodity by Elasticities	Commodity by Flexibilities	Commodity by Inverted Flexibilities
1	Beefh (0.1131)	Beefh (0.5253)	Broiler (0.8629)
2	Pork (0.1050)	Pork (0.5038)	Pork (0.7023)
3	Beefm (0.0618)	Broiler (0.3819)	Beefm (0.5979)
4	Broiler (0.0507)	Beefm (0.2914)	Beefh (0.4602)

Note: Figures in the brackets denote marginal welfare costs computed by the formula in eq. (23) and (30). Column 2 is calculated by eq. (23) and calculation of column 3 and 4 is based on eq. (30).

Another is that these are subject to the similar indirect tax rates in reality. Table 5 reflects this assumption and represents expenditure shares within this group. With this assumption, the marginal welfare costs in the traditional demand case are computed based on eq. (23) while those in the inverse demand case are calculated based on eq. (30).

The second column in table 6 reports the computation results in parentheses and ranking of commodities in the traditional demand case. The third column in table 6 represents the computation results in parentheses and ranking of commodities in the inverse demand case while the fourth column shows those using flexibilities which are obtained by inverting the matrix of directly estimating price elasticities. As seen in table 6, rankings of commodities are not much different between those using estimating elasticities and those using estimating flexibilities. For example, the first and the second ranking commodities are the same in both cases. Those using inverted flexibilities, however, are quite different from others since the last ranking commodity in the traditional demand case becomes the first ranking. It suggests that the inverted flexibilities should not be used in the inverse demand case and that the inverse demand case may change the result of tax reform though not in a great amount. It however deserves some attention and may be more adequate for commodities like food and rents and others. It is not necessarily more restrictive to extend the inverse demand case to every market than to systematically adopt the traditional demand case as is usually done.

V. CONCLUDING REMARKS

In this paper, optimal commodity taxation theory is reformulated based on the framework proposed by Deaton (1979) for the preference structure. The distance function approach has been applied to the Ramsey Rule and the results are well known. However, this approach greatly simplifies the analysis as shown in section 2. It enables the change in the marginal rate of substitution between goods to be broken into scale effects and usual substitution effects in quantity space.

We have also considered the optimal taxation in an inverse demand system, which is thought to be useful to modern econometricians. An inverse demand system was presented as a first attempt in the literature, which is able to easily interpret the meaning of the Ramsey rule. The idea basically comes from Deaton, but we have extended it to the directly estimable demand system. In a way, this insight has been lost in the optimal taxation literature but this case perfectly makes sense in some markets like food, rents, housing, and others. It seems not much more restrictive to extend the inverse demand case to every market than to systematically adopt the traditional demand case. For one case is not more perfect than the other and thus inverse demand case deserves some attention even in the optimal taxation literature.

Furthermore, we have inferred the dependence of theoretical results of the optimal taxes on the specification of market condition, i.e., an ordinary demand system or an inverse demand system. Following Fullerton (1991), the marginal welfare costs of taxes in an ordinary demand system is compared with those in an inverse demand system using Huang's (1994) empirical work. It suggests that reform of optimal taxation be influenced though not in a great amount by the choice of market condition, which specifies either traditional demand systems or inverse demand systems. Although Deaton (1981) and others have suggested that empirical work directed towards providing parameters for evaluating optimal tax formulae should employ the traditional demand case and then obtain flexibilities, if necessary, by inverting matrices of elasticities, there are several problems in their suggestion. One problem among others is that elasticity and flexibility matrices are not the reciprocal of one another as shown in section 4. Since the main objective of this paper is methodological, the empirical issues are left in future research.

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