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ECONOMIC GROWTH AND FLUCTUATIONS WITH THE ENDOGENOUS LENGTH OF BUSINESS CYCLES

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This paper introduces a simple endogenous growth model in which investment in capital and investment in research generate both long run growth and business fluctuations. The main implication of this paper embodies Schumpeter's insight to economic development: economic development takes the form of a sequence of business cycles, each being a response to a discontinuous innovation. The model characterizes comovements, volatility and lagged reactions among aggregate variables which are linked to the endogenous length of cycle.

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I. INTRODUCTION

Recently, a new approach has emerged to model Schumpeters insights of technological change to explain economic growth and fluctuations. While traditional Solow growth models assume that the technology progresses are given exogenously, this new approach endogenizes innovations in general equilibrium models assuming innovations are introduced through R&D effort determined by utility-maximizing (or profit-maximizing) agents. But it should be noted that the existing models of this approach have not considered the process of capital accumulation, which limits their ability to explain the dynamic fluctuation in business cycle.[1] To overcome this limitation, this paper presents a model in

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[1] Most existing growth models with R&D expenditure do not consider the trade-off between capital accumulation and technological progress by taking labor as the only input to the R&D progress. (e.g. Aghion and Howitt(1992), Cheng and Dinopoulos(1992), Grossman and Helpman (1991), Segerstrom(1991), Parante(1994) and Fan(1995)

which two kinds of investments(investment in physical capital and investment in research or infrastructure) generate both long-run growth and fluctuations. Both forms of investment are accumulated over time, compete for household savings, and are jointly determined along with the optimal consumption path. However the two types of investment have different characteristics: investment in physical capital can be immediately put to use to increase productivity capacity, while investment in research or infrastructure pays off only when the research or infrastructure project is completed. Technological improvement is characterized by both lumpiness and discreteness in this paper. This lumpiness implies that a costly process must be completed before any measurable benefits in output can be realized. The length of that process depends on the intensity of the research or infrastructure project. Projects that require a large investment of resources over long time in order to increase the productivity of capital can be exemplified in both R&D and infrastructure contexts. Space research, satellite programs and major medical research are examples of prolonged and costly R&D. Infrastructure examples include the installation of wide systems of communication(telegraph, telephone, internet) or electricity transmission. Many have suggested that these huge investment projects might be responsible for cycles in aggregate economic activity. Therefore this model tries to characterize comovements, volatility and lagged reactions among aggregate variables which are linked to the endogenous length of cycle.

Recent studies such as Jovanovic(1996) and Greenwood and Yorukoglu(1996) have examined the firms behavior related to adopting new technologies. These studies take the arrival and the size of new technologies as exogenous and examine the cyclical implications for productivity of firm-specific costs of adoption. This paper focuses instead on the implications for growth and fluctuations resulting from developing new technology. This paper therefore contributes to the analysis of fluctuations induced by the R&D activity.[2)]

This paper is organized as follows. In part II, the functions used in the model are explained. In part III, movements of economic variables in social optimum are examined. In part IV, the comparative statics show how the length of cycles interacts as some parameter values change.

II. DESCRIPTION OF THE MODEL

The utility function u(C) is twice continuously differentiable, increasing and concave, with $\lim_{C \to 0} u'(C) = \infty$. Utility is discounted exponentially over time at the subjective discount rate ρ. I will assume the constant elasticity of substitution

[2] Bental and Peled(1996) examined the cyclical implications of the trade-off between the size of the capital stock and the intensity of research activity. But their model assumes that the allocation of resources between R&D and production is separate from the saving decision by household.

utility function, $u(C)=\frac{1}{1-\rho}C^{1-\rho}$ for $\sigma\neq1$ and $u(C)=\ln(C)$ for $\sigma=1$.

The amount produced at t units of time after the j^{th} innovation, $Y_j(t)$, is a function of A_j, the technology available during the j^{th} cycle, and $K_j(t)$, the capital stock available at t units of time after the j^{th} innovation. The production function takes the Cobb-Douglas, constant returns to scale form,[3] $Y_j(t)=f(A_j, K_j(t))=A_j^{\alpha}K_j(t)^{1-\alpha}$.

The capital stock is accumulated through investment in capital, $I_j(t)$. The initial capital stock is given and a constant fraction of the capital stock is depreciated at each instant. Then the law of motion for capital accumulation within the j^{th} cycle is $K_j(t)=I_j(t)-\delta K_j(t)$.

Technology advances occur in a series of discrete steps since they need a certain amount of knowledge accumulation. The j^{th} technology available to the economy, A_j, is γ times more effective than the previous technology, or $A_j=\gamma A_{j-1}$ with $\gamma>1$. The initial level of technology, A_1, is positive and given. Knowledge is accumulated through investment in research. Investment in research at t units of time after the j^{th} innovation, $D_j(t)$, produces $h\left(\frac{D_j(t)}{A_j}\right)$ units of knowledge. The function h(.) has decreasing returns in the technology-adjusted research expenditure, $\frac{D_j(t)}{A_j}$. For better tractability of the model, I will assume $h\left(\frac{D_j(t)}{A_j}\right)=\frac{1}{1-\emptyset}\left(\frac{D_j(t)}{A_j}\right)^{1-\emptyset}$ for $\emptyset\neq1$ and $h\left(\frac{D_j(t)}{A_j}\right)=\ln\left(\frac{D_j(t)}{A_j}\right)$ for $\emptyset=1$. Let $H_j(t)$ denote the units of knowledge accumulated at t units of time after the jth innovation. Then the law of motion for knowledge accumulation is $H_j(t)=h\left(\frac{D_j(t)}{A_j}\right)$. When the economy at technology A_j accumulates H* units of knowledge, it reaches the new technology A_j+1.

I assume a small open economy where the economy can borrow and lend freely abroad at a constant world interest rate r. The change of foreign assets h olding, $\dot{S}(t)$, within the j^{th} cycle is the difference between total income (output and interest income) and total expenditure (consumption, investment in capital and in research) $\dot{S}_j(t)=Y_j(t)+rS_j(t)-C_j(t)-I_j(t)-D_j(t)$.

Finally, the budget constraint at the time of innovations in a stationary equilibrium is $S_j(0)+K_j(0)=S_{j-1}(t^*_{j-1})+K_{j-1}(t^*_{j-1})$ equivalently $K_j(0)-K_{j-1}(t^*_{j-1})=-[S_j(0)-S_{j-1}(t^*_{j-1})]$ where t^*_j is the optimally determined terminal time for the jth cycle. This budget constraint shows that the net change in stocks must be equal to zero in the whole economy. One can instantly increase the stock of capital by the exact amount of the change in foreign

[3] I assume that output is distributed to technology and capital. This assumption implies that a constant share (α) of output accrues to technology-holder. This share plays an incentive for research effort and makes technology advances as an outcome of profit maximizing behavior, which plays a critical role in the endogenous growth models.

assets holding.

Ⅲ. SOCIAL OPTIMUM

There exists the representative agent living infinitely in continuous time. His preferences are given by the discounted utility function $\int_{t=0}^{\infty} e^{-\rho t} u(C(t))dt$. This section examines the social optimum, which is the maximum discounted utility of the representative agent. Although time is continuous in the model, technological advances take place in discrete steps. Therefore the analysis of the social optimum involves a combination of discrete and continuous time maximization.

3.1. Investment in Capital and Output

The representative agent has a choice of two assets. One is the domestic physical capital whose return is marginal product of capital minus depreciation $(f_K-\delta)$ and the other is foreign assets whose return is the world interest rate (r). Within the j^{th} cycle, the representative agent holds the capital stock enough to keep the two rates of return equal. If $(f_K-\delta)>r$, then the representative agent sells the foreign assets(or capital inflow occurs) and invests in the domestic capital. If $(f_K-\delta)<r$, then the opposite situation will happen. Therefore, arbitrage requires that

$$f_K-\delta=(1-\alpha)A_j^{\alpha}K_j(t)^{-\alpha}-\delta=r \tag{1}$$

Solving for $K_j(t)$ in equation (1) yields constant amount of capital and investment within the cycles.

$$K_j(t)=A_j\left(\frac{1-\alpha}{r+\delta}\right)^{1/\alpha}=K_j \text{ for } t\in[0,t_j^*] \tag{2}$$

$$I_t(t)=\delta K_j(t)=\delta A_j\left(\frac{1-\alpha}{r+\delta}\right)^{1/\alpha}=I_j \text{ for all } j \text{ and for } t\in[0,t_j^*] \tag{3}$$

At the time of the $j+1^{th}$ innovation, the domestic rate of return on capital jumps to above the world interest rate due to the increase in A. Therefore, at the time of innovation, the representative agent invests an additional, $\triangle K_j$, enough to equalize the two rates of return.

$$\triangle K_j=K_j(0)-K_{j-1}(t_{j-1}^*)=K_j-K_{j-1}=\gamma K_{j-1}-K_{j-1}=(\gamma-1)K_{j-1} \tag{4}$$

The above results are summarized in Property 1.

Property 1. Capital discontinuously rises at the beginning of cycles while remaining constant within the cycle. It grows at the rate γ from cycle to cycle. Investment in capital also grows at the rate γ from cycle to cycle.

Proof of Property 1. From equations (2), (3) and (4),

$$I_{j+1}(t)=\delta K_{j+1}=\delta\gamma K_j=\gamma I_j(t) \text{ for } t\in[0,\ t^*_{j+1}]$$
$$\triangle K_{j+1}=(\gamma-1)K_j=(\gamma-1)\gamma K_{j-1}=\gamma\triangle K_j$$

Corollary 1. Output rises discontinuously at the rate γ at the time of innovations while remaining constant within the cycle.

Proof of Corollary 1. Output can be analyzed along with investment. Using equation (2), outputs in the j^{th} and the $j+1^{th}$ cycles are

$$Y_{j+1}(t)=A^{\alpha}_{j+1}K^{1-\alpha}_{j+1}=(\gamma A_j)^{\alpha}(\gamma K_j)^{1-\alpha}=\gamma Y_j(t) \text{ for } t\in[0,\ t^*_{j+1}] \quad (5)$$

Given the optimal investment strategy just described, the representative agent chooses investment in research, consumption and foreign assets holding to maximize his utility. The optimization problem becomes

$$\text{Max } \int_0^{\infty} e^{-\rho\tau}u(C_j(\tau))$$
$$\text{or Max } \int_0^{t^*_j} e^{-\rho\tau}u(C_j(\tau))d\tau+e^{-\rho t^*_j}\int_0^{\infty} e^{-\rho\tau}u(C_{j+1}(\tau))\,d\tau \quad (6)$$

subject to

$$\dot{H}_j(t)=\frac{1}{1-\varnothing}\left(\frac{D_j(t)}{A_j}\right)^{1-\varnothing} \text{ where } H_j(0)=0 \text{ and } H_j(t^*_j)=H^* \quad (7)$$
$$\dot{S}_j(t)=Y_j+rS_j(t)-C_j(t)-I_j-D_j(t) \quad (8)$$
$$S_j(0)-S_{j-1}(t^*_{j-1})=-(K_j-K_{j-1}) \quad (9)$$

K_j and Y_j are given.

I divide the above maximization problem into two parts: within-cycle problem and across-cycle problem. The within-cycle problem examines the fluctuations of $C_j(t)$, $D_j(t)$ and $S_j(t)$ within a cycle assuming its length, initial and terminal values for foreign assets holding are given by t^*_j, $S_j(0)$ and $S_j(t^*_j)$ respectively. An optimal control model is used to solve the within-cycle problem. The across-cycle problem examines the optimal conditions for t^*_j and $S_j(t^*_j)$ as well as the fluctuations of $C_j(t)$ and $D_j(t)$ at the time of innovation. A dynamic programming model is used to solve the across-cycle problem.

3.2. Within-Cycle Problem

A representative agent chooses $C_j(t)$, $D_j(t)$ and $S_j(t)$ to maxi ize is utility over the j^{th} cycle, assuming that the values for t_j^*, $S_j(0)$ and $S_j(t_j^*)$ are given. Then his problem becomes

$$\text{Max} \int_0^{t_j^*} e^{-\rho\tau} u(C_j(\tau))d\tau$$

subject to

$$\dot{H}_j(t) = \frac{1}{1-\varnothing}\left(\frac{D_j(t)}{A_j}\right)^{1-\varnothing} \text{ where } H_j(0)=0 \text{ and } H_j(t_j^*)=H^*. \tag{10}$$

$$\dot{S}_j(t) = Y_j + rS_j(t) - C_j(t) - I_j - D_j(t) \text{ where } S_j(0) \text{ and } S_j(t_j^*) \tag{11}$$

are given.

The Hamiltonian function associated with this is

$$\Psi(C_j(t), D_j(t), S_j(t), H_j(t)) = e^{-\rho t}u(C_j(t)) + \lambda_{1j}(t) \tag{12}$$
$$[Y_j + rS_j(t) - C_j(t) - I_j - D_j(t)] + \lambda_{2j}(t)\left[\frac{1}{1-\varnothing}\left(\frac{D_j(t)}{A_j}\right)^{1-\varnothing}\right]$$

The FOCs are

$$\frac{\partial\Psi}{\partial C_j(t)} = e^{-\rho t}u'(C_j(t)) - \lambda_{1j}(t) = e^{-\rho t}C_j(t)^{-\sigma} - \lambda_{1j}(t) = 0 \tag{13}$$

$$\frac{\partial\Psi}{\partial D_j(t)} = -\lambda_{1j}(t) + \lambda_{2j}(t)\left[\frac{1}{A_j}\left(\frac{D_j(t)}{A_j}\right)^{-\varnothing}\right] = 0 \tag{14}$$

$$\dot{\lambda}_{1j}(t) = -r\lambda_{2j}(t) \tag{15}$$

$$\dot{\lambda}_{2j}(t) = 0 \tag{16}$$

Eliminating $\lambda_{1j}(t)$ and $\lambda_{2j}(t)$ from equation (14) gives

$$\frac{\dot{D}_j(t)}{D_j(t)} = \frac{r}{\varnothing} \text{ or } D_j(t) = D_j(0)e^{\frac{r}{\varnothing}t} \tag{17}$$

Eliminating $\lambda_{1j}(t)$ from equation (13) gives

$$\frac{\dot{C}_j(t)}{C_j(t)} = \frac{r-\rho}{\sigma} \text{ or } C_j(t) = C_j(0)e^{\frac{r-\rho}{\sigma}t} \tag{18}$$

The difference in the growth rates of consumption and research expenditure comes from two sources: (i) difference in σ and $\varnothing$, the relative curvatures of the functions representing utility and knowledge production, and (ii) difference in the rate of time discount applied to those two functions. Note that if the curvatures are same for both functions ($\sigma=\varnothing$), research grows faster than consumption over the cycle (by the constant ρ/σ) because no rate of time discount is applied to knowledge, in contrast to the time discount utility from consumption.

For the future purpose, I calculate investment in research at the beginning of the j^{th} cycle, $D_j(0)$. Using equation (17), we can integrate equation (10) and solving for $D_j(0)$ gives

$$D_j(0)=H^{*\frac{1}{1-\varnothing}}\left\{(1-\varnothing)\frac{1-\varnothing}{\varnothing}\right\}^{\frac{1}{1-\varnothing}}\left(e^{\frac{1-\varnothing}{\varnothing}rt_j^*}-1\right)^{-\frac{1}{1-\varnothing}}A_j \tag{19}$$

3.3. Across-Cycle Problem

In this section, the optimality conditions for $S_j(t_j^*)$ and t_j^* are derived. Note that they are given in the within-cycle problem. The life-time utility maximization problem can be expressed as a sequence of within-cycle problems as follows

$$\begin{aligned} V(S_j(0)) &= Max \int_0^{\infty} e^{-\rho\tau}u(C_j(\tau))d\tau \\ &= Max \int_0^{t_j^*} e^{-\rho\tau}u(C_j(\tau))d\tau+e^{-\rho\tau}u(C_{j+1}(\tau))d\tau \\ &= \max\ \{\omega(S_j(0),S_j(t_j^*),t_j^*)+e^{-\rho t_j^*}V(S_{j+1}(0))\} \end{aligned} \tag{20}$$

subject to

$$S_{j+1}(0)=S_j(t_j^*)-(K_{j+1}-K_j) \tag{21}$$

This is a dynamic programming model with a state variable, $S_j(0)$, and two control variables, t_j^* and $S_j(t_j^*)$. The function $\omega(S_j(0),S_j(t_j^*),t_j^*)$ denotes the maximized discounted utility over the jth cycle.[4] Equation (21), which is equivalent to the equation (9), is the transition equation for the state variable. The optimality conditions for t_j^* and $S_j(t_j^*)$ are

[4] See Appendix 1 for the derivation of function in terms of a state variable and two control variables.

$$\frac{\partial}{\partial t_j^*}\omega(S_j(0),\ S_j(t_j^*),\ t_j^*) - \rho e^{-\rho t_j^*} V(S_{j+1}(0)) = 0 \tag{22}$$

$$\frac{\partial}{\partial S_j(t_j^*)}\omega(S_j(0),\ S_j(t_j^*),\ t_j^*) + e^{-\rho t_j^*}\frac{\partial S_{j+1}(0)}{\partial S_j(t_j^*)} V(S_{j+1}(0)) = 0 \tag{23}$$

Envelope theorem implies

$$\frac{\partial}{\partial S_j(0)} V(S_j(0) = \frac{\partial}{\partial S_j(0)}\omega(S_j(0),\ S_j(t_j^*),\ t_j^*) \tag{24}$$

Equation (23), with envelope theorem and the differentiation of equation (21), becomes

$$\frac{\partial}{\partial S_j(t_j^*)}\omega(S_j(0),\ S_j(t_j^*),\ t_j^*) + e^{-\rho t_j^*} \frac{\partial}{\partial S_{j+1}(0)}\omega(S_{j+1}(0),\ S_{j+1}(t_{j+1}^*),\ t_{j+1}^*) = 0 \tag{23'}$$

Property 2. Consumption is continuous despite of discontinuous technology improvements.

Proof of Property 2. Consumption grows continuously at the rate of $\frac{r-\rho}{\sigma}$ while output and investment in capital jump discontinuously at the time of innovations. Property 2 implies that any amount of $S_j(t_j^*)$ can be an optimal solution as long as it makes consumption grow continuously over cycles. See Appendix 2 for the detailed proof.

Corollary 2. The growth rate of consumption $\left(\frac{r-\rho}{\sigma}\right)$ is bounded above.

Proof of Corollary 2. The objective functional, $\int_0^\infty e^{-\rho\tau}u(C(\tau))d\tau$, should be bounded above. If it diverges, there may exist more than one consumption path that yield an infinite value for the objective functional. To avoid this, some restrictions on the parameters affecting the growth rate on consumption are necessary.[5] Since consumption grows continuously

$$\begin{aligned}\int_0^\infty e^{-\rho\tau}u(C(\tau))d\tau &= \int_0^\infty e^{-\rho\tau}\frac{1}{1-\sigma}\left(C_0 e^{\frac{r-\rho}{\sigma}\tau}\right)^{1-\sigma} d\tau \\ &= \frac{C_0^{1-\sigma}}{1-\sigma}\ \frac{1}{\frac{r-\rho}{\sigma}(1-\sigma)-\rho}\left| e^{[\frac{r-\rho}{\sigma}(1-\sigma)-\rho]\tau}\right|_0^\infty\end{aligned}$$

[5] For the detailed explanation of this issue, see chapter 5 in Chiang(1992). Jones and Manuelli (1990) discussed this condition in the discrete time model.

The objective functional converges to a finite value only if

$$\frac{r-\rho}{\sigma}(1-\sigma)-\rho<0 \text{ equivalently } \frac{r-\rho}{\sigma}<\frac{\rho}{1-\sigma} \qquad Q.E.D.$$

To examine the optimality condition for t_j^*, we have to solve equation (22). Unfortunately it is very difficult to derive value function V explicitly in terms of t_j^*. Therefore it is necessary that across-cycle problem should be transformed into a tractable structure. After some calculations,[6] the maximization problem is transformed

$$W_1(t_1^*, t_2^*, t_3^*, ...)=\max_{t_1^*}\{w_1(t_1^*)+e^{-rt_1^*}W_2(t_2^*, t_3^*, ...)\} \tag{25}$$

w_j is the net wealth during the j^{th} cycle which is the discounted value of net output $(Y_j-D_j(t)-I_j)$ minus the discounted value of the additional capital demand for the next cycle, $(K_{j+1}-K_j)$. W_j is the net wealth at the beginning of the j^{th} cycle. Equation (25) means that the optimality condition for t_j^* maximizing the discounted life-time utility is equivalent to that maximizing the discounted life-time net wealth, $w_j(t_j^*)+e^{rt_j^*}W_{j+1}(t_{j+1}^*, t_{j+2}^*, ...)$. The FOC for this problem is

$$\frac{\partial}{\partial t_1^*}w_1(t_1^*)-re^{rt_1^*}W_2(t_2^*, t_3^*, ...)=0 \tag{26}$$

The representative agent has the two opposite wealth effects by extending the length of cycle. The first term of equation (26) means that the longer cycle gives him extra wealth from the current cycle. The second term of equation (26) means that the longer cycle gives him less discounted wealth from the future cycles. To make this optimality condition tractable, we will concentrate on the stationary equilibrium in which each cycle has the same length, $t_j^*=t^*$ for $\forall j$.

Property 3. In the stationary equilibrium in which $t_j^*=t^*$ for $\forall j$,

(i) $D_{j+1}(t)=\gamma D_j(t)$ for all j and (ii) $W_2(\cdot, \cdot, \cdot, ...)=\dfrac{\gamma}{1-\gamma e^{-rt^*}}w_1(t^*)$

Proof of Property 3

(i) From equation (17), $D_{j+1}(t)=D_{j+1}(0)e^{\frac{\gamma}{\sigma}t}$. Also from equation (19), $D_{j+1}(0)$

[6] See Appendix3 for the detailed explanation.

$= \gamma D_j(0) = t^*$ for $\forall$j. Therefore, $D_{j+1}(t) = \gamma D_j(t)$ for all j.

$$\text{(ii)}\ W_2(\cdot, \cdot, \cdot, ...) = \left[\int_0^{t^*}(Y_2 - D_2(\tau) - I_2)e^{r\tau}d\tau - e^{-rt^*}(K_3 - K_2)\right]$$
$$+ e^{-rt^*}\left[\int_0^{t^*}(Y_3 - D_3(\tau) - I_3)e^{-r\tau}d\tau - e^{rt^*}(K_4 - K_3)\right] + ...$$
$$= \gamma\left[\int_0^{t^*}(Y_1 - D_1(\tau) - I_1)e^{r\tau}d\tau - e^{rt^*}(K_2 - K_1)\right]$$
$$+ \gamma^2 e^{-rt^*}\left[\int_0^{t^*}(Y_1 - D_1(\tau) - I_1)e^{rt^*}(K_2 - K_1)\right] + ...$$
$$= \gamma w_1(t^*) + \gamma^2 e^{-rt^*} w_1(t^*) + ... = \frac{\gamma}{1 - \gamma e^{-rt^*}} w_1(t^*) \qquad Q.E.D.$$

We can replace $w_1(t_1^*)$ and $w_2(t_2^*, t_3^*, ...)$ in equation (26) by $w(t^*)$ and $\frac{\gamma}{1-\gamma e^{-rt^*}} w(t^*)$ and have a new optimality condition for t^* for thestationary equilibrium.

$$\frac{\gamma}{\gamma t^*} w_1(t^*) - \frac{r\gamma e^{-rt^*}}{1 - \gamma e^{-rt^*}} w_1(t^*) = 0 \tag{26'}$$

IV. COMPARATIVE STATICS

In this section, we examine how the length of cycle (t^*) will change when the required knowledge accumulation (H^*) changes and when the technology improvement from innovations (γ) changes. Lets first define the new function $F(t^*)$, the first-order condition for t^*, equation (26').

$$F(t^*) = \frac{\partial}{\partial t^*} w_1(t^*) - \frac{r\gamma e^{-rt^*}}{1 - \gamma e^{rt^*}} w_1(t^*) = F_1(t^*) - F_2(t^*) \tag{27}$$

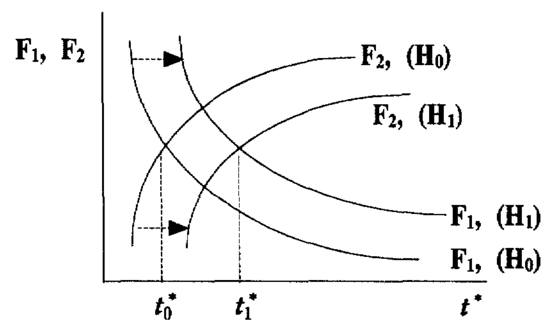
where $F_1(t^*) = \frac{\partial}{\partial t^*} w_1(t^*)$ and $F_2(t^*) = \frac{r\gamma e^{rt^*}}{1 - \gamma e^{rt^*}} w_1(t^*)$

Property 4. As the required knowledge for technology advance becomes larger, the length of cycle becomes longer.

Proof of Property 4. Using the envelope theorem,[7]

[7] The detailed proofs for Property 4 and Property 5 are available on request.

[Figure 1]



$$\frac{\partial t^*}{\partial H^*} = -\frac{\partial F/\partial H^*}{\partial F/\partial t^*} = \frac{1}{\partial F/\partial t^*}\left(\frac{\partial F_1}{\partial H^*} - \frac{\partial F_2}{\partial H^*}\right)$$
$$= -\frac{1}{neg}(pos - neg) = -\frac{pos}{neg} > 0$$

As it is shown in [Figure 1], an increase in $H^*(H_0 < H_1)$ moves both F_1 and F_2 to rightwards and t^* to rightwards $(t_0^* \rightarrow t_1^*)$. As innovation becomes more expensive and costly, the representative agent spends less for investment in research, which delays the arrivals of the next innovations.

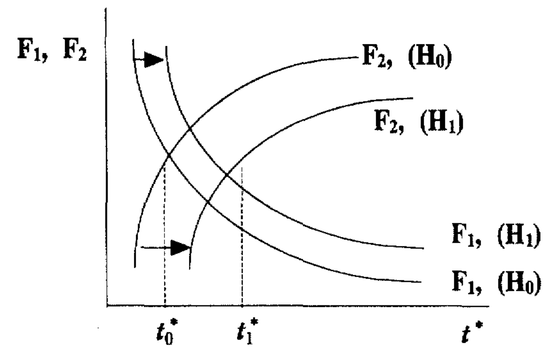
Property 5. As the technology improvement from innovation becomes larger, the length of cycle becomes shorter.

Proof of Property 5. Using the envelope theorem,

$$\frac{\partial t^*}{\partial \gamma} = \frac{\partial F/\partial \gamma}{\partial F/\partial t^*} = -\frac{1}{\partial F/\partial t^*}\left(\frac{\partial F_1}{\partial \gamma} - \frac{\partial F_2}{\partial \gamma}\right)$$
$$= -\frac{1}{neg}(pos - pos) = -\frac{neg}{neg} < 0$$

As it is shown in [Figure 2], an increase in (0<1) moves F_1 to rightwards and F_2 to leftwards but relatively more than F_1, therefore t^* to leftwards $(t_0^* - t_2^*)$. As the return from knowledge accumulation becomes more attractive, the representative agent spends more for investment in research, which makes the arrivals of the next innovations earlier. The economy has faster cycles with faster growth and bigger jumps as the technology improvement from innovation becomes larger.

[Figure 2]



Ⅳ. CONCLUSIONS

This paper introduces a simple endogenous growth model in which investment in capital and investment in research generate both long-run growth and business fluctuations. The main implication of this paper embodies Schumpeter's insight to economic development: economic development takes the form of a sequence of business cycles, each being a response to a discontinuous innovation. The model characterizes comovements, volatility and lagged reactions among aggregate variables which are linked to the endogenous length of cycle. In spite of simple assumptions of the model, it displays some observed business cycle regularities. First, output and investment in capital display greater volatility than consumption (Plosser(1989)). This pattern occurs since consumption grows continuously despite discontinuous technological advances while output and investment in capital rise discontinuously at the time of innovations. Consumption sticks to its trend while investment and output deviate their trend due to the jumps up at the time of innovation. Second, these models explain the comovements between investment of research and investment in capital. Expansion of investment in research is followed by subsequent embodiment in investment in capital (Lach and Rob (1992)). In each cycle, output and investment in capital are initiated when investment in research is completed and the new technology is introduced. This model also examines how the length of cycle is affected by the change of parameter values. The length of cycle becomes longer as the required knowledge accumulation for the next innovation becomes larger. And the length of cycle becomes shorter as the technology improvement from innovation becomes larger.

For the future extensions of this paper, we may think of the closed economy in which the interest rate is changed as capital is accumulated. The endogenous interest rate will affect the propensity of investors either to invest capital in the existing technology or to invest in research to develop an alternative superior technology. Also the endogenous interest rate will affect the households choice either to consume or to save. Both investments compete for households savings, and are jointly determined along with the optimal consumption path. This allows both investments to respond to the interest rate, capital stock, technology level and other variables, and to affect these variables in equilibrium. The analysis of business cycles with endogenous interest rates is studied by Freeman, Hong and Peled forthcoming.

Appendix 1. Derivation of Function $\omega(S_j(0), S_j(t_j^*), t_j^*)$

The maximum utility over the j^{th} cycle is defined as

$$\int_0^{t_j^*} e^{-\rho\tau} u(C_j(\tau))d\tau = \frac{C_j(0)^{1-\sigma}}{1-\sigma}\int_0^{t_j^*} e^{-\rho\tau} e^{\frac{r-\rho}{\sigma}(1-\sigma)\tau} d\tau \qquad \text{(A.1.1)}$$
$$= \frac{C_j(0)^{1-\sigma}}{1-\sigma}\,\frac{1-e^{\varphi j^*}}{-\varphi}$$

where $\frac{1-\rho}{\sigma} - r = \varphi$

We need to show that the initial consumption, $C_j(0)$, is a function of Sj(0), $S_j(t_j^*)$ and t_j^*. Integrate the flow constraint (8) after multiplying both sides by e^{-rt}.

$$\int_0^{t_j^*} C_j(\tau)e^{-r\tau}d\tau = \int_0^{t_j^*} Y_j e^{-r\tau}d\tau - \int_0^{t_j^*} D_j(\tau)e^{r\tau}d\tau - \int_0^{t_j^*} I_j e^{-r\tau}d\tau$$
$$+ S_j(0) - e^{-rt_j^*} S_j(t_j^*) \qquad \text{(A1.2)}$$

Since $C_j(t) = C_j(0)e^{\frac{r-\rho}{\sigma}t}$

$$C_j(0) = \frac{-\varphi}{1-e^{\varphi t_j^*}}\{\int_0^{t_j^*} Y_j e^{-r\tau}d\tau - \int_0^{t_j^*} D_j(\tau)e^{-r\tau}d\tau - \int_0^{t_j^*} I_j e^{-r\tau}d\tau$$
$$+ S_j(0) - e^{-rt_j^*} S_j(t_j^*)\} \qquad \text{(A1.3)}$$

And the discounted value of the total research expenditure during the j^{th} cycle is

$$\int_0^{t_j^*} D_j(t)e^{-rt}dt = \int_0^{t_j^*} D_j(0)e^{\frac{1-\phi}{\phi}rt}dt = H^{*\frac{1}{1-\phi}}(1-\phi)^{\frac{1}{1-\phi}}\left(\frac{1-\phi}{\phi}\right)^{\frac{\phi}{1-\phi}}$$
$$\left(e^{\frac{1-\phi}{\phi}rtj^*}-1\right)^{-\frac{\phi}{1-\phi}} A_j = \Phi_j(t_j^*)$$

Putting all these results into (A1.3), we finally have function.

$$C_j(0)=\frac{-\varphi}{1-e^{\varphi t_j^*}}\left\{\frac{1-e^{rt_j^*}}{r}Y_j-\Phi_j(t_j^*)-\frac{1-e^{rt_j^*}}{r}I_j+S_j(0)-e^{-rt_j^*}\right.$$
$$\left.S_j(t_j^*)\right\}=C(S_j(0),\ S_j(t_j^*),\ t_j^*)\int_0^{t_j^*}e-{}^{\rho\tau}u(C_j(\tau))\,d\tau$$
$$=\frac{C_j(0)^{1-\sigma}}{1-\sigma}\frac{1-e^{\varphi t_j^*}}{-\varphi}=\omega(S_j(0),\ S_j(t_j^*),\ t_j^*) \tag{A1.5}$$

Appendix 2. Proof of Property 2

Using the envelope theorem, the optimality condition for $S_j(t_j^*)$ becomes

$$\frac{\partial}{\partial S_j}(t_j^*)\omega(S_j(0),\ S_j(t_j^*),\ t_j^*)+e^{-\rho t_j^*}\frac{\partial}{\partial S_{j+1}(0)}$$
$$\omega(S_{j+1}(0),\ S_{j+1}(t_{j+1}^*),\ t_{j+1}^*)=0 \tag{A2.1}$$

From Appendix 1,

$$\omega(S_j(0),\ S_j(t_j^*),\ t_j^*)=\frac{C_j(0)^{1-\rho}}{1-\sigma}\frac{1-e^{\varphi t_j^*}}{-\varphi}$$
$$=\frac{C\{S_j(0),\ S_j(t_j^*),\ t_j^*\}^{1-\sigma}}{1-\sigma}\frac{1-e^{\varphi t_j^*}}{-\varphi}$$

With equation (A1.3), the first term of equation (A2.1) becomes

$$\frac{\partial}{\partial S_j(t_j^*)}\omega(S_j(0),\ S_j(t_j^*),\ t_j^*)=\frac{\partial C_j(0)}{\partial S_j(t_j^*)}\frac{\partial}{\partial C_j(0)}\omega(S_j(0),\ S_j(t_j^*),\ t_j^*)$$
$$=\frac{-\varphi}{1-e^{\varphi t_j^*}}(-e^{-rt_j^*})\frac{(1-\sigma)C_j(0)^{-\sigma}}{1-\sigma}\frac{1-e^{\varphi t_j^*}}{-\varphi}=-e^{rt_j^*}C_j(0)^{-\sigma}$$

The second term of equation (A2.1) becomes

$$e^{-\rho t_j^*}\frac{\partial}{\partial S_{j+1}(0)}\omega(S_{j+1}(0),\ S_{j+1}(t_{j+1}^*),\ t_{j+1}^*)$$
$$=e^{-\rho t_j^*}\frac{\partial C_{j+1}(0)}{\partial S_{j+1}(0)}\frac{\partial}{\partial C_{j+1}(0)}\omega(S_{j+1}(0),\ S_{j+1}(t_{j+1}^*),\ t_{j+1}^*)$$
$$=e^{-\rho t_j^*}\frac{-\varphi}{1-e^{\varphi t_{j+1}^*}}\frac{(1-\sigma)C_{j+1}(0)^{-\sigma}}{1-\sigma}\frac{1-e^{\varphi t_{j+1}^*}}{-\varphi}C_{j+1}(0)^{-\sigma}$$

Therefore the optimality condition of $S_j(t_j^*)$ becomes

$$-e^{-rt_j^*}C_j(0)^{-\sigma}+e^{-\sigma t_j^*}=0$$

equivalently $C_{j+1}(0)=C_j(0)e^{\frac{r-\rho}{\sigma}t_j^*}=C_j(t_j^*)$ *Q.E.D.*

Appendix 3. Transformation of the Objective Function

From Corollary2, we found that the objective function is transformed as follows.

$$\int_0^\infty e^{-\rho\tau}u(C(\tau))d\tau=\frac{C_0^{1-\sigma}}{1-\sigma}\frac{1}{\frac{r-\rho}{\sigma}(1-\sigma)-\rho}\left|e^{[\frac{r-\rho}{\sigma}(1-\sigma)-\rho]\tau}\right|_0^\infty$$

$$=-\frac{C_0^{1-\sigma}}{1-\sigma}\frac{1}{\frac{r-\rho}{\sigma}(1-\sigma)-\rho}=-\frac{C_0^{1-\sigma}}{\varphi(1-\sigma)}$$

We can derive C_0 as a function of t_j^*s by adding up equation (A1.2) over cycles.

$$\begin{aligned}\int_0^\infty C(\tau)e^{-r\tau}d\tau&=w_1+e^{-rt_1^*}W_2+e^{r(t_1^*+t_2^*)}W_3+\dots\\&=w_1(t_1^*)+e^{-rt_1^*}w_2(t_2^*)+e^{r(t_1^*+t_2^*)}w_3(t_3^*)+\dots\\&=w_1(t_1^*)+e^{-rt_1^*}w_2(t_2^*,\ t_3^*,\ \dots)\end{aligned}\tag{A3.1}$$

Solving the equation (A3.1) for C_0 gives

$$C_0=\varphi\{w_1(t_1^*)+e^{rt_j^*}W_2(t_2^*,\ t_3^*,\ \dots)\}=C_0(t_1^*,\ t_2^*,\ t_3^*,\ \dots)$$

The objective function is transformed into the new function.

$$\int_0^\infty e^{\rho\tau}u(C(\tau))d\tau=-\frac{1}{\varphi(1-\sigma)}C_0^{1-\sigma}=\frac{\varphi^{1-\sigma}}{\varphi(1-\sigma)}\{w_1(t_1^*)+e^{-rt_j^*}w_2(t_2^*+t_3^*+\dots)\}^{1-\sigma}\tag{A3.2}$$

Equation (A3.2) means that the optimality condition for t_j^* maximizing the discounted life-time utility is equivalent to that maximizing the discounted life-time net wealth, $w_1(t_1^*)+e^{-rt_j^*}W_2(t_2^*,\ t_3^*,\ \dots)$

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