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# DOUBLY IMPLEMENTING THE EQUITABLE AND EFFICIENT SOLUTIONS*

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*In a division problem, where a finite number of agents own a finite number of goods to share, we are interested in implementing equitable and efficient solutions. We propose mechanisms which doubly implement the solutions in Nash and strong Nash equilibria. The mechanisms we propose are simple in the sense that they do not require each agent to report a list of preferences. Each agent only reports a consumption bundle, a price vector, a unit vector, and an integer.*

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## I. INTRODUCTION

We consider the following division problem: there are a finite number of (commonly owned) goods and a finite number of agents whose preferences are defined over the set of allocations.

We are interested in strategic issues in obtaining the desirable (or in some sense socially optimal) allocations rather than in investigating what allocations are desirable in what criteria. When agents behave strategically, knowing the socially optimal allocations does not necessarily guarantee that they can actually obtain those allocations. Suppose there is a solution which optimally divides the goods among agents. Since a solution is responsive to preference profiles in selecting allocations, the information about the true preterence profile is crucial for selecting the optimal outcomes. But, in general, the information is not known to the authority (or the mechanism designer) whose goal is

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achieving the socially optimal allocations. Possessing the information about the preferences, and agent (or a group of agents) may be able to manipulate the outcome for his/her own advantage, which may lead to non-optimal allocations.

In a situation where there is the possibility of manipulation by agents, we are interested in providing a way of obtaining the allocations which would have been recommended by the solution for the true preference profile. To select the right allocations for each profile of preferences, we construct a mechanism: a mechanism consists of a set of strategies for each agent and a function which associates with each strategy profile a feasible allocation. Given an equilibrium concept, if there is a mechanism such that the set of equilibrium outcomes of the mechanism for each preference profile coincides with the set of allocations that would have been recommended by the solution for the preference profile, then we say that the mechanism implements the solution in the equilibrium.

In this paper, we consider solutions which have attractive properties from a normative perspective. One of the solutions which has received much attention for its normative properties is the no-envy solution: the no-envy solution picks the allocations such that each agent prefers her own consumption bundle to others'. By extending the concept of the no-envy solution, we consider a family of no-envy solutions: an extended no-envy solution selects allocations such that each agent prefers her own consumption bundle to some linear combination of others' consumption bundles. We are interested in implementing the intersections of the efficient solution and the family of the no-envy solutions defined as such.

The mechanisms we introduce in this paper have the following special feature. They implement each given solution both in Nash and strong (Nash) equilibria: we say that the mechanism *doubly implements* the solution in Nash and strong equilibria. The desirability of such a mechanism is clear: if a mechanism implements a solution in Nash equilibrium, then it may not work in an environment where agents in some coalitions can cooperate (or form coalitions). And if a mechanism implements a solution in strong equilibrium, then it may not work in an environment where agents in some coalitions cannot cooperate. On the other hand, if a mechanism doubly implements a solution in Nash and strong equilibria, then the mechanism does work regardless of the cooperation possibilities among agents.[1] Thus such a mechanism is needed in a situation where the mechanism designer does not know who can cooperate with whom.[2] Tadenuma and Thomson (1995) is closely related to this paper. They showed that the no-envy solution can be doubly implemented in Nash and strong equilibria by a direct revelation mechanism in an economy with an indivisible good and an infinitely divisible good. The double implementation problem was originally investigated by Maskin (1979). Refer also to

[1] Depending on the cooperation (or coalition formation) possibilities among agents, we can define an equilibrium concept in the same way as we do for Nash equilibrium and strong equilibrium. Then for any game and for any cooperation possibility, the resulting set of equilibrium outcomes is a subset of the set of Nash equilibrium outcomes and a superset of the set of strong equilibrium outcomes. Therefore, if a mechanism implements a solution both in Nash and strong equilibria, then the mechanism implements the solution for any possibility of cooperation.

[2] Suh (1996) has a complete characterization result in a situation where the mechanism designer does know who can cooperate with whom.

Corchon and Wilkie (1996), Suh (1995 and 1997), and Shin and Suh (1996).

The mechanism also works for undominated Nash equilibrium. It is shown that for all preference profiles, every Nash (or strong) equilibrium is also an undominated equilibrium. Hence, no agent uses a dominated strategy in a Nash equilibrium.

Most mechanisms introduced in implementation theory are complex: often each agent is required to announce a list of preferences among other things. Hence in our setting, the announcement each agent makes is an object of an infinite dimensional space. This complexity might not be a problem if we were only concerned with the question of identifying the solutions that are implementable. After knowing that a solution is implementable, we may be further interested in finding a "desirable" mechanism which implements it. One desirable feature of a mechanism is its "simplicity". We propose a series of "simple" mechanisms. In each of the mechanisms, which implements the intersection of the efficient and a generalized version of the no-envy solution, each agent only reports a consumption bundle, a price vector, an unit vector and an integer.

Many papers have been devoted to the issue of finding desirable mechanisms. Saijo, Tatamitani and Yamato (1996), and Thomson (1993b) are among those who have considered the problem of implementing the no-envy solution. Thomson (1993b), by which our paper is motivated, constructed mechanisms which implement the no-envy solution and variants of it in Nash equilibrium. In addition, he provided mechanisms which implement in Nash equilibrium the intersections of the Pareto solution and the no-envy solution, and the Pareto solution and variants of the no-envy solution. The mechanisms introduced in his paper are simple and have a straightforward interpretation. Since in Nash equilibrium no coalitional deviations are allowed, his results apply to environments where group manipulations are not possible. But our interest is in finding mechanisms which are immune to group manipulations.

We introduce the model in Section II. In Section III, we give a series of mechanisms, each of which doubly implements the intersection of each of a family of equitable solutions and the efficient solution in Nash equilibrium and strong equilibrium. In Section IV, we show that the mechanism introduced in Section III also works in undominated Nash equilibrium as well. In Section V, we give concluding remarks.

## II. THE MODEL

There are a finite set $L=\{1, 2, \cdots\cdots, l\}$ of private goods and a finite set $N=\{1, 2, \cdots\cdots, n\}$ of agents. Each agent $i\in N$ has a preference relation $R_i$ defined over $IR^l_+$. Let $P_i$ be the strict preference relation, and $I_i$ be the indifference relation associated with $R_i$. Preferences are continuous, strictly monotonic in $IR^l_{++}$, convex, and satisfies the following boundary condition; for all $i\in N$, for all $R_i\in \Re_i$ and for all $z_i, z'_i\in IR^l_+$, if $z_i\gg 0$, $z'_i \geq 0$ and for at least one $l\in L$, $z'_{il}=0$, then $z_i P_i z'_i$. Let $\Re_0$ be the set of all such preferences an $\Re = \Re_0 \times \cdots\cdots \times \Re_0$.[3]

[3] We use the following notation: For all $x=(x_1, \cdots\cdots, x_l)\in R^l$ and $y=(y_1, \cdots\cdots, y_l)\in R^l$, $x\geq y$ if and only if $x_i\geq y_i$, $x\gg y$ if and only if $x_i > y_i$ and $x>y$ if and only if $x_i\geq y_i$, and $x\neq y$

The social endowment $\Omega \in IR^{l}_{++}$ is fixed throughout the paper. A *feasible allocation* is a list $z=(z_1, \cdots\cdots, z_n) \in IR_+^{nl}$ such that $\sum_{i \in N} z_i \leq \Omega$. Let $Z$ be the set of all feasible allocations.

A *solution* is a mapping $\varphi$ which associates with each preference profile $R \in \Re$ a non-empty subset of $Z$.

A *mechansim* $\Gamma$ is a pair $(S, g)$ of a list of strategy sets $S = S_1 \times \cdots\cdots \times S_n$, where $S_i$ is the strategy set for agent $i$, and an outcome function $g: S \rightarrow Z$, which associates with each strategy profile a unique alternative in $Z$. Given a preference profile $R \in \Re$ and a strategy profile $s \in S$, each agent $i \in N$ evaluates the outcome $g(s)$ according to $R_i$. Hence the preference profile $R \in \Re$ and the mechanism $\Gamma$ define a *game* $(\Gamma, R)$ in normal form.

Given $s \in S$, let $att_i(\Gamma, s) \equiv \{h(s) \in IR_+^{l} \mid h(s_{-i}, s_i')$ for some $s_i' \in S_i\}$ be *agent i's attainable set at* $s_{-i}$. Given $T \subseteq N$, let $s_T = (s_i)_{i \in T} \in S_T = \times_{i \in T} S_i$ and $s_{-T} = (s_i)_{i \in N \setminus T} \in S_{N \setminus T} = \times_{i \in N \setminus T} S_i$.

Given a preference profile $R \in \Re$, the strategy profile $s \in S$ is a ***Nash equilibrium*** of the game $(\Gamma, R)$ if there is no agent $i \in N$ such that for some $s_i' \in S_i$, $g(s_i', s_{-i}) P_i g(s)$. Let $N(\Gamma, R)$ be the set of all Nash equilibria of the game $(\Gamma, R)$ and $NA(\Gamma, R)$ be the set of all Nash equilibrium outcomes. The mechanism $\Gamma$ ***implements the correspondence*** $\varphi$ ***in Nash equilibrium*** if $g(N(\Gamma, R)) = \varphi(R)$ for all $R \in \Re$. The correspondence $\varphi$ in *implementable in Nash equilibrium* if there is a mechanism which *implements* $\varphi$ in *Nash equilibrium.*

Given a preference profile $R \in \Re$, the strategy profile $s \in S$ is a ***strong (Nash) equilibrum*** of the game $(\Gamma, R)$ if there is no coalition $T \subseteq N$ such that for some $\bar{s}_T \in S_T$, $g(\bar{s}_T, s_{N \setminus T}) P_i g(s)$ for all $i \in T$.[4]

Let $S(\Gamma, R)$ be the set of all strong equilibria of the game $(\Gamma, R)$ and $SA(\Gamma, R)$ be the set of all strong equilibrium outcomes. The mechanism $\Gamma$ ***implements*** the conrrespondence $\varphi$ ***in strong equilibrium*** if $g(S(\Gamma, R)) = \varphi(R)$ for all $R \in \Re$. The correspondence $\varphi$ is ***implementable in strong equilibrium*** if there is a mechanism which ***implements*** $\varphi$ ***in strong equilibrium***

The mechanism $\Gamma$ ***doubly implements*** $\varphi$ ***(in Nash and strong equilibria)*** if $NA(\Gamma, R) = SA(\Gamma, R) = \varphi(R)$ for all $R \in \Re$. The correspondence $\varphi$ is ***doubly implementable*** if there is a mechanism which ***doubly implements*** the correspondence.

## III. DOUBLY IMPLEMENTING A FAMILY OF EQUITABLE AND EFFICIENT SOLUTIONS

In this section, we define a set of equitable solutions and provide a general mechanism

[4] We can use the following notion of strong equilibrium instead of ours. A referee pointed out that the following notion is equivalent to the one appeared in the paper: given a preference profile $R \in \Re$ and a mechanism $\Gamma = (S, g)$, the strategy profile $s \in S$ is a strictly ***(Nash) equilibrium*** of the game $(\Gamma, R)$ if there is no coalition $T \subseteq N$ such that for somes $\bar{s}_T \in S_T$ $g(\bar{s}_T, s_{N \setminus T}) R_i g(s)$ for all $i \in T$ and $g(\bar{s}_T, s_{N \setminus T}) P_i g(s)$ for at least one $i \in T$.

which doubly implements the intersection of the Pareto solution and each equitable solution in Nash and strong equilibria. Given $R \in \Re$, an allocation $z \in Z$ is ***efficient*** if there is no $z^* \in Z$ such that $z^*_i R_i z_i$ for all $i \in N$ and $z^*_i P_i z_i$ for at least one $i \in N$. The ***efficient solution*** $P: \Re \rightarrow Z$ is defined as follows: for all $R \in \Re$, $z \in P(R)$ if and only if $z$ is efficient.

Let $D_i \subseteq \triangle^{n-1}$ for all $i \in N$ and $D = (D_1, \cdots\cdots, D_n)$. Given $D \subseteq \triangle^{n(n-1)}$, $z \in Z$ and $i \in N$, let $C^D_i(z) \equiv \{z^*_i \in IR_+ \mid \text{ for some } \delta_i \in D_i, z^*_i = \sum_{j \in N} \delta_{ij} z_j\}$. Given $R \in \Re$ and $D \subseteq \triangle^{n(n-1)}$, an allocation $z \in Z$ is ***D-envy-free*** if for all $i \in N$ and for all $z_i^* \in C^D_i(z)$, $z_i R_i z_i^*$ and $z_i \gg 0$.

The ***D-no-envy solution*** $F^D$ is defined as follows: if for all $R \in \Re$ and for all $z \in Z$, $z \in F^D(R)$ if and only if $z$ is ***D-envy-free***. The ***D-no-envy and efficient solutgion*** $F^D P$ is defined as follows: if for all $R \in \Re$ and for all $z \in Z$, $z \in F^D P(R)$ if and only if $z$ is D-envy-free and efficient.

Many of the following solutions can be obtained as particular cases of this general definition by picking appropriate $D \subseteq \triangle^{n(n-1)}$:

(1) ***an envy-free allocation*** is an allocation such that each agent prefers her own consumption bundle to other's, and is obtained by choosing $D_i = \{e_1, \cdots\cdots, e_n\}$ for all $i \in N$ where $e_i$ is the vector whose $i$th element is one and all others are zeros.[5]

(2) ***An average envy-free allocation*** is an allocation such that each agent prefers her own consumption bundle to the average of all others' consumption bundles, and is obtained by choosing, for all $i \in N$, $D_i = \{(d_1, \cdots\cdots, d_n)\}$ such that $d_j = \left(\frac{1}{n-1}\right)$ for all $j \neq i$ and $d_i = 0$.[6]

(3) ***A strict enby-free allocation*** is an allocation such agent prefers her own consumption bundle to the averages of all possible groups excluding her, and is obtained by choosing, for all $i \in N, D_i = \{\delta_i \in \triangle^{n-1} \mid \delta_i = \frac{1}{\| T \|}((1)_{j \in T}, (0)_{j \notin T})$ where $T_i \in G(i)\}$ and $G(i) \equiv \{H \subseteq N \mid i \notin H\}$.[7]

(4) ***A super envy-free allocation*** is an allocation such that each agent prefers her own consumption bundle to any convex combination of all consumption bundles including her own consumption bundle, and is obtained by choosing, for all $i \in N$, $D_i = \triangle^{n-1}$.[8]

Let $(0, \Omega] \equiv \{z_0 \in IR^l_+ \mid 0 \ll z_0 \leq \Omega\}$. Let $Z^+ \equiv \{z \in IR^{nl}_+ \mid z_{im} > 0$ for all $i \in N$ and for all $m \in L\}$. Let $q: Z^+ \rightarrow N$ be the function defined as follows: given $(z_1, \cdots\cdots, z_n) \in Z^+$, $q(z_1, \cdots\cdots, z_n) = k$, if $z_{km} < z_{im}$ for all $i \neq k$ and for all $m \in L$, and $q(z_1, \cdots\cdots, z_n) = 0$ otherwise. If, for all $i \in N$, we think of $z_i$ as agent i's requested consumption bundle, then $q(z_1, \cdots\cdots, z_n)$ may be interpreted as the agent who is the most modest in stating

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[5] The concept of no-envy was introduced by Foley (1967). Subsequently, the condition has been investigated in various economic domains. Refer to Thomson (1993a) for a nice survey of the literature on no-envy.

[6] This criterion has been investigated by Thomson (1979, 1982) and Baumol (1986).

[7] The criterion is introduced by Zhou (1992).

[8] The criterion is introduced by Kolm (1973).

her consumption bundle. Given $(z_1, \cdots\cdots, z_n) \in Z^+$, let $M(z_1, \cdots\cdots, z_n) \equiv \{z' \in IR_+^l \mid 0 \ll z' \ll (\min\{z_{i1}\}_{i\in N}, \cdots, \min\{z_{il}\}_{i\in N})\}$. Since $M(z_1, \cdots\cdots, z_n)$ is not empty for all $z \in Z^+$, for all $i \in N$ and for all $z \in Z^+$, agent $i$ can choose to be the most modest one by picking a consumption bundle in $M(z_1, \cdots\cdots, z_n)$. Let $\triangle^{l-1} = \{p \in IR^l \mid p_m \geq 0$ for all $m \in L$ and $\sum_{m\in L} p_m = 1\}$. Given $p \in \triangle^{l-1}$ and $z_i \in IR_+^l$, let $B(p, z_i) \equiv \{z_i' \in IR_+^l \mid pz_i' \leq pz_i\}$.

Given $D \subseteq \triangle^{n(n-1)}$, we introduce a mechanism $\Gamma^D$ which implements the $D$-no-envy and efficient solution.

**Mechanism**: $\Gamma^D = (S, h)$.

Given $D = (D_1, \cdots\cdots, D_n) \in \triangle^{n(n-1)}$, the strategy space for each agent $i$ is defined as follows: for all $i \in N$,

$$S_i = (0, \Omega] \times \triangle^{l-1} \times D_i \times \{0, 1, 2, \cdots\cdots\}.$$

Let $s = ((z_1, p_1, \delta_1, t_1), \cdots\cdots, (z_n, p_n, \delta_n, t_n))$ be the generic element of $S = S_1 \times \cdots\cdots \times S_n$. Let $e_i = (0, \cdots\cdots, 0, 1, 0, \cdots\cdots, 0) \in \triangle_{n-1}$ be the vector with the ith component is unit and all others are zero. Given $s = ((z_1, p_1, \delta_1, t_1), \cdots\cdots, (z_n, p_n, \delta_n, t_n)) \in S$, the modulo game winner of s, $\omega(s)$, is defined as follows: $(mod\ n)\sum_{i\in N} t_i = w(s)$.

The outcome function $h$ is defined as follows:

Rule 1. If $p_i = p$ for all $i \in N$ and for some $p \in \triangle^{l-1}$, $z \in Z$,

Rule 1−a. and if $\delta_{w(s)} = e_{w(s)}$, then $h(s) = (z_1, \cdots\cdots, z_n)$,

Rule 1−b. and if $\delta_{w(s)} \neq e_{w(s)}$, then for all $k \in N$,

$$h_k(s) = \begin{cases} \delta_k z & \text{if } k = w(s); \\ 0 & \text{otherwise.} \end{cases}$$

Rule 2. If $p_i = p$ for all $i \in N$ and for some $p \in \triangle^{l-1}$ and $z \notin Z$,

Rule 2−a. and if $p\Omega \geq p\sum_{i\in N} z_i$, then for all $k \in N$,

$$h_k(s) = \begin{cases} z_k & \text{if } k = w(s); \\ 0 & \text{otherwise.} \end{cases}$$

Rule 2−b. and if $p\Omega < p\sum_{i\in N} z_i$, then for all $k \in N$,

$$h_k(s) = \begin{cases} z_k & \text{if } k = q(z_1, \cdots\cdots, z_n); \\ 0 & \text{if } k \neq q(z_1, \cdots\cdots, z_n). \end{cases}$$

Rule 3. If for some agents $i$ and $j$, $p_i \neq p_j$, then, for all $k \in N$,

$$h_k(s) = \begin{cases} z_k & \text{if } k = q(z_1, \cdots\cdots, z_n); \\ 0 & \text{if } k \neq q(z_1, \cdots\cdots, z_n). \end{cases}$$

**Theorem 1.** The Mechanism $\Gamma^D$ doubly implements $F^DP$ in Nash and strong equilibria.

**Proof.**

We will show that $F^DP(R) \supseteq NA(\Gamma^D, R)$ and $F^DP(R) \subseteq SA(\Gamma^D, R)$ for all $R$

**Step 1.** $F^D P(R) \subseteq SA(\Gamma^D, R)$

Let $z=(z_1, \cdots\cdots, z_n) \in F^D P(R)$. We will show that $z \in SA(\Gamma^D, R)$.

Since $z \in P(R)$ for all $i \in N$, there is a price vector $p \in \triangle^{l-1}$ which is orthogonal to the tangent plane at $z_i$ to the indifferent curve of $R_i$ passing through $z_i$. Let $s=((z_1, p, e_1, 0), \cdots\cdots, (z_n, p, e_n, 0)) \in S$. Then by Rule 1-a, $h(s)=(z_1, \cdots\cdots, z_n)$. We will show that $s \in S(\Gamma^D, R)$. First, consider an individual deviation $\bar{s}_i=(\bar{z}_i, \bar{p}_i, \bar{\delta}_i, \bar{t}_i)$ by agent $i \in N$. Obviously agent $i$ can obtain 0 and any dundle in $M(z_1, \cdots, z_n)$. By picking an appropriate integer to be the integer game winner and $\delta_i \in D$, she can obtain any point in the set $C_i^D(z)$. And by picking $\bar{z}_i \in B(p, z_i)$ and $(\bar{p}_i, \bar{\delta}_i, \bar{t}_i) = (p, e_i, i)$, agent $i$ can be the integer game winner and induce Rule 2-a and get $z_i$. Since $B(p, z_i) \supset M(z_1, \cdots\cdots, z_n)$, $att_i(\Gamma^D, s)=\{z_i, C_i^D(z) 0\} \cup B(p, z_i)$. Since $z \in F^D(R)$, $z_i \gg 0$; hence $z_i R_i 0$ for all $i \in N$. Since $z \in F^D(R)$, $z_i R_i z_i^*$ for all $z_i^* \in C_i^D(z)$. Since $z \in P(R)$, $z_i R_i z_i^*$ for all $z_i^* \in B(p, z_i)$. Hence, no agent can be strictly better-off by an individual deviation.

Next, we consider a deviation $\bar{s}_T=(\bar{z}_i, \bar{p}_i, \bar{\delta}_i, \bar{t}_i)_{i \in T}$ from $s_T$ by a coaltion $T$ such that $\| T \| \geq 2$. If Rule 1-a applies to $\bar{s}=(\bar{s}_T, s_{-T})$, then $\sum_{i \in N} h_i(\bar{s}) \leq \Omega$, and $h_i(\bar{s})=\bar{z}_i$ for all $i \in T$ and $h_i(\bar{s})=z_i$ for all $i \in N \backslash T$. We will show that at least one agent in $T$ cannot be strictly better-off. Suppose to the contrary that $h_i(\bar{s}) P_i z_i$ for all $i \in T$. Since $\sum_{i \in T} \bar{z}_i$ $\sum_{i \in N \backslash T} z_i \leq \Omega$, it follows that the allocation $((\bar{z}_i)_{i \in T}, (z_i) i \in N \backslash T)$ is feasible. Since $h_i(\bar{s}) = z_i$ for all $i \in N \backslash T$, $h_i(\bar{s}) R_i z_i$ for all $i \in N$ and for at least on $i \in N$, $h_i(\bar{s}) R_i z_i$; hence $z$ is not efficient. This is a contradiction to the fact $z \in P(R)$.

If Rule 1-b, 2 or Rule 3 applies, then at least one agent in $T$ gets nothing; hence no deviation, which induces Rule 1-b, 2 or Rule 3, can make everyone $T$ strictly better-off.

Therefore, $s \in S(\Gamma^D, R)$; hence $z \in SA(\Gamma^D, R)$.

**Step 2.** $F^D P(R) \supseteq NA(\Gamma^D, R)$

Let $z=(z_1, \cdots\cdots, z_n) \in NA(\Gamma^D, R)$. Then there is a Nash equilibrium strategy $s \in S$ such that $h(s)=z$. Let $s=((\bar{z}_1, p_1, \delta_1, t_1), \cdots\cdots, (\bar{z}_n, p_n, \delta_n, t_n))$.

Since $z$ is a Nash equilibrium outcome, it follows that $z \gg 0$. Suppose to the contrary that for some $k \in N$ and for some $m \in L$, $z_{km}=0$. Then by announcing $p_k' \neq p_j$ for some $j \neq k$ and $z_k' \neq z_0$ for some $z_0 \in M(z_1, \cdots\cdots, z_n)$ to induce Rule 3 and gets $z_0$ as her consumption bundle; hence agent $k$ is strictly better-off. Hence none of Rule 1-b, Rule 2 and Rule 3 applies to $s$. Thus, only Rule 1-a applies to $s$ and $\bar{z}_i = z_i$ for all $i \in N$.

We claim that $z \in P(R)$. Since Rule 1-a applies to $s$, it follows that $\bar{z}_i = z_i$ for all $i \in N$ and $p_i = p$ for all $i \in N$ and for some $p \in \triangle^{l-1}$.

Since $s \in N(\Gamma^D, R)$, $\sum_{i \in N} z_i = \Omega$. If $\sum_{i \in N} z_i < \Omega$, then any agent, say agent $k$, is strictly better-off by announcing $s_k'=(z_k', p_k', \delta_k', t_k')$ such that $z_k' > z_k$, $\sum_{i \neq N} z_i + z_k' \leq \Omega$, $(p_k', \delta_k', t_k') = (p_k, \delta_k, t_k)$ to get $z_k'$ as her consumption bundle. This implies that $s$ is not a Nash equilibrium. This is again a contradiction. Hence, $\sum_{i \in N} z_i = \Omega$.

We will show that $z_i R_i z_i'$ for all $i \in N$ and for all $z_i' \in B(p, z_i)$. Since $att_i(\Gamma^D, s) = \{z_i, C_i^D(z), 0\} \cup B(p, z_i)$ and $s$ is a Nash equilibrium, it follows that $z_i R_i z_i'$ for all $z_i' \in B(p, z_i)$. Therefore, $z_i' \in P(R)$.

Next, we claim that $z \in F^D(R)$, i.e., $z_i R_i z_i^*$ for all $i \in N$ and for all $z_i^* \in C_i^D(z)$. Suppose to the contrary that $z_i^* R_i z_i$ for some $i \in N$ and for some $z_i^* \in C_i^D(z)$. Then, agent $i$ can pick an appropriate $t_i$ to be the modulo game winner and announce $\delta_i \in D_i$ such that $\delta_i z = z_i^*$. Then by keeping other components of her strategy the same as $s_i$, she can induce Rule 1-b and obtain $z_i^*$. Note that this deviation by agent $i$ is possible because $z_i^* \in C_i^D(z)$. Since $z_i R_i^* z_i$, this implies that $s$ is not a Nash equilibrium. This is a contradiction. Therefore, $z \in F^D(R)$.

Q.E.D

**Remark 1.** Since $M(z_1, \cdots\cdots, z_n)$ is an open set, when Rule 2-b or 3 applies to a strat egy profile, no agent has the best response to the strategy profile. Hence, the mechanism does not satisfy the best response property: for all strategy profiles, each agent has her best response to others' strategies. (For the discussion of the best response property, refer to Jackson, Palfrey and Srivastava (1994), and Saijo, Tatamitani and Yamato (1996).) This flaw in the mechanism can be removed by a slight modification of it. Given $z=(z_1, \cdots\cdots, z_n) \in Z^+$, let $q^*(z_1, \cdots\cdots, z_n)$ is a set such that $k \in q^*(z)$, if $z_{km} \leq z_{im}$ for all $i \neq k$ and for all $m \in L$, and $q^*(z_1, \cdots\cdots, z_n) = 0$ otherwise. For a strategy profile $s = ((z_1 p_1 \pi_1 t_1), \cdots\cdots, (z_n p_n \pi_n t_n)) \in S$, if there is only one agent in $q^*(z_1, \cdots\cdots, z_n)$, then apply Rule 2-b or Rule 3 directly. If there are more than one agent in $q^*(z_1, \cdots\cdots, z_n)$, then pick the modulo game winner and apply Rule 2-b or Rule 3. It is easy to check that the mechanism with this modification satisfies the best response property.

**Remark 2.** The mechanism we provided for each $D$ is not non-wasteful. But if $n \geq 3$, then the mechanism can be modified to be non-wasteful; in Rule 1-b, 2 and 3, the outcomes given to the agents other than $k$ are as follows,

$$h_i(s) = \begin{cases} (((\Omega - h_k(s))_m)_{m \neq l}, 0) & \text{if } i = mod(k+1); \\ (0, ((\Omega - h_k(s))_m)_l) & \text{if } i = mod(k+2); \\ 0 & \text{otherwise.} \end{cases}$$

Note that the consumption bundles given by the outcome are worst for all agents except $k$. Hence the same proof as in the previous theorem can be used to show that we can use the modified mechanism in the theorem when $n \geq 3$.

## IV. UNDOMINATED NASH IMPLEMENTATION

The mechanism we introduced in Section 3 not only doubly implements the solutions in question in Nash and strong equilibria, but it also implements those in undominated (Nash) equilibrium. In other words, for all preference profiles, every Nash (or strong) equilibrium is also an undominated equilibrium. Hence, in the mechanisms no strategy profile with a component of dominated strategy can be a Nash equilibrium. For the double implementation in Nash and undominated equilibria, refer to Yamato(1993a and 1993b).

Given a preference profile $R \in \Re$, the strategy profile $s \in S$ is a ***undominated (Nash) equilibrium*** of the game $(\Gamma, R)$ if $s$ is a Nash equilibrium and it is umdominated: for all $i \in N$, there is no $s_i^* \in S_i$ such that $g(s_i^*, s'_{-i}) R_i g(s_i, s'_{-i})$ for all $s'_{-i} \in S_{-i}$ and $g(s_i^*, s'_{-i}) P_i g(s_i^*, s'_{-i})$ for some $s'_{-i} \in S_{-i}$.

Let $UN(\Gamma, R)$ be the set of all undominated equilibria of the game $(\Gamma, R)$ and $UNA(\Gamma, R)$ be the set of all undominated equilibrium outcomes. The mechanism $\Gamma$ ***implements*** the correspondence $\varphi$ ***in undominated equilibrium*** if $UNA(\Gamma, R)) = \varphi(R)$ for all $R \in \Re$. The correspondence $\varphi$ is ***implementable in undominated equilibrium*** if there is a mechanism which ***implements*** $\varphi$ ***in undominated equilibrium.***

**Proposition 1** Given $D \subseteq \triangle^{n(n-1)}$, the mechanism $\Gamma^D$ implements the D-no-envy and efficient solution in undominated Nash equilibrium.

**Proof.** If we show that $NA(\Gamma^D, R) = UNA(\Gamma^D, R)$ for all $R \in \Re$, then by Theorem 1 this implies that $F^D P(R) = UNA(\Gamma^D, R)$ for all $R \in \Re$. Since $UNA(\Gamma^D, R) \subseteq NA(\Gamma^D, R)$ for all $R \in \Re$, we only show that $NA(\Gamma^D, R) \subseteq UNA(\Gamma^D, R)$ for all $R \in \Re$.

Suppose $z = (z_1, \cdots\cdots, z_n) \in NA(\Gamma^D, R)$. We will show that $z \in UNA(\Gamma^D, R)$. Let $s = ((z_1 p_1 \delta_1 t_1), \cdots\cdots, (z_n p_n \delta_n t_n)) \in NA(\Gamma^D, R)$ and $g(s) = z$. For agent $k \in N$ (without loss of generality let $k=1$), suppose, to the contrary, that $s_1$ is a dominated strategy. Then there is $s_1^0 = (z_1^0, p_1^0, \delta_1^0, t_1^0) \neq s_1$ such that

$g(s_1^0, s^*_{-1}) R_1 g(s_1, s^*_{-1})$ for all $s^*_{-1} \in S_{-1}$ and

$g(s_1^0, s^*_{-1}) P_1 g(s_1, s^*_{-1})$ for some $s^*_{-1} \in S_{-1}$.

We will show that $s_1^0 = s_1$ which is a contradiction. Thus, $s_1$ cannot be a dominated strategy. We consider the following four cases separately.

**Case 1.** If $z_1 \neq z_1^0$, then $z_1^0 + \sum_{i \neq 1} z_i \neq \Omega$; hence either Rule 2 or Rule 3 applies to $(s_1^0, s_{-1})$.

If Rule 2-b or Rule 3 applies to $(s_1^0, s_{-1})$, then $z_i \gg g_i(s_1^0, s_{-1})$ for all $i \in N$. Hence $z_i' \gg g_i(s_1^0, s_{-1})$ for all $z_i' \in C_1^D(z)$. Since $g_1(s_1, s_{-1})$ is a Nash equilibrium outcome, we know that $g(s)$ is D-envy-free by the previous results; hence $g_1(s) R_1 z_1'$ for all $z' \in C_1^D(z)$. Thus, $g_1(s_1, s_{-1}) P_1 g_1(s_1^0, s_{-1})$. This is a contradiction.

If Rule 2-a applies to $(s_1^0, s_{-1})$, then we can find $s^*_{-1}$ from $s_{-1}$ by changing only the 4th components of $s_{-1}$ such that $w^* = w(s_1^0, s^*_{-1}) \neq 1$ and $\delta_{w^*} = e_{w^*}$. Since in $s^*_{-1}$ other components of $s_{-1}$ remained the same except the 4th components, Rule 2-a still applies to $(s_1^0, s^*_{-1})$. Since $\delta_{w^*} = e_{w^*}$, Rule 1 still applies to $(s_1, s^*_{-1})$. Then $g_1(s_1, s^*_{-1}) = g_1(s)$ and $g_1(s_1^0, s^*_{-1}) = 0$. Since $g(s)$ is a Nash equilibrium outcome, by the previous result $g_1(s) \gg 0$; hence $g_1(s) P_1 0$. Hence $g_1(s_1, s^*_{-1}) P_1 g_1(s_1^0, s^*_{-1})$. This is a contradication. Thus, $z_1 = z_1^0$.

**Case 2.** If $p_1 \neq p_1^0$, then Rule 3 applies to $(s_1^0, s^*_{-1})$. Then using the same argument as that used in the first half of Case 1, we can show that $g_1(s_1, s_{-1})P_1 g_1(s_1^0, s_{-1})$. This is a contradiction. Thus, $p_1 = p_1^0$.

**Case 3.** If $\delta_1 \neq \delta_1^0$, then consider $s'_{-1} \in S_{-1}$ such that $g_1(s_1^0, s'_{-1})P_1 g_1(s_1, s'_{-1})$.

If Rule 1-a applies to $(s_1^0, s'_{-1})$, then since $z_1 = z_1^0$ and $p_1 = p_1^0$, Rule 1 applies also to $(s_1, s'_{-1})$. Hence $g_1(s_1^0, s'_{-1}) = g_1(s_1, s'_{-1}) = z_1$. This is a contradiction.

If Rule 1-b applies to $(s_1^0, s'_{-1})$, then we can derive a contradiction by finding $s^*_{-1}$ such that $g_1(s_1^0, s^*_{-1}) = 0$, and either $g_1(s_1, s^*_{-1}) = z_1$ if $\delta_1 = e_1$ or $g_1(s_1, s^*_{-1}) = \delta_1 z^*$ for some $z^* \in Z$ such that $\delta_1 z^* \gg 0$, if $\delta_1 \neq e_1$. This can be done by selecting appropriate $\{t_i^*\}_{i \neq 1}$ such that $w(s_1^0, s^*_{-1}) \neq 1$ and $w(s_1, s^*_{-1}) = 1$, and appropriate $\{z_i^*\}_{i \neq 1}$ such that $\{z_1, z_2^*, \cdots\cdots, z_n^*\} \in Z$.

If Rule 2-a applies to $(s_1^0, s'_{-1})$, then again Rule 2-a applies also to $(s_1, s'_{-1})$. Since $z_1 = z_1^0$ and $(s_1^0, s'_{-1})P_1(s_1, s'_{-1})$, it follows that $w(s_1^0, s'_{-1}) = 1$, and $w(s_1, s'_{-1}) \neq 1$; hence $t_1^0 \neq t_1$. We can find $s^*_{-1}$ from $s'_{-1}$ by changing only the 4th components of $s'_{-1}$ such that $w(s_1, s^*_{-1}) = 1$, and $w(s_1^0, s^*_{-1}) \neq 1$. Since in $s^*_{-1}$ other components of $s'_{-1}$ remained the same except the 4th components, Rule 2-a still applies to $g_1(s_1, s^*_{-1})$ and $g_1(s_1^0, s^*_{-1})$. Hence $g_1(s_1, s^*_{-1}) = z_1$ and $g_1(s_1^0, s^*_{-1}) = 0$. Thus, $g_1(s_1, s^*_{-1})P_1 g_1(s_1^0, s^*_{-1})$. This is a contradiction. Thus, $\delta_1 = \delta_1^0$.

If Rule 2-b or Rule 3 apply to $(s_1^0, s'_{-1})$, then, since $z_1 = z_1^0$ and $p_1 = p_1^0$, Rule 2-b applies also to $(s_1, s'_{-1})$. Since $g_1(s_1^0, s'_{-1})P_1 g_1(s_1, s'_{-1})$, it follows that $g_1(s_1^0, s'_{-1}) \neq 0$; hence $g_1(s_1^0, s'_{-1}) = z_1^0$. This means that $q(z_1^0, z_2', \cdots\cdots, s_n') = 1$. But since $z_1 = z_1^0$, we have $q(z_1, z_2', \cdots\cdots, s_n') = 1$ and $g_1(s_1, s'_{-1}) = z_1$ also; hence $z_1 P_1 z_1$. This is a contradiction.

**Case 4.** If $t_1 \neq t_1^0$, then consider $s^*_{-1} \in S_{-1}$ such that (1) Rule 2-a applies to $(s_1, s^*_{-1})$ and $(s_1^0, s^*_{-1})$ and (2) $w(s_1, s^*_{-1}) = 1$ and $w(s_1^0, s^*_{-1}) \neq 1$. Using the same argument as that used in part of Case 3, we can show that such a selection of $s^*_{-1}$ is possible. Then, $g_1(s_1, s^*_{-1}) = z_1$ and $g_1(s_1^0, s^*_{-1}) = 0$. Hence $g_1(s_1, s^*_{-1})P g_1(s_1^0, s^*_{-1})$. This is a contradiction. Thus, $t_1 = t_1^0$

From Case 1, 2, 3 and 4, we conclude that $s_1 = s_1^0$. But this is a contradiction to the relation $s_1 \neq s_1^0$. Therefore, $s$ is undominated.

Q.E.D.

## V. CONCLUDING REMARKS

We have considered the problem of implementing equitable and efficient solutions in environments where agents in some groups can cooperate and in some others cannot, and the information of the cooperation possibilities among agents is not available to the mechanism designer. The focus of this paper is to find a series of mechanisms which work for such environments and have several desirable properties. Similar work dealing with such a situation has been done in other economic domains: the proportional solution in a production economy (Suh (1995)), the ratio correspondence in a public good economy (Corchon and Wilkie (1996)), and the no-envy solution in a division problem with indivisible goods (Tadenuma and Thomson (1995)). It would be interesting to investigate similar questions for other solution concepts.

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