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WILL RANDALL AND STOLL'S BOUND STILL HOLD IN A COMPLETE INVERSE DEMAND SYSTEM?*

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This article provides a theoretically consistent framework for welfare measurement under quantity restrictions and free adjustment of prices in equilibrium. The paper extends Randall and Stoll's and Hanemann's results into a complete inverse demand system. It is found that Randall and Stoll's and Hanemann's results should be modified to fit into the complete inverse demand system.

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I. INTRODUCTION

Researchers involved in applied welfare analysis have conducted demand analysis based on ordinary demands or mixed demands. Ordinary demand is the usual representation of preferences for the individual consumer in the absence of rationing (more generally quantity constraints), who is typically taken as making optimal decisions for given prices and income while mixed demands (Neary and Roberts, Chavas) is the representation of preferences for the consumer in the presence of rationing on some goods. In addition to the two cases of ordinary and mixed demands, another class of demands i.e., inverse demands, can be conducted in welfare analysis. The inverse demand corresponds to the case where all goods are rationed and involves specific assumptions about the way rationed quantities adjust to keep

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utility constant, and also a specific price normalization.[1] As a result, the implications of inverse demands are quite different from other demands.

Since the inverse demand model implies that all consumption goods in the model are predetermined in the short-period enough to prevent to change quantities, there are restrictions on the consumer's response. For some goods, the production process for those goods may be such that market supplies of related goods are determined largely in advance of current prices. As is well known, in an equilibrium under quantity rationing, prices may be exogenous if there is no price adjustment process. However, there is no reason to assume the exogeneity of prices. Furthermore, we can hardly assume that there is always a Walrasian equilibrium on each market. It is especially true when we study the effect of all characteristics of air quality (quantity of environmental and resource goods) on property values or when quantity of nonstorable goods (fish, food, and fruits and so on) is predetermined by production at the market level. For these reasons, a complete inverse demand model may give much more sense than any other specification in these cases.[2] Despite its obvious potential for application into those cases, the inverse demand model has been virtually ignored in both theoretical and empirical works for welfare measurement.

As a first attempt, Karl-Göran Mäler (1974) showed that the concept of the compensating and equivalent variations can readily be extended from conventional price changes to such quantity changes. In a mixed demand model adapted to Mäler's idea, Randall and Stoll (1980) demonstrated that Willig's (1976) bound can be adapted to errors of approximation to the exact welfare effects of exogenous quantity changes, which is called Randall and Stoll's bound.[3] In many empirical studies of environmental economics, Randall and Stoll's results were misleading in the sense that willingness to pay (call it *WTP*) and willingness to accept (call it *WTA*) for changes in quantity of environmental quality characteristics should not much differ without unusual income effects. However, Michael Hanemann (1991) demonstrated that for quantity (of environmental quality characteristics) changes, *WTP* and *WTA* are not presumed to be close in value and the difference between *WTP* and *WTA* depends not only on an income effect but also on a substitution effect.

In this paper, we reexamine Randall and Stoll's bound and Hanemann's analysis. It will be shown that the bound is still hold but the involved concepts should be adapted to the inverse demand models and that while his fundamental result is

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[1] See Madden (1991).

[2] One example of this model may be the case where there are significant transaction costs involved in moving and in buying and selling houses or changing rental agreements in studying the effect of air pollution on property values. Another example may be the case of commercial and recreational fishery due to water quality change and the demand for fish. Note that there have been increasing attempts to investigate systems of inverse demand equations. In the theoretical literature, Theil (1976), Weymark (1980), Anderson (1980), Heien (1982), Chambers and McConnell (1983) provide some theoretical discussions of inverse demand systems.

[3] Robert Willig (1976) derived bounds for the difference between the correct measure of the compensating and equivalent variations, and demonstrated that the difference is likely to be fairly small in value.

correct in the ordinary demand model with quantity restriction (i.e., the mixed demand model), it does not lead us to reach the same result in the inverse demand model with quantity restriction. Further, we show that under appropriate conditions, changes in the area under the inverse demand curve for goods can serve as welfare measures for changes in the quantity constraints and that these measures make no assumptions about prices in deriving the welfare measures.

The layout of the paper is as follows. Section II is concerned with inverse demand models distinct from mixed demand models (ordinary demands with quantity restrictions). Section III develops modified Randall and Stoll's bound in an inverse demand model. Section IV deals specifically with Hanemann's fundamental result adapted to the inverse demand model. Taking a simple example, section V highlights the significance of difference from Hanemann's fundamental result. Section VI closes the paper by concluding remarks.

II. INVERSE DEMAND MODELS

Following Hanemann (1991), this paper first will set up the ordinary demand model with quantity restriction which is in fact Randall and Stoll's economy. It then will derive a complete inverse demand model with quantity restriction from which we obtain modified Randall and Stoll's bound. Consider an individual who maximizes a quasi-concave utility function, $U(x, q)$ where x denotes a quantity vector of conventional market goods and q represents a quantity vector of rationed goods or the supply of environmental goods or amenity. The individual faces quantity constraints on its consumption of goods q but is unconstrained in its consumption of goods x. Suppose that the individual could purchase x at the price px and the hypothetical price of q would be given at p_q. The individual chooses his consumption by solving:

$$V(q, p_x, n) = Max_{\{x\}}[U(x, q) \mid p_x x + p_q q - m = 0] \tag{1}$$

where income is denoted by m and the income net of the cost of q by $n = m - p_q q$. In such a case, conditional demand functions are:[4]

$$q^0 = g_2(p_q, p_x, m) \tag{2}$$

$$x = g_1(q^0, p_x, n) \tag{2'}$$

where q^0 denotes a fixed quantity of environmental good's characteristics or rationed goods and $n = m - p_q q^0$.[5] Note that if the individual has the freedom to

[4] Conditional demand functions were first considered by Pollak (1969). Some useful classical papers on consumer behavior in settings with quantity constraints are Howard (1977), Latham (1980), Neary and Roberts (1980), and Deaton (1981).

[5] Hereafter,variables which are some functions of parameters represent optimal values unless any special superscript is used and any confusion arises.

choose both x and q, that is p_q is assumed to be fixed as p_x is, the optimal demand for x is not a function of q.

Considering the conditional demand function for q, the virtual price chypothetical market pricel of q (p_q) can be implicitly defined in[6]

$$q^0 = g_2(p_q, p_x, m) = g_2[p_q, p_x, e(q, p_x, u)] \tag{3}$$

where $e(.)$ denotes the conditional expenditure function. An explicit expression for the virtual price p_q is

$$p_q = f_2(q^0, p_x, u) = g_2^{-1}[q^0, p_x, e(q, p_x, u)] \tag{4}$$

Substituting the conditional indirect utility function $u = V\ (q,\ p_x,\ n)$ resulting from (1) into (4) yields implicitly 'the individual's marginal-willingness-to-pay function' for q.[7] For the individual, the marginal-willingness-to-pay function may be different from the market price even if the market for q exists. However, when the hypothetical market price p_q equals his marginal evaluation of the last unit received, the marginal-willingness-to-pay function becomes equal to a conditional inverse Marshallian demand function of q.[8] Thus, when it comes to the aggregate behavior from the individual's choice, it makes sense perfectly that the conditional inverse Marshallian demand function would become the marginal-willingness-to-pay function for a representative consumer.[9] The model constituted by equations (2′) and (4) may be called the mixed demand model.[10]

The mixed demand model explained above is the world for Randall and Stoll and Hanemann. We now diverge from their world by putting our feet into a realm of complete inverse demand models. In the inverse demand models, all goods are assumed to be rationed and thus the individual has no freedom to choose both x and q. In order to derive a complete inverse demand model, we take inverses for the $q(.)$ and $x(.)$ for the whole.[11] We would then have:[12]

[6] The concept of virtual prices was first suggested by Rothbarth (1940–41) and it is defined as those prices which would induce an unrationed individual to behavior in the same manner as when faced with a given vector of ration constraints.

[7] I would rather call it "conditional marginal-willingness-to-pay function."

[8] Randall and Stoll (1980) ignored the distinction between them.

[9] Note that the conditional inverse Marshallian demand function has its argument p_x because good x is not a set of rationed goods. It is crucial to distinguish from the complete inverse demand functions.

[10] For more discussion, see chavas (1984).

[11] To obtain a system of inverse demand functions from the ordinary demand system, conditions for global invertibility are required [see Gale and Nikaido (1982); L. Cheng (1985)].

[12] Note that in inverse demands, income is always linearly homogeneous in $f(.)$ and that income net of the cost of $q(=n)$ is not an argument any more because p_q is not a fixed parameter.

$$p_x = f_1(x,\ q,\ m)$$
$$p_q = f_2(x,\ q,\ m) \tag{5}$$

As in the conditional inverse demands, the complete inverse demand functions are marginal willingness to pay functions under the assumption that all goods are rationed and prices are freely adjusted to the Walrasian equilibrium. This is obviously not to be confused with the inverse of the Marshallian demand functions, where only one price at a time is allowed to vary as in (4). When discussing inverse demands in the one good context, the distinction will vanish. For any good, the slope of the inverted Marshallian demand function (4) is simply the inverse of the slope of the ordinary Marshallian demand function. This is generally not true for inverse demand functions; the calculation of the slope property of (5) from knowledge of the ordinary Marshallian demand function will require the inverse of a Jacobian matrix.[13] The case where the slopes of the two are equal will occurs when all uncompensated cross price effects are zero. In this world distinct from Randall and Stoll's, we take a dual approach to derive *WTP* and *WTA* in the following section.

Ⅲ. THE WILLINGNESS-TO-PAY AND-ACCEPT IN INVERSE DEMAND MODELS : A DUAL APPROACH

Suppose now that the individual could purchase q in a hypothetical market at the price p_q, so-called the virtual price, in addition to x at the price p_x. Note that this market for q is entirely hypothetical because q is a fixed quantity of environmental good's characteristics or rationed goods and the individual does not have freedom to choose q. The dual problem to (1), given that all quantities are rationed, is:

$$D(x,\ q,\ u) = Min_{\pi}[\pi_x x + \pi_q q \mid U(x,\ q) = U^0], \tag{6}$$

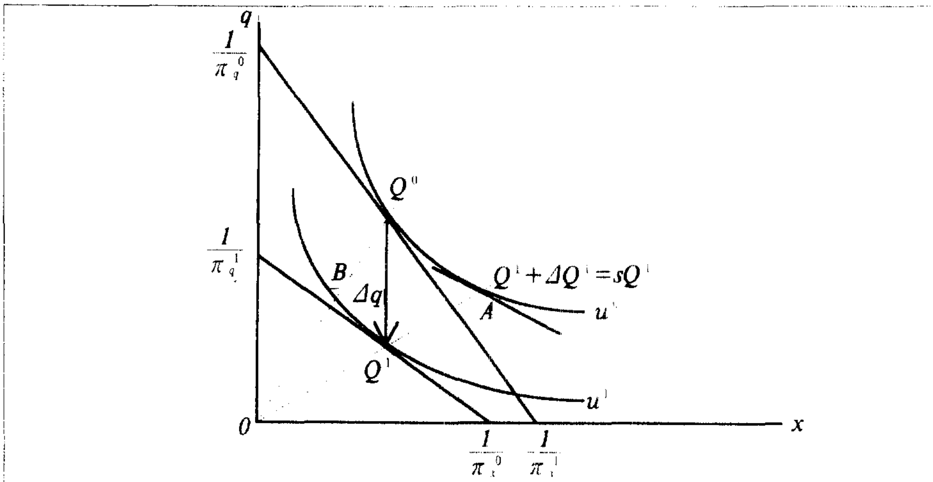
where π_x denotes a normalized price of x by income, π_q is a normalized price of q by income, and D(.) is a distance function or indirect expenditure function.[14] This generates a set of inverse compensated demand functions,
$\pi_x = \pi_x(x,\ q,\ u)$ and $= \pi_q = \pi_q(x,\ q,\ u)$. They give the relation between quanti-

[13] See L.Cheng (1985) for more general discussion. In this case, the reciprocal of the direct price flexibility (f_i) forms the lower limits, in absolute terms, of the direct price elasticity (e_i): $|e_i| \geq |f_i^{-1}|$.

[14] As is well known, the property of the distance function is that it is continuous in its arguments, is decreasing in u and increasing, linearly homogeneous, and concave, first and second differentiable almost everywhere in quantities-see Deaton (1979, 1981). This function has a rather natural interpretation: it represents a scaling factor which scales all quantities up or down to attain a given indifference curve. Note that in the special case when $u = U(q^0)$, we have that $D(u,\ q) = 1$. Hence, we can always write the direct utility function in the equivalent implicit form $D(u,\ q) = 1$.

ty-constrained compensating and equivalent variations (call them *WTP* and *WTA*, respectively).[15] Note that as shown by Deaton (1979), a useful property to be used is:

[**Figure 1**] The welfare changes in inverse demands



$$\frac{\partial D(x, q, u)}{\partial q} = \pi_q(x, q, D(x, q, u)) = \pi_q(x, q, u). \tag{7}$$

Consider the case where q decreases while x is unchanged. The individual is confronted with a change in his bundle of the good q from q^0 to q^1 where $q^1 < q^0$ without loss of generality. Let $\pi_q^0 \equiv \pi(x, q^0, u^0)$ and $\pi_q^1 \equiv \pi(x, q^1, u^1)$ denote the prices ('marginal-willingness-to-pay function') that would have supported q^0 and q^1, respectively. This can be illustrated with the help of Figure 1. Let the base indifference curve be u^0 in the diagram and assume homothetic preferences for simplicity. For our convenience, we represent $U(x, q)$ as $U(Q)$ where $Q=(x, q)$, a vector of quantities. Furthermore, we express $U(Q)$ as $U(sQ^0)$ where $Q^0=(x^0, q^0)$, i.e., the initial consumption bundle, and s is a scalar measure of proportional scale change in x and q. Thus, we may write $U(sQ^0)$ as $U(Q^0, s)$ for some purposes. In this function $U(Q^0, s)$, if $s=1$ then we have the initial utility level.[16] Geometrically, the value of the distance for Q^1 is the ratio $OQ^1/OA(=1/s)$ since it gives the amount by which Q^1 must be divided to bring it to the indifference curve u^0.[17]

Note that, since the individual is presumed to face fixed quantities, the compen-

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[15] It would be better to use *QWTP* (quantity-constrained *WTP*) and *QWPA* (quantity-constrained *WTA*) to distinguish them from the more typical *WTP* and *WPA* in response to a price change. But we use typical terminology to highlight similarity.

[16] Since $U(Q)=U(Q^0, s)$ and an inverse of $U(.)$ gives $D(.)$, s may be defined as $D(Q^0, u)$, which implies that for fixed quantities, it is a utility measure.

[17] Recall that the distance function can be defined implicitly by $U\{Q/D(Q,u)\} = u$. See Deaton (1979) for details.

sation cannot take the form of money income, as is usually considered in measuring welfare changes due to price changes. In the quantity-constrained case, we are only able to compensate him in the form of quantities, which are exogenous to him. For this reason, the compensation scheme in inverse demand systems is quite different from the one in ordinary or mixed demand systems. The former is more pure compensation scheme than income-compensation scheme in the sense that scale compensation (or quantity compensation) gives the consumer the resources sufficient to increase consumption in equal proportions while, for income-compensation, the consumer is offered the resources sufficient to increase consumption in equal proportions, but then is allowed to substitute to a more desired bundle if marginal valuations change. Therefore, the quantity-based compensating variation (*QCV*) may be defined how much scale to compensate to increase the new consumption bundle (q^1) until the consumer is indifferent between the compensated bundle and the initial bundle (q^0). In Figure 1, we may geometrically define *QCV* by OA-OQ^1. Mathematically, using the direct utility function, the quantity-based compensating variation (*QCV*) may be defined implicitly by

$$U\left[\frac{Q^0}{D(Q^0, u^0)}\right] = U\left[\frac{Q^1}{D(Q^1, u^1) - QCV}\right] \tag{8}$$

In our convenient form of preferences, his implicit *WTP* to avoid a change can be rewritten as

$$u^0 = U[x, q^0, D(x, q^0, u^0)] = U[x, q^1, D(x, q^1, u^1) + WTP] \tag{9}$$

Similarly, the quantity-based equivalent variation (*QEV*) can be defined how much scale to take away from the consumer to reduce the initial consumption bundle (q^0) until the consumer is indifferent between the new consumption bundle and the compensated bundle. In Figure 1 again, we may geometrically define *QEV* by OQ^0-OB. Mathematically, *QEV* may be defined implicitly by

$$U\left[\frac{Q^0}{D(Q^0, u^0) + QEV}\right] = U\left[\frac{Q^1}{D(Q^1, u^1)}\right] \equiv u \tag{10}$$

In our convenient form of preferences, his implicit *WTA* to accept a change can be rewritten as

$$U[x, q^0, D(x, q^0, u^0) - WTA] = U[x, q^1, D(x, q^1, u^1)] \equiv u^1 \tag{11}$$

Both *WTP* and *WTA* measure the amount by which the degree of rescaling of Q^0

exceeds the degree of rescaling of Q^1, i.e., a quantity metric welfare affecting quantity change from Q^0 to Q^1. If *WTP* and *WTA* are positive, then q^0 is preferred to q^1. Note that *WTP* and *WTA* depend in no way on how utility is measured, but depend only on the indifference curve indexed by u^t, $t=0, 1$.

As the inverse of indirect utility functions give expenditure functions, the inverse of direct utility functions give distance functions. Thus, *WTP* and *WTA* measures defined in (9) and (11) can be expressed in the explicit forms:

$$WTP = D(x, q^1, u^0) - D(x, q^1, u^1) = D(x, q^1, u^0) - D(x, q^0, u^0) \quad (12)$$

$$WTA = D(x, q^0, u^0) - D(x, q^0, u^1) = D(x, q^1, u^1) - D(x, q^0, u^1) \quad (13)$$

Combining (12) and (13) with (7) and using the Fundamental Theorem of Calculus,

$$WTP = \int_{q0}^{q1} \pi_q(x, q, u^0)dq \quad (14)$$

$$WTA = \int_{q0}^{q1} \pi_q(x, q, u^1)dq \quad (15)$$

These formula express *WTP* and *WTA* as areas under inverse demand curves between the old and new quantity verticals. The only distinction between *WTP* and *WTA* is the level of utility the compensation is designed to reach. It should be emphasized that *WTP* and *WTA* are not monetary measures but unitless scale measures since all prices are normalized by monetary value, income. To recover monetary values, they should be multiplied by income.

IV. RANDALL AND STOLL'S BOUND IN INVERSE DEMAND MODELS

Using a procedure similar to Willig's (1976), Randall and Stoll (1980) derive bounds on the difference between *WTP* and *WPA*. In order to do this in inverse demand models, the area under an inverse demand function for q is defined as

$$A = \int_{q0}^{q1} \pi_q(x, q, s)dp$$

and the scale flexibility is defined as:[18]

$$\frac{\partial \log \pi_q(x, q, s)}{\partial \log s} \tag{16}$$

where s is a scalar measure of consumption bundle (x, q). Assuming that κ is bounded from below by κ^L and from above by κ^U over the range from (x, q^0) to (x, q^1), and using the mean-value theorem, and integrating (7), we obtain if $\kappa^L \leq 1$ and $\kappa^U > 1$:[19]

$$\kappa^L \frac{A^2}{2} \leq WTP - WTA \leq \kappa^U \frac{A^2}{2} \tag{17}$$

which is similar to Hanemann (1991)'s equation 15. The important differences between this bound and Randall and Stoll's bound are κ and A in (17) [ξ and A in Randall and Stoll's]. The term κ is the scale flexibility defined as (16), while Randall and Stoll's ξ is the income elasticity of the conditional inverse Marshallian demand function defined as

$$\xi \equiv \frac{\partial log p_q(p_x, q, m)}{\partial log m}$$

Our term A is the area under an inverse demand function for q, while their A is the area under an conditional inverse Marshallian demand function for q.

As Hanemann (1991) asked, does (17) tell us if *WTP* and *WTA* are likely to be close in value? To answer this question, we examine the relationship between κ and η (income elasticity) to see the likely magnitude of the scale flexibilities. Using techniques by Park and Thurman (1996), we obtain:[20]

$$\kappa = \frac{\eta - 1}{\sigma_{21}} - 1 \tag{18}$$

where η is income elasticity and σ_{21} is the aggregate Hicks elasticity of substitution between q and x. Equation (18) is analogous to the "fundamental result" (equation 17) in Hanemann (1991). The critical differences between (18) and Hanemann's are ξ and σ_{21} in our relation [ξ and σ_0 in Hanemann's]. Our σ_{21} is the Hicks elasticity of substitution, while Hanemann's σ_0 is the Allen-Uzawa elasticity of substitution. Both

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[18] Randall and Stoll called this the "price flexibility of income" in the setting of ordinary demand model with quantity restriction.

[19] See Appendix A for derivation.

[20] See Appendix B for derivation.

of them are the same within two good case. In three good case, however, they will be different assuming the third good does not move to the same direction in quantity.[21)]

As Hanemann (1991) found, the extent of the difference between *WTP* and *WTA* depend not only on income effects but also substitution effects. In the case of zero substitutability between q and x, *the difference between WTP* and *WTA* could be infinite, which is the same conclusion as Hanemann except for the different definition of substitutability and price flexibility. However, the exact relationship is different from Hanemann's and so equation (18) should be used in inverse demand models. Even if we use the same meaning of substitutability (σ) as Hanemann's and $\kappa = \xi$, the interpretation and results are not the same. In the Hanemann's (1991) result, if either $\eta = 0$ (no income effects) or $\sigma_{21} = \infty$ (perfect substitution between q and x) over the relevant range, then Hanemann's $\kappa^{L} = \kappa^{U} = 0$, thereby $WTP = WTA = A$.[22)] In (18), if $\eta = 0$(no income effects) then

$$\kappa = \frac{-1}{\sigma_{21}} - 1 \tag{19}$$

Thus we still have substitution effects which may give a substantial divergence between *WTP* and *WTA*. If $\sigma^{21} = \infty$ (perfect substitution between q and x) over the relevant range, then we have $\kappa = -1$, thus still obtaining the error of approximation for A. Therefore, some of Hanemann's results should be modified to adapt them into inverse demand models since, in some cases, they may give completely misleading interpretation of empirical studies.

V. SIGNIFICANCE OF DIFFERENCE FROM HANEMANN'S RESULTS

This section provides the significance of difference between our result and Hanemann's result by taking a simple example. Consider an individual who maximizes the quasi-concave utility function, $U(x, q)$ where x denotes conventional market goods and q represents the supply of environmental good or amenity. Suppose that the individual could purchase x at the (hypothetical) price p_x and the price of q would be fixed at p_q. We further assume that he has the following utility function:

$$U(x, q) = a\log q + x \tag{20}$$

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[21] For more-than-two good case, both elasticities of substitution are not relevant. As Blackorby and Russell (1989) point out, the indirect (or dual) Morishima elasticity of substitution should be used in the primal space.

[22] By Hanemann's κ (he actually used the notation of ξ), I mean that it has its arguments, price of x, income and quantity of good q, while our κ has its arguments such as quantities of good x and q.

In Hanemann's setup, the demands for x and q are given by:

$$P_q = ap_x/q$$

$$x = m/p_x - a \tag{21}$$

where m denotes income. It follows from (21) that the price flexibility of income for q is:

$$\xi = \frac{\partial log p_q(q, p_x, m)}{\partial log m} = 0 \tag{22}$$

The term A also can be calculated simply as:

$$A = \int_{q0}^{q1} \frac{ap_x}{q} dq \tag{23}$$

Turning to our setup where x is assumed as rationed market goods, (20) gives the following inverse demands for x and q:

$$\pi_q = a/q(a+x),$$

$$\pi_x = 1/(a + x) \tag{24}$$

From (24), the scale flexibility of q can be obtained as:

$$\kappa = -1 - \frac{x}{a+x} = -1 - W_1, \tag{25}$$

where w_1 denotes the budget share of good x. The term A for (24) is given by:

$$A = \int_{q0}^{q1} \frac{a}{q(a+x)} dq \tag{26}$$

To compare with Hanemann's result, we let $m = 1$ and $p_x^* = 1/[a+x^*]$ at the optimum in (21). In this case, (23) and (26) would be equivalent. However, Randall and Stoll's bounds are significantly different between them because (22) and (25) are much different. Furthermore, if we do not assume that $p_x^* = 1/[a+x^*]$ at the optimum in (21), we have different terms of "A" in addition to the different bounds. Therefore, Hanemann's and ours describe, in fact, a completely different world.

As seen in the previous section (section III), the fundamental difference comes from different compensation schemes. While scale compensation in a complete inverse demand model provides the consumer with the resources sufficient to increase consumption in equal proportions, income-compensation in ordinary and mixed demand models offers the consumer the resources sufficient to increase consumption in equal proportions but then allow him to substitute to a more desired bundle if marginal valuations change. These different compensation schemes lead to different concepts of *WTA* and *WTP* and different bounds.

VI. CONCLUSION

In this paper, we have reexamined Randall and Stoll's bound in the inverse demand models. It finds that the bound is still hold but the concepts should be adapted to the inverse demand models. Next, we also reexamined Hanemann's fundamental result that the difference between *WTP* and *WTA* depends not only on an income effect but also on a substitution effect. This paper finds that his result is accurate and still validates analogy to the inverse demand models. However, the exact relationship is not the same as Hanemann's and also some of his interpretation becomes incorrect if it is applied to the inverse demand models. The fundamental difference comes from different compensation schemes appropriate to the models. It leads to different concepts of *WTA* and *WTP* and different bounds. Thus, it suggests that our bound and fundamental result should be used in deriving appropriate welfare measures and correct interpretation of empirical results from complete inverse demand models.

APPENDIX A

Derivation of equation (17)

Since the scale flexibility is defined as

$$\kappa \equiv \frac{\partial log\pi_q(x, q, s)}{\partial logs} \tag{A1}$$

where s is a scalar measure of scale of consumption bundle (x, q), we have

$$\frac{\partial \pi_q}{\partial s} = \kappa \frac{\pi_q}{s} \tag{A2}$$

Integrating both sides. we have:

$$\pi_q(x, q, s) = \pi_q(x, q^*)s^{\kappa} \tag{A3}$$

where q^* is a reference quantity for q. Substituting this into (7) yields:

$$\frac{dD(x, q, u)}{dq} = \pi_q(x, q, u) = \pi_q(x, q, D(x, q, u)) = \pi_q(x, q^*)s^{\kappa} \tag{A4}$$

$$D(x, q^0, u^0) = 1 \tag{A5}$$

where q^0 is a reference quantity demanded and u^0 denotes the initial utility level. Solving the differential equations, we obtain a distance function:

$$D(x, q, u) = [1 + (1-\kappa)\int_{q1}^{q1} \pi_q(x, q)dq]^{1/1-\kappa} = [1 + (1-\kappa)A]^{1/1-\kappa} \tag{A6}$$

Using the Taylor approximation,

$$(1-a)^{1/1-\kappa} \approx 1 + \frac{a}{1-\kappa} + \frac{\kappa a^2}{2(1-\kappa)^2} \tag{A7}$$

we obtain:

$$D(x,\ q^1,\ u^0)-1=A+\frac{\kappa A^2}{2}=WTP \tag{A8}$$

Similarly, we can get:

$$1-D(x,\ q^0,\ u^1)=A-\frac{\kappa A^2}{2}=WTA \tag{A9}$$

Assume that there are upper and lower bounds on κ (κ^U and κ^L, respectively), with neither equal to one. Using the mean-value theorem on $\pi_q(s^2)/\pi_q(s^1)$ and assuming conditions in proposition 3 of Haneman's are satisfied, then we can obtain equation (17). Specifically, the Mean Value Theorem says:

$$\left(\frac{s_2}{s_1}\right)^{\kappa^L}\leq\frac{\pi_q(x,\ q,\ s_2)}{\pi_q(x,\ q,\ s_1)}\leq\left(\frac{s_2}{s_1}\right)^{\kappa^U} \tag{A10}$$

for $s_2\geq s_1$.

Setting $s_2=D(x,\ q^1)$ and $s_1=D(x,\ q^0)=1$, where I drop its argument u for simple expression, we have:

$$D(x,\ q^1)^{\kappa^L}\leq\frac{\pi_q(x,\ q,\ D(x,\ q^1))}{\pi_q(x,\ q,\ 1)}\leq D(x,\ q^1)^{\kappa^U} \tag{A11}$$

Rearranging and substituting from (A4), we obtain:

$$0\leq\pi_q(x,\ q,\ 1)^{-\kappa^U}\leq\frac{\partial D(x,\ q^1)}{\partial q}D(x,\ q^1)^{-\kappa^U} \tag{A12}$$

and

$$0\leq\frac{\partial D(x,\ q^1)}{\partial q}D(x,\ q^1)^{-\kappa^L}\leq\pi_q(x,\ q,\ 1). \tag{A13}$$

Integrating (A12) and (A13) and rearranging, we have:

$$[1+(1-\kappa^L)A]^{1/1-\kappa^L}\leq D(x,\ q^1,\ u^0)\leq[1+(1-\kappa^U)A]^{1/1-\kappa^U} \tag{A14}$$

if neither of κ is equal to one and the terms inside the brackets are positive. Reversing the roles of q^1 and q^0 in (A14), we have:

$$[1-(1-\kappa^L)A]^{1/1-\kappa^L} \leq D(x, q^0, u^1) \leq [1-(1-\kappa^U)A]^{1/1-\kappa^U} \quad \text{(A15)}$$

Applying Taylor approximation of (A7) to equations (A14) and (A15), and invoking definitions of *WTP* and *WTA*, we get the required bounds of (17).

APPENDIX B

Derivation of equation (18)

Let $\varepsilon^*=[\varepsilon_{ij}^*]$ be the matrix of price elasticities of the Hicksian demands,

$$\varepsilon_{ij}^* = \frac{\partial \ln f_i^*(p, u)}{\partial \ln p_j}, \quad i, \ j=1,\cdots\cdots,n \quad \text{(B1)}$$

where $f_i^*(\cdot)$ denotes the Hicksian demand functions. The price homogeneity condition implies

$$\sum_j \varepsilon_{ij}^* = 0, \quad i,j=1,\cdots\cdots,n \quad \text{(B2)}$$

and symmetry implies that

$$\varepsilon_{ij}^* w_i = \varepsilon_{ij}^* w_j, \quad i,j=1,\cdots\cdots,n \quad \text{(B3)}$$

where w_i is the budget share of good *i*. The adding-up condition gives

$$\sum_i w_i = 1 \quad \text{(B4)}$$

Let $\eta = [\eta_i]$ be the vector of income elasticities of ordinary demand. Engel aggregation requires that:

$$\sum_i \eta_i w_i = 1 \quad \text{(B5)}$$

We define $\varepsilon = [\varepsilon_{ij}]$ as the matrix of uncompensated price elasticities of the

Marshallian demands, i.e., $q=f(p, m)$:

$$\varepsilon_{ij}=\frac{\partial \ln f_i(p,m)}{\partial \ln p_j} \tag{B6}$$

Then, we can obtain the Slutsky equation in elasticity form as follows:

$$\varepsilon_{ij}=\varepsilon_{ij}^{*}-\eta_i w_j \tag{B7}$$

Consider the case where $n=2$. Combining (B2)-(B5) and (B7), we can write

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11}^{*}-w_1\eta_1 & -\varepsilon_{11}^{*}-(1-w_1)\eta_1 \\ \dfrac{-\varepsilon_{11}^{*}w_1-w_1(1-w_1\eta_1)}{1-w_1} & \dfrac{\varepsilon_{11}^{*}w_1-(1-w_1)(1-w_1\eta_1)}{1-w_1} \end{bmatrix} \tag{B8}$$

Let $B=[b_{ij}]$ be the matrix of uncompensated price flexibilities of inverse demand. Since $d\log(p)=\varepsilon^{-1}d\log(q)-\varepsilon^{-1}\eta\, d\log(m)$, the matrix B can be obtained by inversion of ε:

$$B=\varepsilon^{-1}$$

$$=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$=\begin{bmatrix} -w_1+\dfrac{(1-w_1)(1-w_1\eta_1)}{\varepsilon_{11}^{*}} & -(1-w_1)-\dfrac{(1-w_1)^2\eta_1}{\varepsilon_{11}^{*}} \\ -w_1\left(1+\left(\dfrac{1-w_1\eta_1}{\varepsilon_{11}^{*}}\right)\right) & (1-w_1)\left(-1+\left(\dfrac{w_1\eta_1}{\varepsilon_{11}}\right)\right) \end{bmatrix} \tag{B9}$$

Using the relation of k_i to b_{ij}, we can derive the explicit expression for the scale flexibility:[22]

$$k_1=-1+\frac{(1-w_1)(1-\eta_1)}{\varepsilon_{11}^{*}}=-1-\frac{(1-w_1)(1-\eta_1)}{\varepsilon_{12}^{*}} \tag{B10}$$

Using $\varepsilon_{ij}^{*}=\sigma_{ij}w_j$, we obtain the required expression, (18). For the multi-good cases, see Park and Thurman (1996).

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[22] $k_i=\sum_j b_{ij}$ by definition of k_i and homogeneity.

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