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EXACT MAXIMUM LIKELIHOOD ESTIMATION OF FRACTIONAL MODELS AND TIME SERIES NATURE OF THE REAL GNP OF KOREA*

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An exact maximum likelihood estimation algorithm for estimating univariate ARFIMA(p, d, q) model is suggested. The transformation method based on Cholesky decomposition is useful in that it can easily incorporate well-established techniques of estimating ARMA(p, q) model and also avoids the problems of Sowell (1992a). For the development of an exact maximum likelihood estimation method that is used throughout the study, the algorithm of Ansley(1979) for general ARMA(p, q) model is incorporated. A small simulation shows that any one of maximum likelihood estimation method and the method of Li and McLeod(1986) does not clearly dominate the other in precision and dispersion, although the latter has known truncation problem, in the samples of length 100. The performances of maximum likelihood estimation when the true fractional differencing parameter is close to the boundary of 0.5 are relatively poor, however that can be improved by reparameterizing d. Estimating ARFIMA models by exact maximum likelihood method developed in this study, a time series nature of quarterly real GNP series of Korea is analyzed. When two intervention variables are introduced under the assumption of known events, all three types of the models, deterministic trend model, stochastic trend model, and more general fractional model, seem to estimate the series reasonably well. However, the likelihood ratio test and spectral analysis show that the stochastic trend representations appear to be less adequate in capturing the low frequency behavior of GNP than fractional representations and deterministic trend representations.

I. INTRODUCTION

Recently ARFIMA(p, d, q) (autoregressive fractionally integrated moving

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average) model is receiving more attention as an alternative to standard ARIMA (p, d, q) (autoregressive integrated moving average) model as the former can depict more general type of long-run dependence in the data than the latter can do. Granger(1980) showed that univariate fractionally integrated series can be obtained by aggregating simple, possibly dependent, dynamic microeconomic variables. Most of important macro-variables are thought to have the data generating process of that nature, for instances GNP, unemployment, interest rates, personal consumption, exchange rates, etc. Numerous studies have been followed to estimate the fractional integration parameter in macroeconomic series. However, the estimation techniques employed in earlier studies appear to have more limitations than the maximum likelihood estimation method suggested in Sowell(1992a).

Especially, ARFIMA(p, d, q) models are general enough to incorporate both of the competing stochastic trend model and deterministic trend model as two extremes. The integer value restriction of $d=1.0$ means stochastic trend model and $d=0.0$ means deterministic trend model in a ARFIMA(p, d, q) representation. The traditional distinction and testing between difference stationarity and trend stationarity become less justified as the integration order d can take real values. The unit root tests with fractionally integrated series are known to have different limiting distributions due to flexible integration parameter and naturally conventional unit root tests have quite low power against fractionally-integrated alternatives. [see Sowell(1990) and Diebold and Rudebusch(1989)] As the correct specification of a series is very important for the testing, it can be rewarding to employ ARFIMA(p, d, q) modeling for model selection, which explicitly takes account all parameters including integration parameter d, as illustrated in Sowell (1992b).

For the estimation of fractionally integrated series Sowell(1992a) suggests a maximum likelihood estimation method that does not suffer from truncation or approximation unlike other methods. His simulation shows that, at least, unconditional maximum likelihood method appears to perform better than the estimation method of Geweke and Porter-Hudak(1983) and Fox and Taqqu(1986). However, in general ARFIMA(p, d, q) form, the estimation algorithm of Sowell is somewhat complicated for programming and also conveys a few other problems. It can not handle the model with repeated AR roots as the partial fraction decomposition simply can not be applied. The calculation of autocovariances becomes cumbersome when roots are complex, and likely to have some degree of inaccuracy when at least one AR root is approaching unit circle boundary, because truncation is inevitable to evaluate infinite summations. [see p. 174 equation (9)]

As an alternative to Sowell, an exact maximum likelihood method is pursued in this study, which is not only simple and flexible to use in that we can incorporate well-established algorithms of ARMA estimation, but does not suffer those problems of Sowell. And then a small sample performance of maximum likelihood method is investigated, also with that of another time-domain estimation method of Li and McLeod(1986)[1]. This comparison will provide more convincing evaluation of maximum likelihood estimation. Additionally, as an application of ARFIMA model, more appropriate representations for the real GNP of Korea are investigated. The issues of difference stationarity and trend stationarity in the series will be addressed in a more general context of considering ARFIMA alternative.

Next section presents a method for evaluating the maximum likelihood for ARFIMA(p, d, q) model employing the Cholesky decomposition of covariance matrix, which can be easily combined with many conventional ARMA algorithms. In this study, the algorithm of Ansley(1979) for general ARMA(p, q) model is incorporated. A small simulation shows that maximum likelihood method does not dominate the method of Li and McLeod, vice versa, in bias and root-mean-squared error in the entire range of d, even though the latter has known truncation problem. Also, it is shown that the bias of maximum likelihood method, which becomes larger as true fractional integration parameter d approaches boundary of 0.5, can be improved by reparameterizing d away from boundary into safe region. Section III shows that the GNP series can be reasonably estimated in terms of all three models; stochastic trend model, deterministic trend stationary model, and ARFIMA(p, d, q) model with no restriction. Those models are estimated with two intervention variables under the assumption of known events, the second oil shock and political crisis that largely overlap in time. However, the results of likelihood ratio test for restrictions and spectral analysis reveal that, not like the other two representations, the stochastic representations do not appear to capture the low frequency behavior of real GNP series. The final section summarizes the results and concludes.

II. FRACTIONAL TIME SERIES MODEL

2.1 Aspects of Fractional Model

Fractional time series model, often called long-memory model, is distinguished from usual ARMA(p, q) model in that it can parsimoniously capture wide range of long-run dependence with a single real valued integration parameter. The autocorrelation function $\rho(\cdot)$ of fractional time series exhibits a hyperbolically decaying pattern, while that of stationary ARMA(p, q) series shows a geometrically decaying pattern. Thus the former is called long memory process and the latter short memory one. A time series $y_t=\{y_1,\cdots, y_T\}$ following a long

[1] In Sowell(1992a), the method of Li and McLeod was ignored as inferior method due to truncation problem. However, no simulation evidence was presented. This study shows that small simulation results do not support his assertion in later section.

memory process of ARFIMA(p, d, q) is denoted as

$$\phi(L)(1-L)^d y_t = \theta(L)\varepsilon_t, \qquad -0.5 < d < 0.5 \tag{1}$$

where $\varepsilon_t \sim iid(0, \sigma^2)$, AR polynomial $\phi(L) \equiv (1-\phi_1 L - \cdots - \phi_p L^p)$ and MA polynomial $\theta(L) \equiv (1-\theta_1 L - \cdots - \theta_q L^q)$ have all roots out side the unit circle and no common zeros, and $(1-L)^d$ is the fractional differencing operator.

An ARFIMA(p, d, q) process can have various characteristics according to the values of parameter d. [see Hosking(1981), Granger and Joyeux(1980) for details] The autocorrelation at lag k, $\rho(k)$, is proportional to k^{2d-1} implying that the value of d is crucial to the memory property of a series. As a corollary the autocorrelations do not have a finite sum for $0 < d < 0.5$, and have a finite sum for $d < 0$, which corresponds to strong positive dependency between distant observations for $0 < d < 0.5$ and mild negative dependency between distant observations for $-0.5 < d < 0$. For $d < 0.5$, the process is covariance stationary as it has finite variance, while invertibility requires that $d > -0.5$. On the other hand, for $0.5 \leq d \leq 1.0$, the process is covariance nonstationary as the variance becomes infinite. One peculiar feature of ARFIMA(p, d, q) series is the mean-reverting behavior for $d < 1.0$. For the series the effect of any shock persists, but eventually the effect dies out. This is quite different from the unit root process for $d = 1.0$ where any shock has a permanent effect on the process. [see Cheung and Lai(1993)] The frequency domain feature of ARFIMA(p, d, q) series is characterized by the low-frequency of spectral density function. As the frequency λ approaches zero, the spectral density of ARFIMA(p, d, q) series converges to $c\lambda^{-2d}$, where c is the constant related to a stationary ARMA(p, q) series for $d = 0.0$. Thus, a variety of spectral shapes near the origin can be produced according to the values of d.

ARFIMA(0, d, 0) process, a process driven only by fractionally-integrated errors,

$$(1-L)^d y_t = \varepsilon_t, \tag{2}$$

can be useful for the analysis of general ARFIMA(p, d, q) models. The difference operator $(1-L)^d$ in equation (2) can be rewritten as, by using the binomial expansion,

$$(1-L)^d = \sum_{j=0}^{\infty} h_j L^j, \qquad h_j = \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)}. \tag{3}$$

The notation $\Gamma(\cdot)$ represents gamma function. The binomial expansion is possible as long as $d > -1.0$. The estimation method of Li and McLeod(1986) suggests transforming the original series in the optimization process by employing the finite version of equation (3). Obviously truncation is inevitable and this aspect was pointed out as the source of inferiority by Sowell(1992a). The inverse of $(1-L)^d$ exists if $d < 0.5$ and then we have

$$y_t = (1-L)^{-d}\varepsilon_t \tag{4}$$

$$= u_t, \tag{5}$$

where u_t represents fractionally-driven errors. Similarly, the following can be obtained.

$$(1-L)^{-d} = \sum_{j=0}^{t} h_j^* L^j, \qquad h_j^* = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}. \tag{6}$$

It can be shown that the sequence $\{h_j^*\}$ is square-summable as long as $d < 0.5$. The condition guarantees that the process in equation (2) has a finite variance and is covariance-stationary. [see Granger and Joyeux(1980) and Hamilton(1994) pp. 448-452] As the process is covariance-stationary for $d < 0.5$, we can construct a positive definite symmetric covariance matrix in Toeplitz form. To obtain the autocovariances, first the spectral density of y_t, equivalently u_t, is expressed as

$$f_y(\lambda) = |1-e^{-i\lambda}|^{-2d}\sigma^2/(2\pi) \tag{7}$$

$$= (2\sin(\lambda/2))^{-2d}\sigma^2/(2\pi). \tag{8}$$

The autocovariances at lag period s are denoted by

$$\gamma(s) = \int_{-\pi}^{\pi} e^{is\lambda} f_y\, d\lambda \tag{9}$$

$$= (\sigma^2/\pi)\int_0^{\pi} \cos(s\lambda)(2\sin(\lambda/2))^{-2d} d\lambda \tag{10}$$

$$= \frac{(-1)^s\Gamma(1-2d)}{\Gamma(1-d+s)\Gamma(1-d-s)}\sigma^2, \qquad s = 0, 1, 2, \cdots \tag{11}$$

$$= \frac{\Gamma(1-2d)\Gamma(d+s)}{\Gamma(1-d+s)\Gamma(1-d)\Gamma(d)}\sigma^2. \qquad s = 0, 1, 2, \cdots \tag{12}$$

The derivation of the autocovariances can be found in Brockwell and Davis (1987, p. 468) and Granger and Joyeux(1980). Equation (12), which is more convenient to evaluate, is obtained by applying the equation (8.334.3) of Gradshteyn and Ryzhik(1980). From the results we can construct $(T\times T)$ covariance matrix $\sigma^2\Omega$,

$$\sigma^2 \Omega = [\gamma(|i-j|)], \tag{13}$$

where i, j represents ith row and jth column respectively. As the symmetric matrix Ω in equation(13) is positive definite for $d < 0.5$, the lower triangular matrix can be obtained by Cholesky factorization. The inverse of Ω is decomposed as

$$\Omega^{-1} = (CC')^{-1}, \tag{14}$$

where C is the lower triangular matrix. Then it is easily shown that C^{-1} can be applied to transform $\{y_t\}$ into $\{y_t^*\}$, a process driven by white noise errors as the following,

$$Y_T^* = C^{-1} Y_T, \tag{15}$$

where $Y_T^* = (y_1^* \cdots y_T^*)'$ and $Y_T = (y_1 \cdots y_T)'$ respectively. This transformation will provide a tool to handle general ARFIMA(p, d, q) models in the following subsection.

2.2 Estimating ARFIMA(p, d, q) Model

The transformation procedure of equation (15) can be extended to general ARFIMA(p, d, q) models. If $d < 0.5$, the stationary and invertible ARFIMA model (1) can be rewritten as

$$y_t = \phi^{-1}(L)\theta(L)(1-L)^{-d}\,\varepsilon_t \tag{16}$$

$$= \psi(L)u_t. \tag{17}$$

Given d, as the autocovariance matrix of u_t can be calculated, the transformation of y_t is straightforward. Note that the fractionally-driven errors are changed into nicely-behaving ones by the transformation, and the transformed series becomes usual ARMA(p, q) series. Assume this stationary and invertible mean-adjusted ARMA(p, q) series is with covariance matrix $\sigma^2\Phi$, where the matrix Φ has the Cholesky decomposition $\Phi = DD'$. This implies that the original series y_t can be transformed twice to obtain a sequence $\{\varepsilon_t\}$ of *iid* normal random variables, that is

$$\varepsilon_T = D^{-1}C^{-1}y_T \tag{18}$$

$$= D^{-1}y_T^* \tag{19}$$

where $\varepsilon_T = (\varepsilon_1 \cdots \varepsilon_T)'$. Accordingly, the likelihood function of ARFIMA(p, d, q) takes the form of

$$l(d, \phi, \theta, \sigma^2 \mid Y_T) = (2\pi\sigma^2)^{-T/2} \mid C \mid^{-1} \mid D \mid^{-1} \exp\left(-\frac{1}{2\sigma^2} \varepsilon_T' \varepsilon_T \right), \; Y_t \in R^n, \quad (20)$$

where the determinants $|C|$ and $|D|$ are the products of their diagonal elements respectively. A maximum likelihood can be obtained by evaluating the log likelihood[2],

$$-\frac{T}{2}\left[\log(2\pi) + 1 + \log\left(\frac{1}{T} \sum_{t=1}^{T} \tilde{\varepsilon}_t^2 \right) \right], \quad \tilde{\varepsilon}_t = (|C||D|)^{1/T} \varepsilon_t . \quad (21)$$

Many existing algorithms for estimating ARMA(p, q) process can be easily combined into the estimation process and executed by slightly modifying the log likelihood function of equation (21). For the estimation of ARFIMA(p, d, q) models throughout the study, an exact maximum likelihood estimation method for general ARFIMA(p, d, q) process is used, which incorporates the exact maximum likelihood estimation method of Ansley(1979) for ARMA(p, q) model. The matrix Φ in the variance-covariance matrix for ARMA structure is constructed following the algorithm of Ansley(1979) and its lower triangular matrix D is used appropriately for equation (19) and (21). The above likelihood for ARFIMA(0, d, 0) process is exactly the same as that of Sowell(1992a). Due to the consecutive data transformation, the maximum likelihood estimation suggested in this study may be more time-consuming. However, the effort for writing computer program can be minimized by partially rewriting the existing source code for ARMA(p, q) estimation.[3] And the problems of Sowell noted earlier can be avoided because we do not need to follow his algorithm any more.

The estimation method of Li and McLeod(1986) that approximates maximum likelihood method transforms the data using equation(3). The log likelihood of Li and McLeod can be considered as an approximation of equation (21), but $|C|$ should be replaced by 1. The reason is that the first observation gets the full weight of 1 and thus all the diagonal elements of matrix are 1's. Technically, the transformation is executable as long as $d > -1$, and even when the series do not

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[2] The estimator $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2$ is employed for the derivation of log likelihood,

$Log\, l(d, \phi, \theta, \sigma^2 \mid Y_T) = -\frac{T}{2} \log 2\pi - \frac{T}{2} \log\left(\frac{1}{T} \sum_{t=1}^{T} \tilde{\varepsilon}_t^2 \right) - \frac{T}{2}$.

[3] In general, user-specific econometric packages are not flexible enough to rewrite the source code. In this study the author used Gauss, a matrix-oriented compiled language and interpreter, and Gaussx, an add-on program to Gauss.

have a finite variance, and also can save more computation time compared with the other exact maximum likelihood methods. The estimator is consistent and asymptotically normal sharing the same large-sample distribution as that of exact maximum likelihood estimator for $-0.5 < d < 0.5$. If we visualize the differences of weights given to the observations by each of the binomial expansion and the inverse of the Cholesky decomposition discussed above, the weight functions show differences in magnitude according to the values of d, however, the patterns are very similar. In that sense, a comparison of finite-sample performances of Li and McLeod and the maximum likelihood estimation method discussed above can be interesting. Clearly, the observations are getting ever decreasing weights as the distance becomes longer, and after some periods they will surely contribute only negligibly. This point makes clear that the transformation matrix will be a band matrix with some band width. A preliminary study shows that the band width of around 150 is acceptable for larger size data when $-0.5 < d < 0.5$.

2.3 Small Sample Properties

In large samples, the maximum likelihood estimator presented here and the estimator of Li and McLeod are consistent and asymptotically normal. However, it is difficult to figure out which estimator is more reliable in finite samples before we perform some experiments. The long-run dependence of data is not likely to be fully captured in small samples. A Monte-Carlo simulation is employed to investigate small sample properties by comparing relative performances of the two estimators.

Fractionally integrated series are obtained as the following. First a length of standard normal random variates are generated and then this white noise series is transformed into fractionally-driven errors. The procedure is to premultiply the lower triangular matrix C obtained from the Cholesky decomposition of the matrix Ω in equation (13) to the white noise vector. This exact method is also used in Diebold and Rudebusch(1991) and, originally in terms of autocorrelation matrix, Hipel and McLeod(1978). Next the fractionally-driven errors are fed into ARMA(p, q) process generating scheme. This procedure is repeated to replicate multiple sets of ARFIMA(p, d, q) series. To generate nonstationary fractionally-integrated series for d outside of $-0.5 < d < 0.5$, the parameter d was segmented into partials such that the lower triangular matrices of the partials could be applied consecutively. For instance, if $d = d_1 + d_2$, then C_1 and C_2 are premultiplied sequentially to the vector of white noise errors.

In this study, the log-likelihood function of equation (21) is maximized using the Marquardt algorithms. [Marquardt(1963)] The true parameter values are used as the starting values for iterations. As a parallel to the study of Sowell(1992a), ARFIMA(0, d, 0) model is considered first. Remind that the exact maximum likelihood estimation method suggested here and that of Sowell is equivalent for

ARFIMA(0, d, 0) model. Table 1 reports the results for this model. Wide range of d from 0.8 to -0.9 are considered to see the properties in extreme condition, where nonstationary and noninvertible cases are included on purpose. For each d, 300 samples of length 100 are generated and then estimated. The bias and RMSE (square root of mean-squared error) are calculated. The estimation method of Li and McLeod can be applied as long as $d \rangle -1.0$, as the binomial expansion is possible for $d \rangle -1.0$. In the range from 0.4 to -0.4, the magnitudes of the bias and RMSE are reasonably small and exhibit a flat U-shape pattern. Outside of the region, the magnitude increases gradually. Note that the magnitude becomes much larger as d approaches -1.0 than as d approaches 1.0. In contrast the maximum likelihood method employing the inverse matrix of the Cholesky decomposition is applicable as long as $d \langle 0.5$. However, for $d \geq 0.5$, the Cholesky factorization does not hold as the covariance matrix is no longer positive-definite. The bias becomes much larger as d approaches 0.5, the boundary value where nonstationarity begins. The small RMSEs are due to the boundary estimates close to 0.5. As d approaches -1.0, the magnitude of bias and RMSE is slowly increasing. Obviously, these features, which are not evident in Sowell(1992a), deserve an attention. Table 2 presents the results for ARFIMA(1, d, 0) model. The performances of the estimation method of Li and McLeod and maximum likelihood method are very similar for either $d=-0.3$ or $d=0.0$ with all values of AR parameter ϕ. However, the bias of maximum likelihood estimation becomes substantially larger than that of Li and McLeod for $d=0.3$. First differencing or reparameterization would be helpful in situations like that.

Reparameterization of d can significantly improve the performances of estimation close to the boundary. In table 1, the numbers in the parentheses are obtained by moving the location of d into the neighborhood of -0.2 during estimation. For that purpose, a given d is segmented as $\overline{d}+d^*$ and initial transformation related to $\overline{d}$, which is fixed, was applied first. Next, the likelihood estimation is concentrated for d^*. To evaluate the likelihood function correctly, the two determinants of lower triangular matrices related to $\overline{d}$ and d^* are used respectively. The bias and RMSE obtained in this way are slightly larger than those of Li and McLeod. This can make more reliable the estimation for the series in level as exemplified in the following section. Note that the reparameterization for Li and McLeod is more simple.

We can conclude that, as long as the estimates are concerned, any of the two estimation methods can not dominate the other in the whole range of parameter d with the samples of length 100. Without reparameterization, the Li and McLeod appears to work well in the positive and mild negative range for d. In contrast, maximum likelihood estimation works well in the negative and very small positive range for d. However, the problem that the Li and McLeod employs approximate likelihood due to truncation still bothers when information criteria for model selection are considered.[4] Thus, in the following section, the exact maximum likelihood method for general ARFIMA(p, d, q) suggested in this paper will be applied in order to reveal the time series nature of the real GNP of Korea.

[Table 1] Estimated Parameter Bias and Square Root of the Mean-Squared Error for the ARFIMA(0, d, 0) Model[a] (T = 100, 300 replications)

	Li and McLeod		maximum likelihood	
d	bias	RMSE	bias	RMSE
0.8	−0.038	0.091	n.a.(−0.041)	n.a.(0.092)
0.7	−0.031	0.088	n.a.(−0.033)	n.a.(0.090)
0.6	−0.025	0.086	n.a.(−0.027)	n.a.(0.089)
0.5	−0.020	0.084	n.a.(−0.023)	n.a.(0.089)
0.4	−0.015	0.083	−0.046 (−0.018)	0.080 (0.086)
0.3	−0.010	0.082	−0.026 (−0.016)	0.078 (0.087)
0.2	0.006	0.082	−0.013 (−0.014)	0.078 (0.088)
0.1	−0.004	0.082	−0.006	0.080
0.0	−0.003	0.082	−0.003	0.081
−0.1	−0.004	0.082	−0.005	0.083
−0.2	−0.008	0.082	−0.009	0.084
−0.3	−0.010	0.084	−0.009	0.087
−0.4	0.010	0.089	−0.006	0.089
−0.5	−0.007	0.086	−0.005	0.088
−0.6	0.015	0.091	0.006	0.089
−0.7	0.081	0.131	0.011	0.091
−0.8	0.094	0.141	0.015	0.092
−0.9	0.083	0.134	0.014	0.091

[a] Technically the estimation method of Li & McLeod(1986) employing the binomial expansion can be applied as long as $d > -1$ and the maximum likelihood method as long as $d < 0.5$. The numbers in the parentheses were obtained by artificially moving the location of parameter d to the neighborhood of −0.2. In the process data were transformed by segmenting d, several times if required.

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[4] For ARFIMA(p, d, q) models, the method of Li and McLeod, which employs approximate likelihood instead of exact one, tends to produce larger maximum likelihood values than those of exact maximum likelihood estimation. This implies that the method of Li and McLeod may be less reliable for model selection as we do not know how the quantity of information criteria, which is the sum of log likelihood and penalty value for increasing the number of parameters, will be affected in finite samples.

[Table 2] Estimated Parameter Bias and Square Root of the Mean-Squared Error for the ARIMA(1, d, 0) Model[a] (T = 100, 300 replications)

d	ϕ	Li and McLeod				maximum likelihood			
		d		ϕ		d		ϕ	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
−0.3	−0.7	−0.012	0.093	0.020	0.083	−0.010	0.094	0.019	0.082
	−0.2	−0.025	0.127	0.021	0.144	−0.022	0.133	0.017	0.145
	0.3	−0.091	0.248	0.066	0.254	−0.057	0.210	0.029	0.221
	0.8	0.012	0.186	−0.039	0.158	−0.031	0.183	−0.055	0.158
0.0	−0.7	−0.012	0.093	0.019	0.083	−0.014	0.094	0.019	0.083
	−0.2	−0.019	0.128	0.018	0.145	−0.025	0.150	0.022	0.157
	0.3	−0.078	0.242	0.057	0.251	−0.083	0.241	0.059	0.246
	0.8	0.011	0.190	−0.034	0.160	0.003	0.174	−0.028	0.147
0.3	−0.7	−0.022	0.095	0.021	0.084	−0.047	0.090	0.031	0.086
	−0.2	−0.031	0.132	0.024	0.148	−0.059	0.124	0.048	0.145
	0.3	−0.095	0.246	0.068	0.253	−0.126	0.227	0.101	0.232
	0.8	0.007	0.186	−0.035	0.156	−0.043	0.130	0.000	0.099

[a] For each set of data, initial 600 realizations were discarded to remove possible start-up effect.

Frequently a time series is well estimated in different forms of representations. The real GNP series of Korea can be well represented by either difference-stationary ARIMA(p, 1, q) model or trend-stationary ARMA(p, q) one. In the circumstances more investigation beyond mere estimation of ARFIMA(p, d, q) model to the series is demanded. Accordingly, in the next section attempts are made to narrow down the competing representations. In the process the fact that ARFIMA(p, d, q) model can nest the other two types of models as the two restrictions serves as the basis for further analysis.

Ⅲ. MODELING THE TREND BEHAVIOR OF REAL GNP

3.1 Competing Models and Estimates

The log of the quarterly real GNP of Korea in figure 1 shows a strong upward trend for the periods from 1970:I to 1991:I when the economy sustained a rapid growth.[5] A visual inspection of the plotting indicates that there might

[5] The autocorrelation plot of the series exhibits a slowly decaying hyperbolic pattern.

have been structural breaks in the series. However, it is not an easy task to handle the structural breaks. Frequently the break points and nature of interventions are unknown and appear obscure. That is why the detection of structural breaks and statistical tests in the presence of structural breaks are another research topic in econometrics field. However, sometimes, known events can be used effectively in modeling without much difficulty. In this study, *a priori* known events of oil price shock and unprecedented political instability are modeled in the framework of conventional intervention analysis in time domain model. [see Wei(1990) pp. 184-204]

Historically, the world economy experienced two major oil price shocks in the 1970s. Those are the oil price increase in 1973-1974 and 1979-1980. Especially, the world oil price rose steeply from the end of 1979:I and reached the highest level at 1981:I. After the peak, the price continuously declined until mid-1985. The scarce oil would have decreased the marginal productivity of capital and labor. The supply shock, which entailed upward shifts in aggregate supply curve, could have effects on both of inflation and output. The relative size of the effects on the two variables can be different in general depending on the condition of an economy reflected in the shapes of aggregate demand and aggregate supply. A preliminary estimation did not support the inclusion of an intervention variable to capture the effect of the first oil shock on output. The period of 1979:II-1981:I exhibiting the downturn of the real GNP seems to correspond to the second oil shock period. Another factor that must have contributed to the phenomenon is the political instability in Korea from the end of 1979 to the middle of 1980. The chilling effect of political crisis on output is not likely to be separable from the effect of the oil shock, as the two periods almost overlap.

For explicit modelling of intervention effect, the following two intervention variables are introduced into competing trend models;

$$I_{1t} = \begin{cases} 0, & t < T_2 \\ 1, & t \geq T_2 \end{cases} \tag{22}$$

and

$$I_{2t} = \begin{cases} 0, & t < T_1 \\ t-37, & T_1 \leq t < T_2, \\ 0, & t \geq T_2 \end{cases} \tag{23}$$

where T_1 and T_2 represents 1979:II and 1981:II respectively. I_{1t}, the step function, is expected to capture the change in level, the crash. I_{2t} is to capture the downward trend line over the period of the second oil shock and political crisis as envisaged in figure 1. Modeling the trend behavior of real GNP needs the following types of models.

$$y_t = c + \mu t + \omega_1 I_{1t} + \omega_2 I_{2t} + \phi^{-1}(L)\theta(L)\varepsilon_t, \tag{24}$$

$$(1-L)y_t = \mu + \omega_1 I^*_{1t} + \omega_2 I^*_{2t} + \phi^{-1}(L)\theta(L)\varepsilon_t, \tag{25}$$

$$(1-L)y_t = \mu + \omega_1 I^*_{1t} + \omega_2 I^*_{2t} + \phi^{-1}(L)\theta(L)(1-L)^{-d}\varepsilon_t, \tag{26}$$

$$y_t = c + \mu t + \omega_1 I_{1t} + \omega_2 I_{2t} + \phi^{-1}(L)\theta(L)(1-L)^{-d}\varepsilon_t, \tag{27}$$

where c is an intercept term, and each of μ in (24) and (25) represents trend coefficient and drift coefficient. I^*_{1t}, a pulse function, and I^*_{2t} represents the first differences of the two intervention variables I_{1t} and I_{2t} respectively. Equation (24) expressed in level describes a stationary ARMA(p, q) movement around a deterministic time trend. On the other hand, Equation (25) of stochastic trend model with a unit root depicts a stationary ARMA(p, q) process when the mean is adjusted after the series is first differenced. These two competing models can be nested into fractional model of either equation (26) or (27) as restricted cases. Basically, equation (26) in first difference implies equation (27) in level, vice versa. Clearly, by subtracting the parameter d in equation (27) from another parameter d in equation (26), we get negative one. The restriction of $d = 0.0$ for equation (26) implies equation (25) and the restriction of $d = -1.0$ in equation (26) implies equation (24). Equivalently, if there is a unit root in MA part of equation (26), which is the result of over-differencing, then the equation (26) implies equation (24).

Information criteria, AIC and SBC, are often valuable as a selection rule for unknown order (p, q). In the context of ARFIMA(p, d, q) model, the question is whether these criteria can be used for distinguishing ARFIMA(p, d, q) process from ARIMA(p, 1, q) one. A preliminary experiment shows that AIC and SBC quantities obtained, by maximum likelihood estimation, from the artificially created series in first difference for those competing models overwhelmingly favored ARFIMA representation irrespective of true model.[6] The result implies that the rule of AIC and SBC can not be relied upon as model selection criteria when different types of models such as equations (24) and (25) are compared with ARFIMA models. Sowell(1992b) does not seem to pay attention to the property

[6] For instance, when ARMA(1, 1) and ARFIMA(1, d, 0) models are fitted to 1,000 replications of estimated ARMA(1, 1) model in first difference, AIC and SBC select ARFIMA(1, d, 0) about 75 percent of time. Similar results were found in many cases when true model is ARMA model and that is tested against ARFIMA model with the same order. As those criteria do not appear to be well-balanced for model selection in this study, it is reasonable to use them only for order selection.

during the model selection process, and his results leave something to be reinvestigated.[7]

Table 3 reports the results obtained by estimating equation (25) and (26) in first-difference form to the real GNP series. Note that we use information criteria only for deciding proper order of (p, q). For ARFIMA model, AIC selects ARFIMA(0, d, 2) and SBC selects ARFIMA(0, d, 0). For ARMA model, ARMA(2, 1) and ARMA(2, 2) appear to have a unit MA root respectively, which means models in level are appropriate. Accordingly, AIC selects ARMA(1, 1) and SBC selects ARMA(1, 0). Table 4 report the results for applying equation (24) and (27) in level form to the data. For ARFIMA model, AIC selects ARFIMA(2, d, 0) and SBC selects ARFIMA(0, d, 0). For ARMA model, AIC and SBC unanimously select ARMA(2, 0), which also implies ARMA(2, 1) model in first difference with a unit MA root. The models with higher AR or MA order were also investigated, but was not successful as appropriate representations. For ARFIMA models one model expressed in equation (26) has its counterpart expressed in equation (27). In that sense selected ARFIMA models based on AIC shows a discrepancy. When d is very close to 0.5 or smaller than -0.5, the likelihood may become less reliable, even though the estimation is possible, and thus the quantities of information criteria may become adversely affected. Therefore, we consider all the chosen models by one criteria or another and try to investigate their nature.

Table 5 presents the parameter estimates of thirteen models including the models selected in the above, where estimates of additional models are also reported for analysis. The estimates of parameter d for ARFIMA(p, d, q) model in level are estimated by reparameterizing d. Those estimates of d are in the range $0.0 < d < 0.5$, and the symmetric relations with the estimates of d for ARFIMA(p, d, q) model in first difference are reasonably well-established. All the estimates of c, μ, ω_1, and ω_2 exhibits only minor changes for different representations and have significant t-values. The implied trend line is shown in Figure 1. The possbility of different slopes of upward trend before and after the second oil shock can be ruled out, as the difference of slope coefficients is negligible.[8] As long as the estimates are concerned, the real GNP series seems to be compatible with various types of models with different properties and implications.

In the next chapter, the competing models are narrowed down by applying

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[7] From the table 1, he has selected two competing models; ARFIMA(3, d, 2) based on AIC and ARIMA(0, 1, 2) base on SBC. My suggestion is that four models, instead of the two, need to be considered. Those are ARFIMA(3, d, 2) and ARIMA(3, 1, 2) selected as competing models by AIC, and ARIMA(0, 1, 2) and ARFIMA(1, d, 0) by SBC.

[8] Additional mutiplicative dummy that would detect the change in slope was introduced and estimated. The estimated coefficient of the dummy variable was very small and also was not significantly different from zero.

[Table 3] AIC, SBC of Competing Models in First Difference of the Log of Quarterly Real GNP (s.a.) (9 ARFIMA models and 9 ARMA models are considered.)

Order of	Information	Order of AR parameters (p)		
MA parameters (q)	criteria	0	1	2
		ARFIMA models[a]		
0	AIC	−529.467	−527.793	−529.570
	SBC	−519.743	−515.639	−514.985
1	AIC	−527.571	−528.411	−528.550
	SBC	−515.417	−513.826	−511.535
2	AIC	−529.696	−528.034	−526.551
	SBC	−515.111	−511.019	−507.104
		ARMA models[b]		
0	AIC	−510.395	−525.296	−523.296
	SBC	−503.102	−515.573	−511.142
1	AIC	−525.029	−525.455	−529.218
	SBC	−515.036	−513.301	−514.633
2	AIC	−523.049	−523.011	−528.569
	SBC	−510.895	−508.426	−511.553

[a] For ARFIMA model, AIC selects ARFIMA(0, d, 2) and SBC selects ARFIMA(0, d, 0).
[b] ARMA(2, 1) and ARMA(2, 2) appear to have a unit MA root respectively, which means models in level are appropriate. Accordingly, AIC selects ARMA(1, 1) and SBC selects ARMA(1, 0).

the likelihood ratio test and also their behaviors in the frequency domain are investigated.

3.2 Tests for Model Selection and Implication

As the stochastic trend model of equation (25) and the deterministic trend model of equation (24) can be nested into the ARFIMA(p, d, q) model in first difference of equation (26), the two former models can be considered as restricted models and the latter model as general model. In this situation, the restrictions

[Table 4] AIC, SBC of Competing Models in Level of the Log of Quarterly Real GNP (s.a.) (9 ARFIMA models and 9 ARMA models are considered.)

Order of MA parameters (q)	Information criteria	Order of AR parameters (p) 0	1	2
		ARFIMA models[a]		
0	AIC	−535.860	−536.641	−537.560
	SBC	−523.648	−521.985	−520.461
1	AIC	−533.898	−535.560	−537.552
	SBC	−519.343	−518.461	−518.011
2	AIC	−537.549	−537.129	−535.876
	SBC	−520.450	−517.588	−513.892
		ARFIMA models[b]		
0	AIC	−512.457	−535.980	−539.548
	SBC	−502.686	−523.767	−524.892
1	AIC	−525.063	−537.244	−539.362
	SBC	−512.849	−522.589	−522.264
2	AIC	−537.936	−538.056	−537.409
	SBC	−523.280	−520.958	−517.867

[a] For ARFIMA model, AIC selects ARFIMA(2, d, 0) and SBC selects ARFIMA(0, d, 0). However, note that the likelihood may not be reliable for comparison purpose when d is close to or over 0.5.

[b] For ARMA model, AIC and SBC unanimously selects ARMA(2, 0).

can be tested by employing the likelihood ratio test. The restriction of 0.0 on the parameter d in equation (26) implies stochastic trend model and the restriction of -1.0 means deterministic trend model.

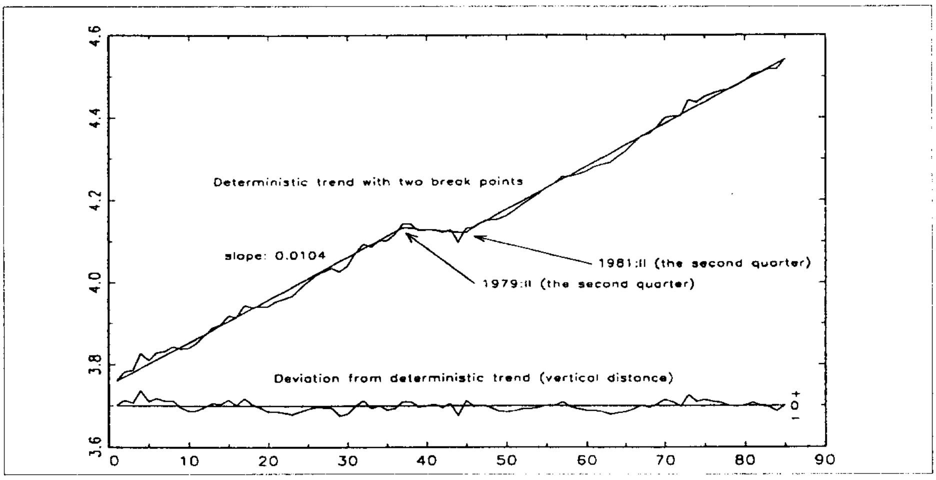
First the validity of ARMA(2, 0) model in level, which has been unanimously selected by the two information criteria, is investigated. Note that the ARMA(2, 1) model in first difference with a unit MA root is equivalent to the model. Now, we can test whether this model can be considered as a restricted model of ARFIMA(2, d, 0) in first difference. Figure 2 shows the empirical density of the likelihood ratio statistic of ARMA(2, 1) in first difference against ARFIMA(2, d, 0) in first difference for the null hypothesis of $d=-1.0$ (support for the deterministic trend model). The empirical density is obtained by 1,000 replication of

[Table 5] Parameter Estimates for ARFIMA Models and ARMA Models of Interest; Log of Quarterly Real GNP[a]

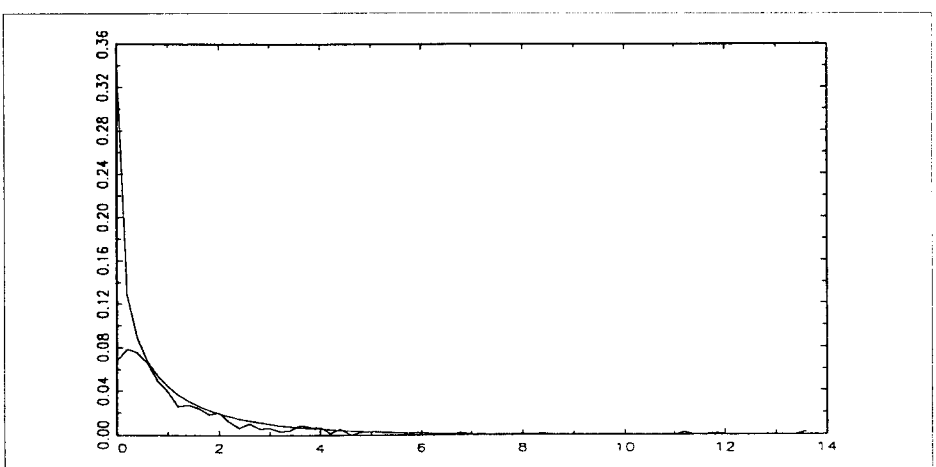
	d	c	μ	ω_1	ω_2	ϕ_1	ϕ_2	θ_1	θ_2
first difference									
ARFIMA(2, d, 0)	−0.795		0.010	−0.092	−0.012	0.188	0.268		
	(−1.669)		(41.814)	(−7.267)	(−7.120)	(0.431)	(1.618)		
ARFIMA(1, d, 1)	−0.424		0.010	−0.094	−0.012	−0.778		−0.623	
	(−3.348)		(34.304)	(−6.609)	(−6.654)	(−3.019)		(−1.795)	
ARFIMA(0, d, 0)	−0.538		0.010	−0.094	−0.012				
	(−5.466)		(40.919)	(−7.516)	(−7.099)				
ARFIMA(0, d, 2)	−0.731		0.010	−0.093	−0.012			−0.166	−0.272
	(−4.442)		(44.824)	(−7.827)	(−7.314)			(−0.967)	(−2.093)
ARMA(2, 1)			0.010	−0.092	−0.012	0.429	0.290	1.000	
			(37.587)	(−6.037)	(−6.065)	(3.276)	(2.234)	(n.a.)	
ARMA(1, 0)			0.010	−0.096	−0.013	−0.452			
			(12.494)	(−4.089)	(−4.628)	(−4.328)			
ARMA(1, 1)			0.010	−0.093	−0.011	0.478		0.934	
			(43.749)	(−8.710)	(−7.361)	(3.837)		(15.234)	
ARMA(0, 1)			0.010	−0.097	−0.012			0.496	
			(16.718)	(−5.264)	(−5.641)			(5.018)	
Level									
ARFIMA(2, d, 0)	0.106	3.747	0.010	−0.092	−0.012	0.272	0.270		
	(0.165)	(341.192)	(44.408)	(−7.635)	(−7.284)	(0.460)	(1.709)		
ARFIMA(0, d, 0)	0.448	3.752	0.010	−0.094	−0.012				
	(4.751)	(466.096)	(43.521)	(−7.684)	(−7.206)				
ARFIMA(0, d, 2)	0.223	3.749	0.010	−0.092	−0.012			−0.204	−0.291
	(1.441)	(445.068)	(47.884)	(−8.211)	(−7.505)			(−1.215)	(−2.237)
ARMA(2, 0)		3.749	0.010	−0.092	−0.012	0.366	0.263		
		(634.030)	(46.402)	(−7.845)	(−7.458)	(3.282)	(2.345)		
ARMA(0, 2)		3.749	0.010	−0.092	−0.012			−0.389	−0.383
		(876.776)	(61.792)	(−10.612)	(−8.843)			(−3.644)	(−3.642)

[a] t-statistics are in parentheses. The series was seasonally adjusted by Census X-11 method. The above thirteen models are considered for a comparison. Some of the models in first difference are the counterparts of the models in level, vice versa, however they were estimated independently. For estimating ARFIMA(p, d, q) model in level for $d > 0$ more precisely, data was transformed in the estimation process such that the parameter d can be estimated in the neighborhood of -0.2.

[Figure 1] The Plotting of the Logarithm of Quarterly Real Gross National Product of Korea from 1970:I to 1991:I (s.a.) (The deterministic trend line in the figure was estimated with ARMA(2, 0) model in level. The trend changes very slightly with the models considered in this study.)



[Figure 2] Empirical Density of the Likelihood Ratio of ARMA(2, 1) in First Difference against ARFIMA(2, d, 0) in First Difference (for the null hypothesis H_0: $d = -1.0$) (The empirical density was obtained by 1,000 replications of the ARMA(2, 1) model estimated for the first difference of the log of real GNP. The smooth line is the chi-squared density with one degree of freedom.)



the ARMA(2, 1) model of size 84 estimated for the first difference of the log of real GNP and subsequent construction of likelihood ratio values after the estimations of both models to the replicated data. The empirical density does not seem to follow the chi-squared density with one degree of freedom. The empirical critical values at the size of test 0.05 and 0.10 are 3.707 and 2.232 respectively. As the likelihood ratio statistic value is 0.352, we can not reject the null hypothesis of -1.0 at both levels of significance. For a chi-squared distribution with one degree of freedom 0.352 implies a p-value of 0.553. Figure 3 presents the empirical density of the maximum likelihood estimate of d, obtained by estimating ARFIMA(2, d, 0) model in first difference to the simulated ARMA(2, 1) series. The density with sample mean -0.915 imitates a normal distribution, even though there are some outliers. The empirical critical values at the size of test 0.05 and 0.10 are -0.368 and -0.481 each. The estimated d of -0.795, which is listed in the first model of table 5, is way to the left of these tail quantities. For a normal distribution -0.795 implies a p-value of 0.333. All the same, we can not reject the null hypothesis of $d = -1.0$. The result means that the trend stationary representation of ARMA(2, 0) in level for the real GNP series is compatible with the fractional model of ARFIMA(2, d, 0).

Next we examine one competing stochastic trend model of ARMA(1, 0) selected based on SBC. As another stochastic trend model of ARMA(1, 1) is selected based on AIC, ARFIMA(1, d, 1) model instead of ARFIMA(1, d, 0) model, which can nest both models, is considered as general model. The likelihood ratio statistic value is 7.166. The estimated small sample density for the likelihood statistic for the null hypothesis of $d = 0.0$ (support for stochastic trend model) is given in Figure 4. The density does not seem to follow the chi-squared distribution with two degrees of freedom. The empirical critical values at the size of test 0.05 and 0.10 are 8.735 and 6.693 respectively. We can not reject the null hypothesis of $d = 0.0$ at 5 percent level, but can reject the null at 10 percent level. For a chi-squared distribution with two degrees of freedom 7.166 implies a p-value of 0.028. Figure 5 exhibits the empirical density of the maximum likelihood estimate of d, obtained by estimating ARFIMA(1, d, 1) model in first difference to the simulated ARMA(1, 0) series. The density seems to approximate a normal density. For a normal distribution -0.424 implies a p-value of 0.9996. The empirical critical values at the left-tail area of 0.05 and 0.10 are -0.495 and -0.404 each. As the estimated d is -0.424, the null hypothesis of $d = 0.0$ can not be rejected at 5 percent level, however the null can be rejected at 10 percent level. As the tests with the size of 10 percent may not be large for a balanced testing, the results are less favorable to the stochastic model of ARMA(1, 0).

Now we check the stochastic trend model of ARMA(1, 1) selected by AIC. For the model, the estimated MA coefficient of 0.934, as shown in Table 5, cancels out large portion of the effect of first differencing. Accordingly, the model is somewhat close to ARMA(1, 0) model in level. This kind of ambiguity seems to be reflected in the tests. The empirical density given in Figure 6 appears to deviate severely from the chi-squared distribution with one degree of freedom. The likelihood ratio statistic value of 4.956 falls in the middle of two empirical critical values, 6.081 at the size of test 0.05 and 4.595 at the size of test 0.10. Also the empirical density of d plotted in Figure 7 is clearly different from normal distribution. Note that the estimated d is -0.424 as before. The empirical critical values at the size of test 0.05 and 0.10 are -0.438 and -0.492 respectively. Thus, empirically, the null hypothesis of $d=0.0$ is rejected at 5 percent level. It would be safe to conclude that the stochastic model of ARMA(1, 1) does not represent the real GNP series better than the ARFIMA(1, d, 1) model does. The bimodal nature of the density implies that many times the simulated series behaves like the ARMA(1, 0) process in level, equivalently the ARMA(1, 1) process in first difference with a unit MA root. The spectral shape of the stochastic model of ARMA(1, 1) shows a similar low frequency behavior found in estimated ARMA(2, 1) model in first difference with a unit MA root as presented in Table 5.

As noted earlier, for ARFIMA(p, d, q) model a variety of spectral shapes near the origin can be produced according to the values of d. Thus we investigate how well each of the thirteen estimated models in table 5 can capture the low frequency behavior of data in frequency domain. First, under each estimated model, the real GNP series is adjusted such that its mean becomes zero. In the process, the estimated coefficients of constant, time trend, and intervention variables under each model are used to provide the most favorable condition to the adjusted series. With the series one periodogram can be constructed for each model. Also, the spectral density can be obtained, which is implied either by estimated ARFIMA(p, d, q) portion or by estimated ARMA(p, q) portion according to the model. All the periodograms are obtained by employing the FFT (fast Fourier transform) algorithm. In calculating the estimated density, the estimated variance under each model is used.

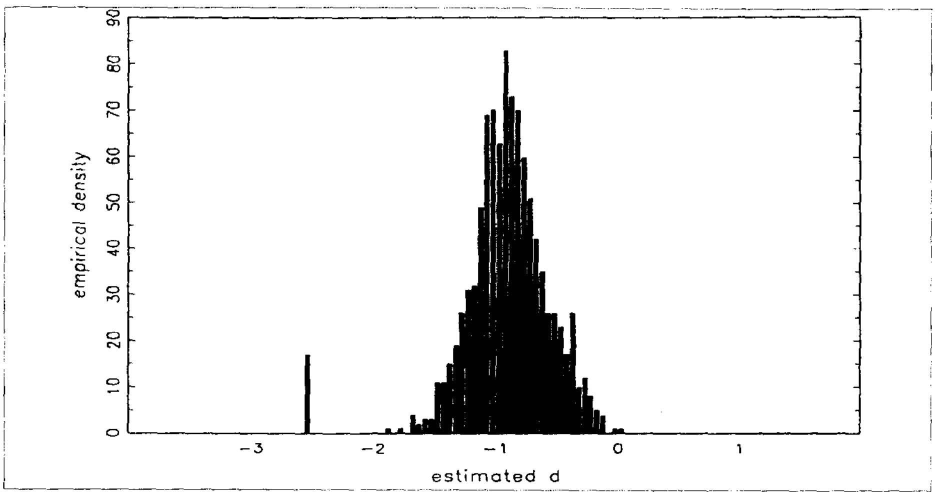
Figure 8 is the plotting of the periodogram over the estimated spectral density for ARFIMA(0, d, 2) model in first difference. All the other periodograms of ARFIMA models considered exhibit a common pattern. Approaching the origin the periodograms are gradually decreasing to zero. The spectral densities seem to well capture the common characteristic of the low frequency behavior of the periodograms. Among them, ARFIMA(0, d, 2) model that also catches the middle frequency behavior reasonably well appears to perform slightly better than the other.

Figure 9 is for ARMA(1, 0) model in first difference. Surprisingly, there seems to be no change in the periodogram compared to that of the above ARFIMA(0, d, 2) model in first difference. We note that the spectral densities of ARMA(1, 0) model, which is presented in Figure 9, and ARMA(0, 1) model do not capture the zero frequency behavior of the periodograms. The periodograms are gradually decreasing to zero approaching the origin. However, the spectral densities do not die out approaching the origin. According to Granger and Joyeux(1980), for fractional series, it is possible to select a model of the usual ARMA(p, q) type, with integer d, which will closely approximate the spectrum of fractional series at all frequencies except those near zero. In that sense, those two stochastic models are highly likely to be the mere approximations of fractional series. Additionally, the estimated ARMA(2, 1) model in first difference with a unit MA root is equivalent to ARFIMA(2, d, 0) model in first difference with $d = -1.0$. Thus the estimated spectral density of ARMA(2, 1) is very close to that of ARFIMA(2, d, 0) model. As mentioned earlier, the estimated spectral density of ARMA(1, 1) model in first difference is severely affected by its MA root, which is somewhat close to the unit circle, and imitates that of typical ARFIMA model approaching the origin.

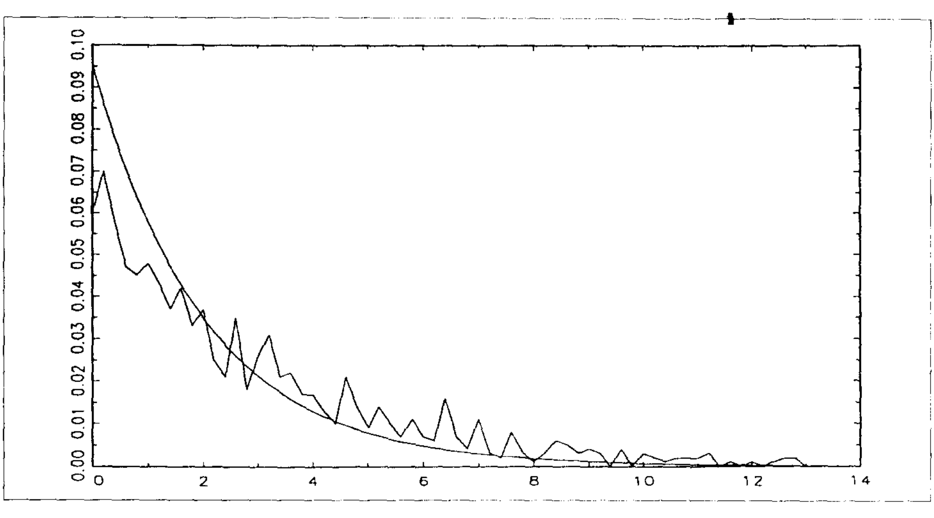
Finally, Figure 10 and Figure 11 are for ARFIMA(2, d, 0) and ARMA(2, 0) models in level respectively. Clearly, the periodograms exhibit patterns of gradually exploding tendency approaching low frequencies. Following the tendency of periodograms, the spectral density functions of all ARFIMA(2, d, 0), ARFIMA (0, d, 0), ARFIMA(0, d, 2), and ARMA(2, 0) models in level are all continuously moving upward in low frequencies, which implies that all these models can describe, at least, the low frequency nature of the real GNP series. Exceptionally, the estimated spectral density of ARMA(0, 2) model in level is not satisfactory as the estimated spectral density is not ever increasing approaching zero frequency.

The above analysis reveals that the real GNP series can be interpreted as either stationary fractional series or stationary series around deterministic trend. Surprisingly, the stochastic trend model considered in this study does not seem to perform well in capturing low frequency behavior of the real GNP series. The implication is that shocks given to the series are not likely to have permanent effect which is postulated in unit root hypothesis. On the contrary, the effect of a shock can persist for a long period of time, but eventually the effect will die out. However, the permanent effect of the second oil shock and political crisis on output are captured in all models with highly significant t-values on the two intervention variables. Although it is difficult to tell whether the real GNP series is a long-memory series or short-memory ones, the mean-reversion property, both of long-memory and short-memory series have in common, tells that temporarily the series can deviate from the deterministic trend, possibly moving equilibrium, due to economic shocks, but it will eventually return to the trend. In terms of economic policy, the result implies that the short-run economic policy, mainly of stabilization policy, can be pursued more or less independently from long-term economic policy, mostly of growth policy.

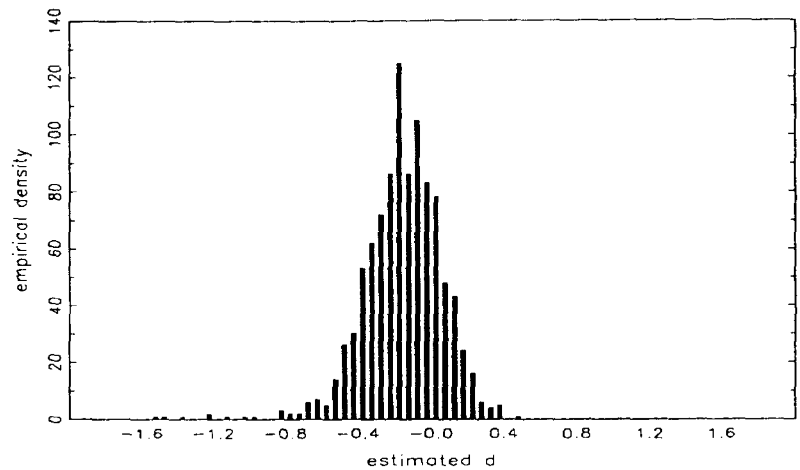
[Figure 3] Empirical Density of the Maximum Likelihood Estimate of d (where the true parameter value is $d = -1.0$) (The empirical density was obtained by 1,000 replications of the ARMA(2, 1) model estimated for the first difference of the log of real GNP.)



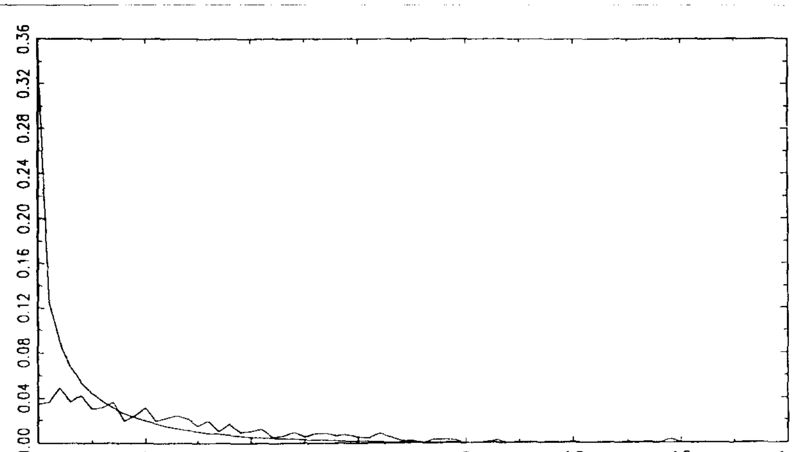
[Figure 4] Empirical Density of the Likelihood Ratio of ARMA(1, 0) in First Difference against ARFIMA(1, d, 1) in First Difference (for the null hypothesis H_0: $d = 0.0$) (The empirical density obtained by 1,000 replications of the ARMA(1, 0) model estimated for the first difference of the log of real GNP. The smooth line is the chi-squared density with two degrees of freedom.)



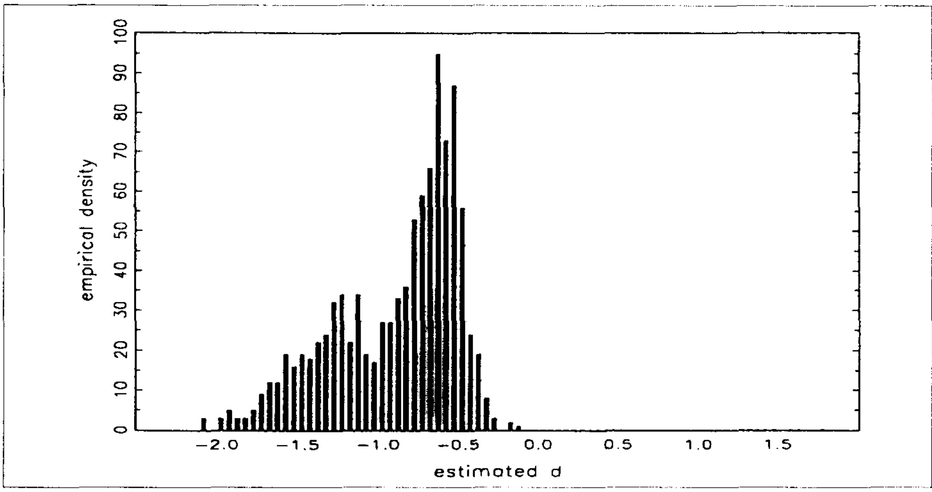
[Figure 5] Empirical Density of the Maximum Likelihood Estimate of d (where the true parameter value is $d = 0.0$) (The empirical density was obtained by 1,000 replications of the ARMA(1, 0) model estimated for the first difference of the log of real GNP.)



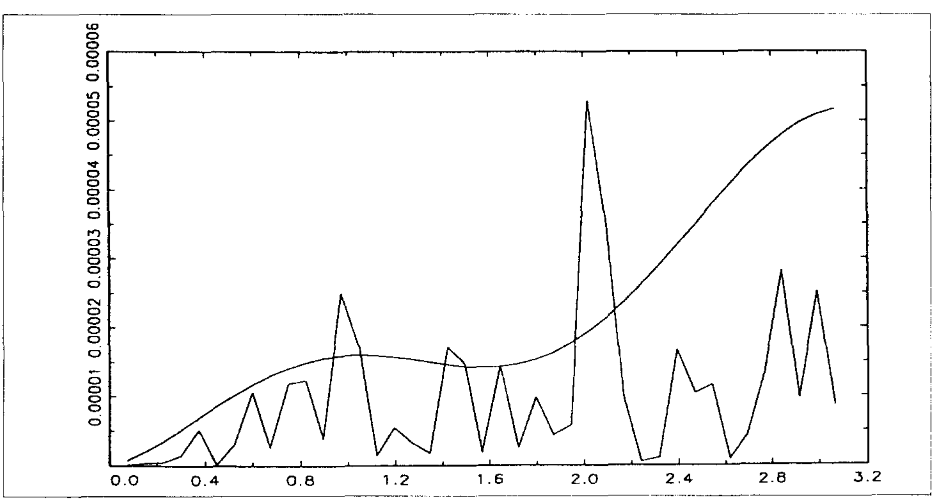
[Figure 6] Empirical Density of the Likelihood Ratio of ARMA(1,1) in First Difference against ARFIMA(1, d, 1) in First Difference (for the null hypothesis H_0: $d = 0.0$) (The empirical density was obtained by 1,000 replications of the ARMA(1,1) model estimated for the first difference of the log of real GNP. The smooth line is the chi-squared density with one degree of freedom.)



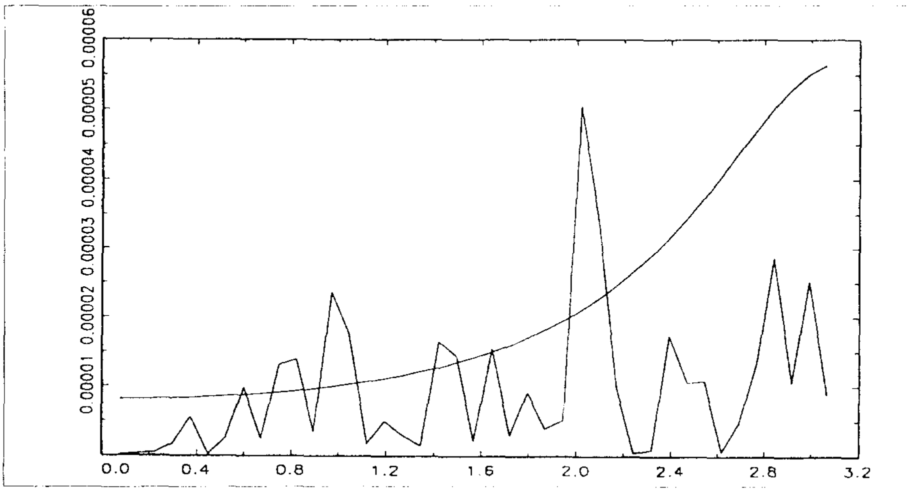
[Figure 7] Empirical Density of the Maximum Likelihood Estimate of d (where the true parameter value is d = 0.0) (The empirical density was obtained by 1,000 replications of the ARMA(1, 1) model estimated for the difference of the log of real GNP.)



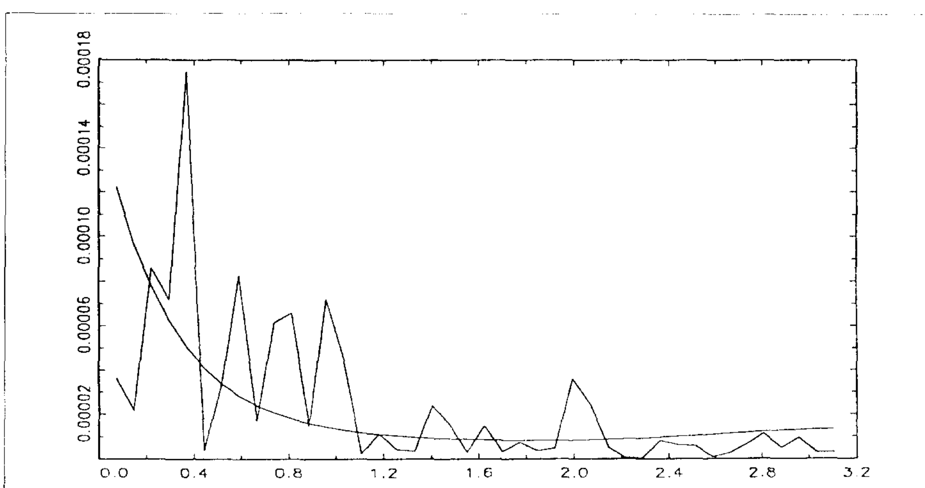
[Figure 8] Spectral Density of the Estimated ARFIMA(0, d, 2) Model Plotted over the Periodogram for the First Difference of the Log of Quarterly Real GNP



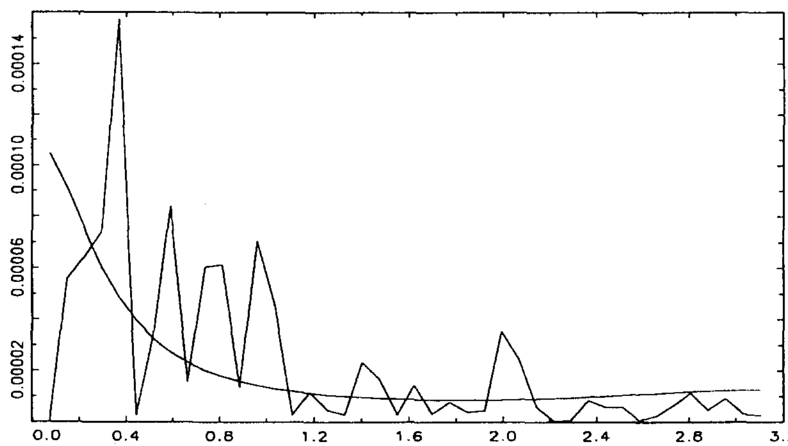
[Figure 9] Spectral Density of the Estimated ARMA(1, 0) Model Plotted over the Periodogram for the First Difference of the Log of Quarterly Real GNP



[Figure 10] Spectral Density of the Estimated ARFIMA(2, d, 0) Model Plotted over the Periodogram for the Level of the Log of Quarterly Real GNP



[Figure 11] Spectral Density of the Estimated ARMA(2, 0) Model Plotted over the Periodogram for the Level of the Log of Quarterly Real GNP



Ⅳ. SUMMARY AND CONCLUSION

This paper has presented an exact maximum likelihood estimation method for ARFIMA(p, d, q) model that is more convenient and has fewer problems than the method of Sowell(1982a). The transformation method employing the inverse of Cholesky factorization can be easily adapted to conventional algorithms for estimating ARMA(p, q) model. For the estimation of ARFIMA(p, d, q) models in this study, exact maximum likelihood estimation algorithm of Ansley(1979) for ARMA(p, q) model has been incorporated. A Monte Carlo simulation has shown that maximum likelihood estimation method has unsatisfactory small sample properties when d is getting close to the boundary of 0.5, which can be somewhat improved by reparameterizing d. As long as the bias and RMSE are concerned, the evidence is that any one of maximum likelihood estimation method and the method of Li and McLeod, which has alleged defect of truncation problem, do not outperform the other in all parameter range for d with samples of size 100.

ARFIMA(p, d, q) model can nest both of the competing stochastic trend model and deterministic trend model. Its ability to capture a wide range of long-run dependence in the data can provide a tool for more general model selection. The log of quarterly real GNP series of Korea, which exhibits a strong upward trend, is analyzed for better representations. With the assumption of known events, the second oil price shock and political crisis, two intervention variables are used. The three types of models, fractional model, stochastic trend model, and deterministic trend model, estimate the series reasonably well. However, the likelihood ratio test and subsequent frequency domain analysis have shown that the stochastic trend model is less adequate when compared with fractional model and deterministic trend model. The other two models appear to capture the low frequency behavior of the real GNP series equally well. The mean-reversion property implied by the two models can have many important implications in theory and policy.

Even though the above results seem to be robust to some choice of break points, the intrinsic questions surrounding structural breaks such as number of breaks, choice of break points, and response patterns are not fully answered in this paper. A more challenging work on structural breaks in connection with fractional time series modelling might be rewarding. Additional observations would make it clearer which of the two models, fractional model or deterministic model, better estimates the real GNP series. The testing procedure for restriction and model selection followed in this study can be applied to other macro-variables immediately. Also the estimation method suggested in this study can be employed for the test of fractional cointegration.

REFERENCES

Ansley, C. F. (1979), "An Algorithm for the Exact Likelihood of a Mixed Autoregressive-Moving Average Process," Biometrika, 66, pp. 59-65.

Brockwell, P. J. and R. A. Davis (1987), *Time Series: Theory and Methods*, Springer-Verlag, New York, NY.

Cheung, Y. W. and K. S. Lai (1993), "A Fractional Cointegration Analysis of Purchasing Power Parity," *Journal of Business and Economic Statistics*, 11, pp. 103-112.

Diebold, F. X. and G. D. Rudebusch (1991), "On the Power of Dickey-Fuller Tests against Fractional Alternatives," *Economic Letters*, 35, pp. 155-160.

Fox, R. and M. S. Taqqu (1986), "Large-Sample Properties of Parameter Estimates for Strongly Dependent Stationary Gaussian Time Series," *Annals of Statistics*, 14, pp. 517-532.

Geweke, J. F. and S. Porter-Hudak (1982), "The Estimation and Application of Long Memory Time Series Models," *Journal of Time Series Analysis*, 4, pp. 221-238.

Gradshteyn, I. S. and I. M. Ryzhik (1980), *Tables of Integrals, Series and Products*, Academic Press, New York, NY.

Granger, C. W. J. (1980), "Long Memory Relationships and the Aggregation of Dynamic Models," *Journal of Econometrics*, 14, pp. 227-238.

Granger, C. W. J. and R. Joyeux (1980), "An Introduction to Long-Memory Time Series Models and Fractional Differencing," *Journal of Time Series Analysis*, 1, pp. 15-29.

Hamilton, J. D. (1994), *Time Series Analysis, Princeton University Press*, Princeton, NJ.

Hipel, K. W. and A. I. McLeod (1978), "Preservation of the Rescaled Adjusted Range," Part I. A reassessment of Hurst phenomenon, *Water Resources Research*, 14, pp. 491-508.

Hosking, J. R. M. (1981), "Fractional Differencing," *Biometrika*, 68, pp. 165-176.

Li, W. K. and A. I. McLeod (1986), "Fractional Time Series Modelling," *Biometrika*, 73, 1, pp. 217-21.

Marquardt, D. W. (1963), "An Algorithm for Least Squares Estimation of Non-Linear Parameters," *Journal of the Society of Industrial and Applied Mathematics*, 11, pp. 431-441.

Sowell, F. B. (1990), "The Fractional Unit Root Distribution," *Econometrica*, 58, pp. 495-505.

Sowell, F. B. (1992a), "Maximum Likelihood Estimation of Stationary Univariate Fractionally Integrated Time Series Models," *Journal of Econometrics*, 53, pp. 165-188.

Sowell, F. B. (1992b), "Modeling Long-Run Behavior with the Fractional ARIMA Model," *Journal of Monetary Economics*, 29, pp. 277-302.

Wei, W. W. S. (1990), *Time Series Analysis:Univariate and Multivariate Methods*, Addison-Wesley Publishing Company, New York, NJ.