Customer Loyalty and Multi-stop Shopping*

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Abstract: In this paper, we study how customer loyalty impinges on the existence of a multi-stop shopping equilibrium through its effects on the price-setting behaviors of firms. A particular focus lies on the incentive of firms when they operate in two distinct modes of retail, the department store and the shopping mall, relying on whether the individual retailers in the firm are allowed to set their prices independently or not. We show that there exists a multi-stop shopping equilibrium in which customers visit another store although paying higher price. More importantly, the minimum level of customer loyalty for such an equilibrium goes beyond the level that induces customers to do multi-stop shopping. We argue that this gap is required to discourage the individual retailer, being tempted to raise the price independently in the shopping mall, knowing that the loyal customers are not readily leave her. In light of the existing literature, these results provide a new understanding about multi-stop shopping: (i) multi-stop shopping customers are not motivated entirely by the incentive to search for the lowest available price; (ii) the incentive of firms under different modes of retail is critical in the existence of multi-stop shopping. (195 words)

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1 Introduction

In July 2017, *USA Today* and *NBC News* published articles reporting that an increasing number of consumers nowadays visit multiple stores in different locations on the same trip to obtain the goods on the shopping list.[1] This so-called *multi-stop shopping* or *multiple-store shopping* (henceforth MSS) is not only a recent trend in consumer behavior but also a well-documented behavior in the marketing literature. According to Gijsbrechts et al. (2008), about 75 % of all grocery shoppers regularly visit more than one store each week, and similar numbers are reported in Fox and Hoch (2005) and in Drèze and Vanhuele (2006). As MSS incurs an additional traveling cost, it seems puzzling why customers behave against the temptation to buy everything on their shopping list in one store, thus saving on the traveling cost.

The typical view in the literature (Fox and Hoch, 2005; Lal and Rao, 1997) regards MSS to arise from an opportunistic cherry picking incentive, i.e. the incentive to search for the lowest available price in each product category. The economics literature on MSS (Brandão et al., 2014; Kim and Serfes, 2006; Thill, 1985, 1986, 1992) also takes the similar view, thereby comparing the traveling cost and the price advantage of making another trip to other stores within a multi-product variant of the celebrated Hotelling model. However, such a view does not fully explain MSS. As Gijsbrechts et al. (2008) point out, the stability and the regularity of MSS patterns do not square well with cherry picking customers hunting for temporary price promotions, and Urbany et al. (1996) report that such customers account for only 10 to 35 percent of the whole group of MSS customers.

In this paper, we investigate a role of customer loyalty as one of the most crucial non-cherry-picking motives for MSS behaviors. By customer loyalty, we mean psychological benefits that a customer expects to derive from her purchasing decision, thus reflecting the concerns for branding, reputation, quality, and loyalty program. Accordingly, customer loyalty may vary across products and stores.[2] This opens up the possibility that a customer

[1] Meyer, Z. (2017, July 19). Grocery shopping is no longer a one-stop experience. *USA Today*, Retrieved from https://www.usatoday.com/story/money/2017/07/19/grocery-shopping-no-longer-one-stop-experience/471484001/, White, M.C. (2017, July 28). Americans no longer want one-stop grocery shopping. *NBC News*, Retrieved from https://www.nbcnews.com/business/consumer/americans-no-longer-want-one-stop-grocery-shopping-n787166

[2] Our definition distinguishes customer loyalty from the customer's preference, which is basically defined over the products. This is consistent with the celebrated work of Jacoby and Kyner (1973), arguing that customer loyalty involves a psychological processes in which a customer compares and evaluates products/stores in various dimensions and may choose a certain product/store even when the prices dictate otherwise. Therefore, customer loyalty according to our definition is distinct from two closely related concepts, repeated purchasing behaviors and customer preferences. Specifically, customer loyalty is not defined

is loyal to different stores in different product categories,[3] and thus the customer makes additional shopping trips to buy a particular product from a particular store. To illustrate, consider a customer now in a Dunkin' Donuts store may choose to take a shopping trip to a nearby Starbucks store for coffee, although the Dunkin' Donuts store is also selling coffee. For donuts, however, the same customer would like to make a purchase in Dunkin' Donuts. That is, this customer's MSS behavior arises because her loyalty to Dunkin' Donuts lies only in its donuts, while she is loyal to Starbucks only in its coffee.

Nevertheless, we may not jump to a conclusion that customer loyalty simply implies the existence of MSS behaviors. The existence of MSS is not as obvious as it seems. The illustration we have just provided is silent about the firms' price-setting incentive in response to the presence of customer loyalty. Isn't it possible for Dunkin' Donuts to set its price for coffee low enough to steal some customers from the Starbucks store? If this is the case, then the customer's behavior would not exhibit MSS even with her loyalty to Starbucks for coffee.

When each firm consists of multiple individual retailers, figuring out the firms' price-setting incentive becomes more challenging. Suppose hypothetically that both Dunkin' Donuts and Starbucks consist of two individual retailers selling different products. Each store's incentive would be the same as before if it organizes its retailers in the form of a *department store* like Macy's, where the retailers coordinate their product prices and share the profit. However, if the individual retailers set their own product prices independently, as in a *shopping mall* like Tysons Corner Center, the donut seller in Dunkin' Donuts may want to raise its price. Then, MSS customers find it less attractive to make another trip for buying donuts. Despite some losses in the demand, it may be more profitable for the donut seller if the demand is rather inelastic. All these hypothetical stories illustrate what the previous studies in the marketing literature on customer loyalty and the existence of MSS fail to incorporate.

The goal of this paper is thus to study how customer loyalty impinges on the existence of MSS, especially through its effect on the firms' price-setting behaviors. Particularly, we attempt to answer the following questions: Is the mere presence of customer loyalty sufficient for the existence of MSS? What conditions does the customer loyalty have to satisfy in order to give rise to MSS? What is the effect of customer loyalty on the firms' behaviors that

in purely behavioral terms, thus being distinct from repeated purchasing. It is also distinct from preference because the notion of customer loyalty in this paper contains evaluative processes. See Jacoby and Kyner (1973) for detailed discussion about the concept of customer loyalty.

[3]According to the terminology of Zhang et al. (2013), customer loyalty is a store- and product category-specific trait.

lies behind the condition? Does the effect rely on whether a firm is a department store or a shopping mall? The first two questions are about the existence of a MSS equilibrium while the remaining two are about the effect of customer loyalty on the firms' price-setting behaviors.

To address these questions, we adapt the celebrated Hotelling model to allow for two products and customer loyalty. Specifically, each store sells two different goods and customers have a unit demand for each good. We also assume that store 1 has customer loyalty in good 1 while store 2 has in good 2. Therefore, a customer gains an additional value when purchasing good 1/good 2 in the store 1/store 2, respectively. Furthermore, both stores consist of two individual retailers or shops, but the two stores differ in the organization of their constituents. One store is a shopping mall and the other is a department store[4]

Our main result is that there indeed exists a MSS equilibrium in which multi-stop shoppers take another trip to buy a good in the store they are loyal to, although the posted price of such a good is higher, and more importantly, the minimum level of the customer loyalty guaranteeing the existence of such an equilibrium is higher than the level that provides an incentive for customers to visit both stores (Theorem 1). This result answers the two sets of questions we ask previously about the customer loyalty and MSS.

First of all, the mere presence of customer loyalty is not sufficient for the existence of a MSS equilibrium. If a customer cannot gain from the customer loyalty more than the traveling cost when visiting both stores, then there is no MSS equilibrium (Lemma 1). In particular, with no customer loyalty, there exists no MSS equilibrium (Corollary 1). This is also an implication of Brandão et al. (2014) when the number of goods in their model is set to two. Motivated by this non-existence result, they provide a cautionary warning that analyzing multi-product competition using the Hotelling model may be a substantial restriction when assuming only two goods. Nevertheless, our result indicates that one may disregard the warning if taking account of the customer loyalty.

Secondly, and more importantly, our result indicates that there is a gap between the minimum level for guaranteeing the existence of a MSS equilibrium and the minimum level for covering the expense of customers to do MSS, and we shall refer to it as *the loyalty gap*. This seemingly unnecessary gap is required to regulate each store's and its individual retailers' price-setting incentives that we illustrate in our hypothetical example. A close examination of the loyalty gap reveals how the customer loyalty affects each firm's price-

[4]Our terminology regarding the two different retail modes of price-setting behaviors of multi-product firms is due to Brandão et al. (2014).

setting incentive, depending on different retail modes of a firm.

Surprisingly, the loyalty gap is required only to discourage the individual retailer in the shopping mall, who enjoys the benefit of customer loyalty, from raising its price. In other words, the other retailer in the shopping mall who has no loyal customers, and the two retailers in the department store would not deviate from the MSS equilibrium even when there is no loyalty gap. The intuition behind is not hard to see in our illustrative example. The coffee seller of Dunkin' Donut would decrease the price in order to steal customers from Starbucks. The donut seller, knowing that customers would not leave readily, may want to raise the price. If they are organized in the form of a department store, the two sellers share the profit, and thus the coffee seller's loss due to the customer loyalty would be the donut seller's gain. However, the resulting gain is larger than the loss. The presence of multi-stop shoppers divides the markets, thus enabling the department store to charge different prices in different markets. Similarly to the prediction of the third-degree price discrimination, the overall profit of the department store with the presence of multi-stop shoppers is higher than the profit without it. Due to these cross-subsidization and the price discrimination effects, these sellers and the department as a whole have no incentive for deviation.

In the shopping mall, with the absence of cross-subsidization, the customer loyalty has differential effects on the individual retailers. When deviating from the MSS equilibrium, an individual retailer faces a lower price elasticity of the demand for his product as the two different markets for coffee and for donuts are integrated into the one for the bundle of coffee and donuts. For example, the coffee seller in Dunkin' Donuts have to consider the donuts prices in both stores as well as the coffee price in Starbucks. For this coffee seller, the customer loyalty is a burden. He needs to set the price prohibitively low enough to compensate these customers for the loyalty value they gain by purchasing the coffee from the Starbucks. For the donut seller, however, the customer loyalty is a subsidy. Due to the psychological gains that the customers obtain from their loyalty, the donut price perceived by them is lower than the price posted by the donut seller in Dunkin' Donut. This partially offsets the negative effects of having more elastic demand, and may increase the donut seller's profit relative to the profit in the MSS equilibrium by raising the posted price more than the loss in the demand (Lemma 2). However, this effect does not grow as the value of customer loyalty increases further. As the customer loyalty increases, it also costs the donut seller a huge proportion of the demand that comes from the loyal customers, and the temptation from charging a higher price loses its attractiveness. The loyalty gap for firms is exactly the size of customer loyalty to make the donut seller indifferent between sticking to the rather elastic, higher demand curve of the MSS equilibrium and switching to a more inelastic, lower demand. If the level of customer loyalty goes beyond the gap, the former option dominates the latter one.

The rest of paper proceeds as follows. In Section 2, we describe our model and do preliminary analysis of the optimal pricing behavior of the two stores which differ in their modes of retail. In Section 3, we present our main results and we conclude with a brief discussion in section 4.

2 Model

2.1 Environment

We consider a variant of the model of Brandão et al. (2014) which is a multi-product version of Hotelling. There are two firms and the spatial distance between them is normalized to unity. As in Hotelling's model, we consider a unit interval $[0, 1]$ and each firm is located at the end of this interval. Let L(Left) and R(Right) denote the two firms located at 0 and 1, respectively. Each firm consists of two shops selling two different goods, good 1 and good 2, thus in what follows, we shall refer to a firm as a shopping center or a store for short. There is a continuum of customers uniformly distributed on this unit interval. Each customer has a unit demand for each good, thus she always purchases both goods. All customers share the same preference. Specifically, the purchase of both goods yields the intrinsic value of $V > 0$. In addition to the intrinsic value V, every customer gains some additional values by choosing to use a particular store or to buy one particular product. This so-called *customer loyalty* is captured by a matrix $A = \begin{pmatrix} a_{1L} & a_{2L} \\ a_{1R} & a_{2R} \end{pmatrix}$ where a_{ij} is the value derived from buying good $i = 1, 2$ at store $j = L, R$. For simplicity, we assume that $a_{1L} = a_{2R} = a > 0$ and $a_{2L} = a_{1R} = 0$. Customers perceive good 1 to have a higher value (better quality) when it is on sale in store L, while good 2 is expected to have a higher value when it is on sale in store R. In order to buy a good from store j, a customer whose location is $x \in [0, 1]$ (henceforth consumer x) must take a trip to the store by incurring a traveling cost $t > 0$ per distance. To be specific, the traveling cost is tx if she buys from store L, and it is $t(1 - x)$ if she buys from store R.

Let p_{ij} be the price of good i charged by each store j for $i = 1, 2$ and $j = L, R$. As each customer's demand for both goods is perfectly inelastic, she chooses among four possibilities: (i) buying both goods from store L, (ii) buying both goods from store R, (iii) buying good

1 from L and good 2 from R, or (iv) buying good 1 from R and good 2 from L. Let L, R, LR, and RL denote each possibility. Hence, the payoff of customer $x \in [0,1]$ in each case can be expressed as follows:

$$u_L(x) = V - tx + a - p_{1L} - p_{2L}, \quad u_R(x) = V - t(1-x) + a - p_{1R} - p_{2R}$$
$$u_{LR}(x) = V - t + a - p_{1L} + a - p_{2R}, \quad u_{RL}(x) = V - t - p_{1R} - p_{2L}$$

We shall refer to those cases where customer x drops by both stores, as *two-stop shopping*. Accordingly, the other cases are called *one-stop shopping*. Note that the presence of the customer loyalty $a > 0$ reduces the cost, psychologically though, paid by a customer. The customer may find the effective price for good i at store j is not the price p_{ij} on the price tag (set by the store), but $p_{ij} - a_{ij}$. Based on this observation, we shall work with the notion of the effective price of good i at store j, $\mu_{ij} = p_{ij} - a_{ij}$, instead of p_{ij}. Specifically, the effective prices are $\mu_{1L} = p_{1L} - a$, $\mu_{2L} = p_{2L}$, $\mu_{1R} = p_{1R}$ and $\mu_{2R} = p_{2R} - a$.

Now that we use new notations, the customer x's payoffs are expressed as follows:

$$u_L(x) = V - tx - \mu_{1L} - \mu_{2L}, \quad u_R(x) = V - t(1-x) - \mu_{1R} - \mu_{2R}$$
$$u_{LR}(x) = V - t - \mu_{1L} - \mu_{2R}, \quad u_{RL}(x) = V - t - \mu_{1R} - \mu_{2L}$$

where the payoffs in the last two cases can be further summarized as follows:

$$u_{TS}(x) = \max\{u_{LR}(x), u_{RL}(x)\} = V - t - [\min\{\mu_{1L}, \mu_{1R}\} + \min\{\mu_{2R}, \mu_{2L}\}] .$$

For notational convenience, we shall use $\mu_L = \mu_{1L} + \mu_{2L}$, $\mu_R = \mu_{1R} + \mu_{2R}$, $\mu_{LR} = \mu_{1L} + \mu_{2R}$, $\mu_{RL} = \mu_{1R} + \mu_{2L}$, and $\mu_{TS} = \min\{\mu_{1L}, \mu_{1R}\} + \min\{\mu_{2R}, \mu_{2L}\}$. Notice that μ_{TS} must be either μ_{LR} or μ_{RL}, otherwise no consumer would travel to both ends.

The demand for store $j = L, R$ selling good $i = 1, 2$ depends upon whether there exist customers who would travel to both stores in search for a cheaper price or for a loyalty value. If there are such customers, we refer to them as *two-stop shoppers*, while calling the other customers who purchase only in one location as *one-stop shoppers*. The two-stop shoppers can further be classified into two different categories, depending upon their consumption-traveling behaviors. If two-stop shoppers buy good 1 from store L, good 2 from store R, they are *LR-type two-stop shoppers* (in short, LR-type shoppers). As these two-stop shoppers are seeking to buy goods that bring them with the loyalty values, we shall also refer them as *loyalty-seeking two-stop shoppers*. Otherwise, they are *RL-type two-stop shoppers* or *non-*

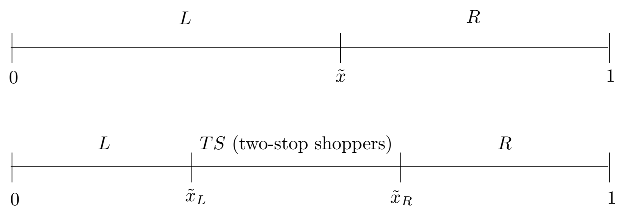


Figure 1: One-stop and Two-stop Shopping Scenario

loyalty-seeking two-stop shoppers. Furthermore, we shall refer to the demand for each store and its constituent shops in the presence of two-stop shoppers as the demand in the two-stop shopping scenario. When there are no two-stop shoppers, we say that the demand under such a situation is in the one-stop shopping scenario. These two possible scenarios are illustrated in Figure 1.

In these scenarios, the demand for each good can be determined by identifying the location of a customer who is indifferent between any two adjacent choices. Specifically, in the one-stop shopping scenario, the indifferent customer between buying all from L and from R is $\tilde{x} = \frac{1}{2} + \frac{\mu_R - \mu_L}{2t}$ by solving for $u_L(\tilde{x}) = u_R(\tilde{x})$. All the customers lie on the left side of $\tilde{x}$ would buy both goods at L, while those on the right side of $\tilde{x}$ would buy at R. Similarly, in the two-stop shopping scenario, we can identify the customers $\tilde{x}_L = 1 - \frac{\mu_L - \mu_{TS}}{t}$ and $\tilde{x}_R = \frac{\mu_R - \mu_{TS}}{t}$. Note that $\tilde{x}_L < \tilde{x}_R$ must hold in the two-stop shopping scenario, otherwise the situation goes back to the one-stop shopping scenario. In other words, the necessary and sufficient condition for the existence of two-stop shoppers is determined by $\tilde{x}_L < \tilde{x}_R$, which is

$$\sum_{i=1,2} |\mu_{iL} - \mu_{iR}| = |\mu_{1L} - \mu_{1R}| + |\mu_{2L} - \mu_{2R}| > t \tag{1}$$

or, equivalently, $|p_{1L} + a - p_{1R}| + |p_{2L} - p_{2R} - a| > t$. Let $\mathcal{P}_{TS}$ denote the set of vectors $\mu = (\mu_{1L}, \mu_{2L}, \mu_{1R}, \mu_{2R}) \in \mathbb{R}^4_+$ satisfying the condition (1), while denoting its complement by $\mathcal{P}_{OS}$. The demand for good i at store L is thus

$$q_{iL} = \begin{cases} \tilde{x} & \text{if } \mu \in \mathcal{P}_{OS} \\ \min(\tilde{x}_R, 1) & \text{if } \mu \in \mathcal{P}_{TS} \text{ and } \mu_{iL} < \mu_{iR} \\ \max(0, \tilde{x}_L) & \text{if } \mu \in \mathcal{P}_{TS} \text{ and } \mu_{iL} > \mu_{iR} \\ \frac{\max(0,\tilde{x}_L)+\min(\tilde{x}_R,1)}{2} & \text{if } \mu \in \mathcal{P}_{TS} \text{ and } \mu_{iL} = \mu_{iR} \end{cases}$$

where we assume that the half of the consumers buy good i at L and the other half buys it at

R when there is a tie $\mu_{iL} = \mu_{iR}$.[5] The demand for good i at store R is simply $q_{iR} = 1 - q_{iL}$.

2.2 Modes of Retail and Pricing Rules

Both shopping centers L and R consist of two different shops, which we refer to as shop $1L$, shop $2L$, shop $1R$ and shop $2R$. However, these shopping centers differ from each other in their organization, particularly in their modes of retail. Each shopping center may take either of the following modes of retail: *a department store* or *a shopping mall*. In the department store, the two shops are under the control of one single headquarter and thus they behave like a single firm. In contrast, the two shops in the same shopping mall behave in a non-cooperative manner: each shop chooses the price of its product independently of the other shop. For our later purpose of studying how this difference interacts with the level of customer loyalty, we assume that store L is a department store and store R is a shopping mall. In this subsection, as a preliminary step for it, we shall investigate how a store's pricing behavior differs, depending upon its mode of retail and the different demand scenarios.

Consider firstly the department store L. As its constituent shops coordinate their prices, the department store's pricing rule is determined by choosing p_{1L} and p_{2L} altogether, given its opponent's prices p_{1R} and p_{2R}. In the one-stop shopping scenario, its profit maximization problem is $\max_{p_{1L}, p_{2L}} (p_{1L} + p_{2L})\tilde{x}$. Expressed in terms of the effective prices, the problem is

$$\max_{\mu_{1L}} (\mu_L + a) \left(\frac{1}{2} + \frac{\mu_R - \mu_L}{2t} \right)$$

and the optimal pricing rule is computed as $\mu_L = \frac{1}{2}[t + \mu_R - a]$, or equivalently,

$$p_L = \frac{1}{2}[t + p_R].$$

As the latter expression shows, the customer loyalty has no effect on the pricing behavior. This is not surprising. In the one-stop shopping scenario, a customer buys all goods at one location, and the department may determine the price for a bundle of two goods by coordinating the prices of its constituent shops. It is as if the two markets, one for good 1 and the other for good 2, are integrated into one. That is, the department store L is competing against its opponent R in the market for one good, that is to say, a bundle of good 1 and good 2. Consequently, only the bundle prices p_L and p_R do matter. As the values of customer loyalty are equal across stores, their effect on the demand and thus on

[5] This tie-breaking rule has no effect on our results as the same assumption does in Brandão et al. (2014).

the pricing behavior vanishes.

In the two-stop shopping scenario, however, this is no longer true. As there are two different markets (one for good 1 and the other for good 2), the department store's profit relies on the individual prices and their competitive advantage (in terms of the effective prices). This implies that the pricing behavior of the department store also depends on whether the two-stop shoppers are of LR-type or of RL-type. When the two-stop shoppers are of LR-type, the department store's profit maximization problem is

$$\max_{\mu_{1L},\mu_{2L}} (\mu_{1L}+a)\left[\frac{\mu_{1R}-\mu_{1L}}{t}\right]+\mu_{2L}\left[1-\frac{1}{t}(\mu_{2L}-\mu_{2R})\right]$$

and the resulting pricing rules are $\mu_{1L}=\frac{\mu_{1R}-a}{2}$ and $\mu_{2L}=\frac{t+\mu_{2R}}{2}$. That is,

$$p_{1L}=\frac{p_{1R}+a}{2}, \quad p_{2L}=\frac{p_{2R}+t-a}{2}.$$

The above pricing rules witness that the value of customer loyalty matters. When the two-stop shoppers are of RL-type, the pricing rules can be obtained as $\mu_{1L}=\frac{\mu_{1R}+t-a}{2}$ and $\mu_{2L}=\frac{\mu_{2R}}{2}$. Equivalently,

$$p_{1L}=\frac{p_{1R}+t+a}{2}, \quad p_{2L}=\frac{p_{2R}-a}{2}.$$

The price of good 1, p_{1L} is increasing in the loyalty value a, while the price of good 2 is decreasing. The presence of two-stop shoppers now renders the department store L compete against its opponent R in two different markets. In the market for good 1, the higher loyalty value attracts more customers to L from R, thus allowing L to raise its price of good 1. On the other hand, in the market for good 2, the department store L cannot enjoy the benefit of customer loyalty when selling good 2. It thus needs to lower the price to attract customers who are loyal to good 2 being sold in its opponent R.

Now, we turn to the pricing rules of the shopping mall R. As the two shops in the shopping mall set their prices independently, we need to analyze the pricing rule of each shop $i=1,2$ given the prices (p_{1L},p_{2L}) of the department store L, and the price p_{jL}, $j\neq i$ of the other shop in the shopping mall R. By adapting Brandão et al. (2014) to allow for customer loyalty a, we track down how the demand changes when the price switches back and forth between the one-stop and the two-stop shopping scenarios. To be specific, we partition the domain of μ_{iR} into five different regions: (D1) all customers buy good i at R; (D2) the two-stop shopping scenario in which the two-stop shoppers buy good i at R

$(\mu_{iR} < \mu_{iL})$; (D3) the one-stop shopping scenario; (D4) the two-stop shopping scenario in which the two-stop shoppers buy good i at L $(\mu_{iR} > \mu_{iL})$; (D5) no customer buys good i at R. Accordingly, the demand for shop iR selling good i in the shopping mall R, given $(\mu_{iL}, \mu_{jL}, \mu_{jR})$, is

$$q_{iR} = \begin{cases} 1, & \mu_{iR} \in D_1 = [-a(i-1), -t + \mu_{iL} + \max\{0, \mu_{jL} - \mu_{jR}\}] \\ 1 - \tilde{x}_L = \frac{1}{t}(\mu_{iL} - \mu_{iR}), & \mu_{iR} \in D_2 = (-t + \mu_{iL} + \max\{0, \mu_{jL} - \mu_{jR}\}, -t + \mu_{iL} + |\mu_{jL} - \mu_{jR}|) \\ 1 - \tilde{x} = \frac{1}{2} + \frac{1}{2t}(\mu_L - \mu_R), & \mu_{iR} \in D_3 = [-t + \mu_{iL} + |\mu_{jL} - \mu_{jR}|, t + \mu_{iL} - |\mu_{jL} - \mu_{jR}|] \\ 1 - \tilde{x}_R = 1 - \frac{1}{t}(\mu_{iR} - \mu_{iL}), & \mu_{iR} \in D_4 = (t + \mu_{iL} - |\mu_{jL} - \mu_{jR}|, t + \mu_{iL} - \max\{0, \mu_{jR} - \mu_{jL}\}) \\ 0, & \mu_{iR} \in D_5 = [t + \mu_{iL} - \max\{0, \mu_{jR} - \mu_{jL}\}, +\infty) \end{cases}$$

We shall now derive each shop's optimal pricing rule in the relevant ranges (D_2, D_3, D_4). The profit maximization problem and the corresponding pricing rule of shop iR can be computed as follows:

D_2: $\max_{\mu_{iR}}(\mu_{iR} + a(i-1))(1 - \tilde{x}_L)$ and $\mu_{iR} = \frac{\mu_{iL} - a(i-1)}{2}$, i.e. $p_{iR} = \frac{p_{iL} - a}{2} + a(i-1)$.

D_3: $\max_{\mu_{iR}}(\mu_{iR} + a(i-1))(1 - \tilde{x})$ and $\mu_{iR} = t + \mu_L - \mu_R$, i.e. $p_{iR} = t + p_L - p_R$.

D_4: $\max_{\mu_{iR}}(\mu_{iR} + a(i-1))(1 - \tilde{x}_R)$ and $\mu_{iR} = \frac{t + \mu_{iL} - a(i-1)}{2}$, i.e. $p_{iR} = \frac{t + p_{iL} - a}{2} + a(i-1)$.

As in the case of the department store L in the two-stop scenario, the price of each shop increases in the value of customer loyalty if the product of the shop has customer loyalty, and the price decreases otherwise.

3 The Multi-stop shopping equilibrium

3.1 Existence

In this subsection, we investigate whether there exists a two-stop shopping equilibrium in the presence of customer loyalty. Particularly, we focus on the size of customer loyalty a (relative to the unit traveling cost t) that gives rise to a two-stop shopping equilibrium.

We first note that for there to be a two-stop shopper, the total loyalty value she gains from visiting both L and R must be no smaller than the traveling cost incurred by doing so. Otherwise, there would exist no two-stop shoppers.

Lemma 1. *If $2a \leq t$, there exists no two-stop shopping equilibrium.*[6]

The above lemma implies that the mere presence of customer loyalty does not guarantee the existence of a two-stop shopping equilibrium. In particular, we can easily see the following:

Corollary 1. *With no customer loyalty ($a = 0$), there exists no two-stop shopping equilibrium.*

This corollary is consistent with Brandão et al. (2014) when there are only two products[7]. This nonexistence of a multi-stop shopping equilibrium motivates Brandão et al. (2014) to warn that analyzing multi-product competition based on the model assuming only two goods may be a substantial restriction. Nevertheless, the following theorem demonstrates that one may disregard it as long as one includes the customer loyalty.

Theorem 1. *For $a/t > \frac{6\sqrt{2}-4}{7}$ (approximately, $a/t > 0.64$), there exist a unique two-stop shopping equilibrium, at which two-stop shoppers purchase each good in pursuit of the additional value from their loyalty (LR-type). In particular, there are two kinds of equilibria depending on the value of a/t.*

(1) $a/t \geq 2$: All customers are two-stop shoppers, $\tilde{x}_L = 0$ and $\tilde{x}_R = 1$.

(2) $\frac{6\sqrt{2}-4}{7} < a/t < 2$: A proportion of customers are one-stop shoppers, i.e. $0 < \tilde{x}_L < \tilde{x}_R < 1$.

(a) prices: $p_{1L} = p_{2R} = \frac{t+a}{3}$ and $p_{1R} = p_{2L} = \frac{2t-a}{3}$

(b) demands: $q_{1L} = q_{2R} = \frac{1}{3} + \frac{a}{3t}$, and $q_{1R} = q_{2L} = \frac{2}{3} - \frac{a}{3t}$.

(c) profits: $\pi_L = \frac{(t+a)^2+(2t-a)^2}{9t}$, $\pi_{1R} = \frac{(2t-a)^2}{9t}$, and $\pi_{2R} = \frac{(t+a)^2}{9t}$.

The above theorem shows that a multi-stop shopping equilibrium exists even when assuming two goods, as long as the value of customer loyalty is higher than a certain level, which is $a/t > \frac{6\sqrt{2}-4}{7}$.

The minimum level of customer loyalty $\frac{6\sqrt{2}-4}{7} \approx 0.64$ that supports the existence of a two-stop shopping equilibrium presents a puzzle. By Lemma 1, we see that if $2a \leq t$, there exists no two-stop shopping equilibrium because the two-stop shopping customers cannot cover their traveling expenses by the additional gains from their loyalty. Hence, a

[6]When $2a \leq t$, there exists an one-stop shopping equilibrium. As our focus lies in the existence of a two-stop shopping equilibrium, and also it can be easily seen in light of Brandão et al. (2014), we shall leave this to the readers.

[7]See Proposition 3 of Brandão et al. (2014)

natural conjecture would be that a two-stop shopping equilibrium always exists whenever $a/t > 0.5$. Surprisingly, however, it turns out that it is not true. The minimum level for the existence of two-stop shopping, $\frac{6\sqrt{2}-4}{7} \approx 0.64$, is higher than $a/t = 0.5$. This seemingly unnatural result has its relevance to the optimal responses of the individual retailers in the presence of customer loyalty, which we shall elaborate in the next section. Therefore, the gap $\Delta = \frac{6\sqrt{2}-4}{7} - 0.5$ in the value of customer loyalty is required to provide an incentive for the stores and their individual retailers *not* to adjust their product prices further for deviation to the one-stop shopping scenario. Therefore, we shall refer to this gap as *the loyalty gap* throughout the rest of this paper.

Remark 1. The equilibrium prices in Theorem 1 explains the empirical stability and regularity of multi-stop shopping that the existing literature fails to do. The existing literature regards multi-stop shopping behaviors of the customers to arise from their incentive to search for the lowest available price in each product category. However, this explanation does not square well with the empirical evidences (Gijsbrechts et al., 2008; Urbany et al., 1996), because only a small proportion of multi-stop shopping customers hunt for temporary price promotions. On the contrary, we assume that the multi-stop shopping behaviors of customers are motivated also by their loyalty to a specific product of a specific store. This explains why multi-stop shopping customers do not easily change their behaviors in response to temporary price promotions. Indeed, when $\frac{6\sqrt{2}-4}{7} < a/t < 2$, the posted prices of goods purchased by the multi-stop shoppers in the two-stop shopping equilibrium, p_{1L} and p_{2R}, are higher than the prices of the other goods:

$$p_{1L} = p_{2R} = \frac{t+a}{3} > p_{1R} = p_{2L} = \frac{2t-a}{3}.$$

The multi-stop shoppers buy more expensive goods because they conceive that these goods ($1L$ and $2R$) are actually cheaper because of their loyalty to these goods. In other words, the effective prices that are perceived by these multi-stop shoppers is lower:

$$\mu_{1L} = \mu_{2R} = \frac{t+a}{3} - a = \frac{t-2a}{3} < \mu_{1R} = \mu_{2L} = \frac{2t-a}{3}.$$

3.2 Modes of retail and price-setting incentive for multi-product firms

In this subsection, we aim to explain the loyalty gap that arises for the existence of a two-stop shopping equilibrium by investigating the price-setting incentives of stores and their constituent shops. A particular focus lies on the differences between the department store and the shopping mall. This subsection also constitutes the proof of Theorem 1 as well.

First of all, we present the necessary and sufficient condition for the department store L to induce two-stop shopping scenario.

Lemma 2. *Suppose that the shopping mall R's effective prices satisfy* $\max\{\mu_{1R}, \mu_{2R}\} < 2t$*. Then, the department store L prefers to induce the two-stop shopping scenario instead of the one-stop shopping scenario if and only if $|\mu_{1R} - \mu_{2R}| > t - a$.*

The above lemma implies that whenever the two-stop shopping scenario is feasible (condition (1) holds), the department store L would stick to the two-stop shopping scenario. In particular, *the department store does not deviate to the one-stop shopping scenario from a two-stop shopping equilibrium (either LR-type or RL-type) as long as the profile of the equilibrium effective price $(\mu^*_{1L}, \mu^*_{2L}, \mu^*_{1R}, \mu^*_{2R})$ satisfies the condition* (1). Then, we may show easily the following:

Corollary 2. *There exists no RL-type two-stop shopping equilibrium.*

Proof. Suppose otherwise that there exist a RL-type two-stop shopping equilibrium. Then, the equilibrium (effective) price profile $(\mu^*_{1L}, \mu^*_{2L}, \mu^*_{1R}, \mu^*_{2R})$ is $\mu^*_{1L} = \mu^*_{2R} = \frac{2}{3}(t-a)$ and $\mu^*_{1R} = \mu^*_{2L} = \frac{t-a}{3}$. Note that $\max\{\mu^*_{1R}, \mu^*_{2R}\} = \mu^*_{2R} = \frac{2}{3}(t-a) < 2t \iff -a < 2t$ but $\mu^*_{2R} - \mu^*_{1R} = \frac{2}{3}(t-a) - \frac{t-a}{3} = \frac{t-a}{3} \not> t-a$. By Lemma 2, the department store L would deviate to induce the one-stop shopping scenario. □

As there is no RL-type two-stop shopping equilibrium. We may now focus on the price-setting incentives of the department store and the shopping mall in the LR-type two-stop shopping equilibrium to check for deviation.

Corollary 3. *If $2a > t$, the department store L would not deviate from a two-stop shopping equilibrium. The equilibrium profit of L is higher than the deviation profit under the one-stop shopping scenario.*

Proof. It suffices to check whether the equilibrium profile, $\mu^*_{1L} = \mu^*_{2R} = \frac{t-2a}{3}$ and $\mu^*_{1R} = \mu^*_{2L} = \frac{2t-a}{3}$, satisfies the conditions in Lemma 2. Specifically, we first see that $\max\{\mu^*_{1R}, \mu^*_{2R}\} = \mu^*_{1R} = \frac{2t-a}{3} < 2t$. Moreover, we observe that

$$\mu^*_{1R} - \mu^*_{2R} = \frac{2t-a}{3} - \frac{t-2a}{3} = \frac{t+a}{3} > t-a,$$

where the last inequality follows from $2a > t$. □

In other words, as long as the two-stop customers may cover their traveling expenses by the loyalty gains, the department store L would not deviate without being provided any further incentive. To understand the intuition behind this result, we examine more closely the incentives of the individual retailers in the department.

Let π^n_{1L} and π^n_{2L} be the profits of individual retailers in the department store, when they do not share the overall profit of the department store. In the LR-type two-stop shopping equilibrium, $\pi^n_{1L} = \frac{(t+a)^2}{9t}$ and $\pi^n_{2L} = \frac{(2t-a)^2}{9t}$. As expected, the increase in the loyalty value a has positive effects on π^n_{1L} and negative effects on π^n_{2L}, $\frac{\partial \pi^n_{1L}}{\partial a} = \frac{2(t+a)}{9t} > 0$ and $\frac{\partial \pi^n_{2L}}{\partial a} = \frac{-2(2t-a)}{9t} < 0$. However, the overall effect on the department store as a whole is positive:

$$\frac{\partial \pi_L}{\partial a} = \frac{2(2a-t)}{9t} > 0$$

where $\pi_L = \pi^n_{1L} + \pi^n_{2L}$ is the profit of the department. As the individual retailers share this profit, the increase in a is beneficial to both.

On the other hand, if deviating to the one-stop scenario, they earn the profit $\pi^d_L = \frac{t^2}{2t}$ with the bundle price $p^d_{1L} + p^d_{2L} = t$. Notice that $p^d_{1L} + p^d_{2L} = t = p^*_{1L} + p^*_{2L}$ and the demand $\tilde{x}_R - \tilde{x} = \tilde{x} - \tilde{x}_L$. Therefore, the overall demand and the associated price for the bundle are the same, but the profit in the two-stop shopping scenario is higher. This seems puzzling, but arises naturally according to the logic similar to the theory of third-degree price discrimination. In the two-stop shopping scenario, there are two markets, one for each good. The department store maximizes its profit in each market, given the prices posted by the shopping mall. In the one-stop shopping scenario, there is only one market, the market for the bundle that consists of good 1 and good 2. Therefore, in the former, the profit must be higher. The additional profit comes from the difference between the prices posted in the two different markets, $(p^*_{1L} - p^*_{2L})$.[8]

Now, turning to the shopping mall R, we investigate the price-setting incentives of the individual retailers, $1R$ and $2R$, in the (LR-type) two-stop shopping equilibrium. That is, we check for each retailer's incentive when the rest (the other retailer in the shopping mall and the department store) follows the equilibrium strategy profile.

We first note that there is no incentive for an individual retailer to deviate by inducing RL-type two-stop shopping. To see this, consider the retailer $1R$. As he already expects the effective price of $2R$ is lower than that of $2L$ ($\mu^*_{2R} < \mu^*_{2L}$), he cannot induce RL-type two-stop shopping. Moreover, in the range of the effective prices that induces LR-type two-stop

[8] The exact expression for the additional profit is $p^*_{1L}(\tilde{x}_R - \tilde{x}) - p^*_{2L}(\tilde{x} - \tilde{x}_L)$. Using $\tilde{x} - \tilde{x}_L = \tilde{x}_R - \tilde{x}$, we may write it as $(p^*_{1L} - p^*_{2L})(\tilde{x} - \tilde{x}_L)$ (or, equivalently, $(p^*_{1L} - p^*_{2L})(\tilde{x}_R - \tilde{x})$).

shopping scenario (that is, in D_4), the equilibrium effective price μ_{1R}^* is the unique maximizer as it is demonstrated in the previous section. Hence, the only remaining choice to $1R$ for deviation is to induce the one-stop shopping demand scenario. Similarly, the retailer $2R$, expecting $\mu_{1L}^* < \mu_{1R}^*$, has nothing but to induce the one-stop shopping demand scenario if he wish to deviate.

Moreover, we note that when deviating, an individual retailer in the shopping mall faces a more inelastic demand curve. The price elasticity of the demand is smaller in the one-stop shopping scenario than it is in the two-stop shopping scenario. Let ϵ_{iR}^{OS} and ϵ_{iR}^{TS} be the price elasticity of the demand faced by the retailer iR ($i = 1, 2$) respectively in the one-stop shopping and in the two-stop shopping scenario.

For the retailer $1R$, a simple calculation gives $\epsilon_{1R}^{OS} = \frac{3p_{1R}}{5t-a-3p_{1R}}$ and $\epsilon_{1R}^{TS} = \frac{3p_{1R}}{4t-2a-3p_{1R}}$. When p_{1R} passes the threshold at which the two-stop shopping scenario switches to the one-stop shopping scenario, the elasticity drops. That is, at the threshold $p_{1R} = t - a$, $\epsilon_{1R}^{OS} = \frac{3(t-a)}{2(t+a)} < \epsilon_{1R}^{TS} = \frac{3(t-a)}{t+a}$. Similarly, to the retailer $2R$, at the threshold between the one-stop and the two-stop shopping scenarios, i.e. $p_{2R} = \mu_{2R} + a = a$, the elasticity drops: $\epsilon_{2R}^{OS} = \frac{3a}{2(2t-a)} < \epsilon_{2R}^{TS} = \frac{3a}{2t-a}$.

The intuition behind this drop is the following: The two different markets, one for good 1 and the other for good 2, in the two-stop shopping scenario becomes integrated into the one market for a bundle of good 1 and good 2 in the one-stop shopping scenario. Then, for example, the individual retailer $1R$ in the shopping take account of not only the price of its opponent $1L$ who sells the same good but also the prices of good 2 posted by $2L$ and $2R$. This makes the demand more inelastic.

Lemma 3. *Suppose that $a/t > 0.5$. The individual retailer $1R$ in the shopping mall would not deviate to the one-stop shopping demand scenario. Specifically, the deviation price would be $\mu_{1R}^d = t - a$ such that the demand under the one-stop shopping scenario coincides with that under the two-stop shopping scenario ($1 - \tilde{x} = 1 - \tilde{x}_R$), and the resulting profit π_{1R}^d is strictly less than the equilibrium profit π_{1R}^*.*

The result is associated with the changes in the price elasticity of the demand when $1R$ switches to the one-stop shopping scenario. The retailer $1R$, who does not have any loyal customers, has an incentive to set the price low enough to steal the demand from $1L$ by inducing the one-stop shopping scenario. However, when he does, the elasticity of his demand becomes less elastic. In addition, within the price ranges that induces one-stop shopping $\mu_{1R} = p_{1R} \in [0, t - a]$, a further decrease in the price would not increase

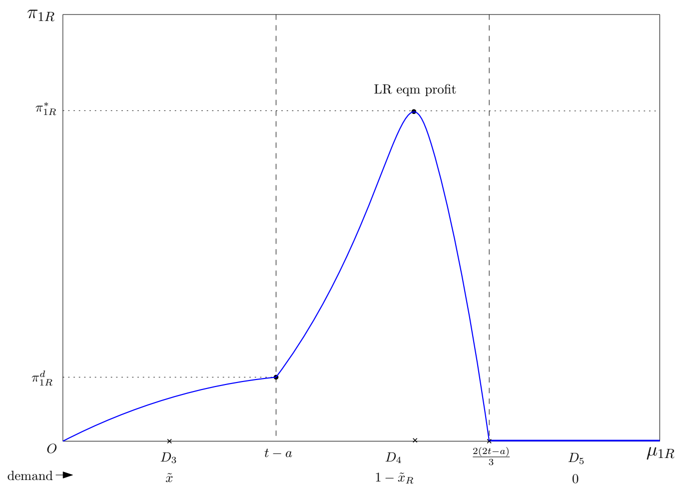


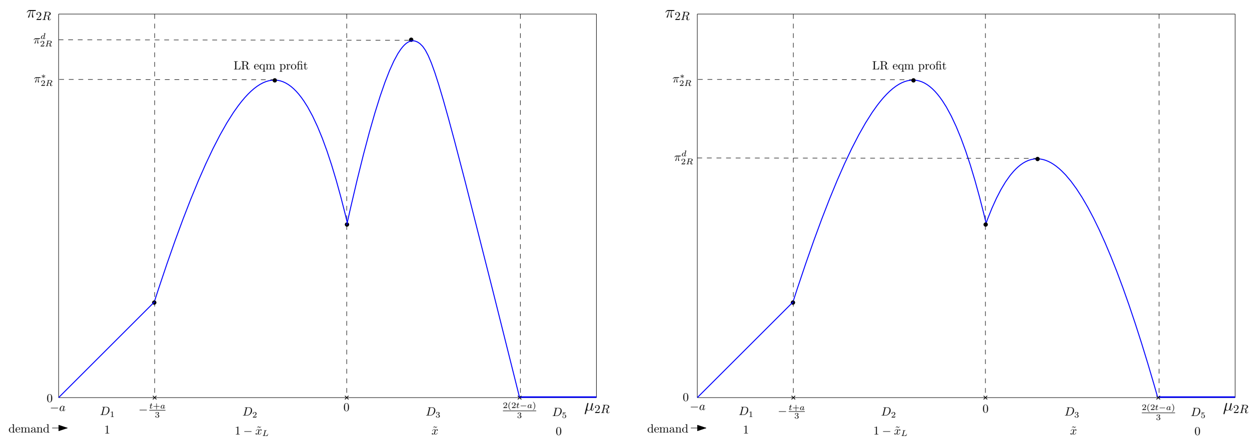
Figure 2: The profit of $1R$ when deviating to μ_{1R}

the demand much. Hence, $1R$ would decrease the price just to make two-stop shopping customers indifferent between one-stop shopping and two-stop shopping.

On the other hand, for the individual retailer $2R$, the same logic does not apply. Expecting that loyal customers would not readily leave her, the retailer $2R$ has an incentive to raise its price. As in the case of $1R$, the decrease in the price elasticity of the demand makes the deviation more costly. On the contrary to the case of $1R$, however, to the retailer $2R$, the customer loyalty is a subsidy. The price perceived by the customers is less than the price she charges. The customer loyalty a affects the retailer $2R$'s price-setting incentive through two different channels. First of all, it offsets the effects of the changing price elasticity of the demand. When calculating the effective price elasticity of the demand at the threshold $p_{2R} = a$ (i.e. $\mu_{2R} = 0$), this loyalty-adjusted elasticity does not fall. Therefore, without fearing the sudden drop in the elasticity of demand, the retailer $2R$ may consider deviating to the one-stop shopping scenario.

Secondly, and more importantly, the subsidy effect on the retailer $2R$'s profit is larger in the two-stop shopping scenario, although it benefits her in both demand scenarios. The reasoning follows easily from the standard theory about the incidence of the subsidy. In the one-stop shopping scenario, the demand becomes less elastic, thus benefiting customers more, while benefiting the retailer $2R$ less. As the size of customer loyalty increases, the deviation to the one-stop shopping scenario thus becomes less attractive to the retailer $2R$.

Although the two effects point out different directions, the overall effect of the customer loyalty on the retailer $2R$'s incentive for deviation is obvious. Note that the first effect does



(a) $0.5 < a/t \leq \frac{6\sqrt{2}-4}{7}$

(b) $a/t > \frac{6\sqrt{2}-4}{7}$

Figure 3: The profit of $2R$ when deviating to μ_{2R}

not depend on the size of a, while the second one increases in a. We thus can expect that there is a threshold at which the sizes of these two effects coincide. Indeed, such a threshold determines the loyalty gap as the following lemma demonstrates.

Lemma 4. *If $0.5 < a/t \leq \frac{6\sqrt{2}-4}{7}$, the retailer $2R$'s deviation profit π^d_{2R} to one-stop scenario (by raising its product price) is larger than the two-stop shopping equilibrium profit π^*_{2R}. Otherwise, if $a/t > \frac{6\sqrt{2}-4}{7}$, then the retailer $2R$ would have no profitable deviation.*

4 Concluding Remarks

In this study, we aim to understand the well-documented phenomenon of multi-stop shopping behaviors of customers through the lens of the customer loyalty. Specifically, we adapt the celebrated Hotelling model to allow for two products and customer loyalty. Moreover, we consider different modes of retail for stores as in Brandão et al. (2014).

The innovation that differentiates this study from the vast literatures on customer loyalty or from those on multi-product competition, is our focus on the firms' price-setting behaviors rather than solely on the consumer behavior. Specifically, we show that there exists a multi-stop shopping equilibrium and the minimum level of customer loyalty guaranteeing such an equilibrium, is above the level required to induce customers to make another trip. The key to understanding this gap lies in the optimal responses of firms, knowing that customers are loyal to a particular product in their stores.

In particular, we find that the gap is required to discourage only an individual retailer who has loyal customers when the store is organized to take the form of a shopping mall. In the department store, such a gap is not required because of the cross-subsidization across individual retailers. In the shopping mall, however, each retailer may deviate to face only the one-stop shoppers. This deviation is not profitable for the retailer having no loyal customers, because the presence of customer loyalty puts additional pressure on him to set the price to a prohibitively low level. For the retailer who owns loyal customers, however, the customer loyalty works as a subsidy, and thus makes it more bearable to face lower and more elastic demands from deviation in exchange for a higher price. This subsidy effect does not last as the customer loyalty gets larger by making it costly to give up multi-stop shoppers.

Lastly, we conclude by addressing possible concerns regarding the two assumptions we impose in this study: (i) symmetric loyalty values and (ii) different modes of retail for the two stores. We assume throughout this paper that the loyalty value a customer gains when purchasing good 1 at store L is identical to the value when purchasing good 2 at store R. Our main results about the multi-stop shopping equilibrium carries over to the case of asymmetric loyalty values, only to make the exposition harder because the region of the customer loyalty values for the existence of a multi-stop shopping equilibrium is characterized in two dimensions.

The other assumption in this study is that one store is a department store while the other is a shopping mall. This assumption is sufficient to serve our main interest about how the customer loyalty affects the existence of a multi-stop shopping through different retail modes and the individual retailers' price-setting incentives. The case where both stores share the same retail modes, either department stores or shopping malls, may be of interest for those who study further about how the profit of each store changes depending on the retail mode of its opponent. However, we shall leave this question for the future, for it is tangential to this paper's main interest.

References

Brandão, A., J. Correia-da Silva, and J. Pinho (2014): "Spatial competition between shopping centers," *Journal of Mathematical Economics*, 50, 234–250.

Drèze, X. and M. Vanhuele (2006): "Deconstructing store switching," Tech. rep.

Fox, E. J. and S. J. Hoch (2005): "Cherry-picking," *Journal of Marketing*, 69, 46–62.

Gijsbrechts, E., K. Campo, and P. Nisol (2008): "Beyond promotion-based store switching: Antecedents and patterns of systematic multiple-store shopping," *International Journal of Research in Marketing*, 25, 5–21.

Jacoby, J. and D. B. Kyner (1973): "Brand loyalty vs. repeat purchasing behavior," *Journal of Marketing research*, 1–9.

Kim, H. and K. Serfes (2006): "A location model with preference for variety," *The Journal of Industrial Economics*, 54, 569–595.

Lal, R. and R. Rao (1997): "Supermarket competition: The case of every day low pricing," *Marketing science*, 16, 60–80.

Thill, J.-C. (1985): "Demand in space and multipurpose shopping: a theoretical approach," *Geographical Analysis*, 17, 114–129.

——— (1986): "A note on multipurpose and multistop shopping, sales, and market areas of firms," *Journal of Regional Science*, 26, 775–784.

——— (1992): "Spatial duopolistic competition with multipurpose and multistop shopping," *The Annals of Regional Science*, 26, 287–304.

Urbany, J. E., P. R. Dickson, and R. Kalapurakal (1996): "Price search in the retail grocery market," *The Journal of Marketing*, 91–104.

Zhang, Q., M. Gangwar, and P. Seetharaman (2013): "Store Loyalty as a Category Specific Trait-What Drives It?" Unpublished Manuscript.

APPENDIX

A Omitted Proofs

We shall compute a candidate two-stop shopping equilibrium for LR- and RL-type, respectively. By combining the pricing rules in Section 2.2, we obtain the following:

(LR) $\mu^*_{1L} = \mu^*_{2R} = \frac{t-2a}{3}$ and $\mu^*_{1R} = \mu^*_{2L} = \frac{2t-a}{3}$.

(RL) $\mu^*_{1L} = \mu^*_{2R} = \frac{2}{3}(t-a)$ and $\mu^*_{1R} = \mu^*_{2L} = \frac{t-a}{3}$.

where $t > a$ holds in the case of RL-type to make sure that $\tilde{x}_L < 1$ and $\tilde{x}_R > 0$. Moreover, in the case of LR-type, as the price cannot be negative, $2t \geq a$.

If $a > 2t$

Proof of Lemma 1. Suppose to the contrary that there exists a two-stop shopping equilibrium. Let $\mu^* = (\mu^*_{1L}, \mu^*_{2L}, \mu^*_{1R}, \mu^*_{2R})$ denote the equilibrium effective price vector. Such an equilibrium must be either of LR-type ($\mu^*_{1L} < \mu^*_{1R}$ and $\mu^*_{2L} > \mu^*_{2R}$) or of RL-type ($\mu^*_{1L} > \mu^*_{1R}$ and $\mu^*_{2L} < \mu^*_{2R}$). In each case, the corresponding candidate equilibrium profile must satisfy the following: (1):

$$(LR) \quad |\mu^*_{1L} - \mu^*_{1R}| + |\mu^*_{2L} - \mu^*_{2R}| = \frac{2}{3}(t+a) > t; \tag{2}$$

$$(RL) \quad |\mu^*_{1L} - \mu^*_{1R}| + |\mu^*_{2L} - \mu^*_{2R}| = \frac{2}{3}(t-a) > t \tag{3}$$

However, none of the above inequalities hold. In the inequality (2), $\frac{2(t+a)}{3} \leq \frac{2t+t}{3} = t$ due to $a \leq 2t$. In (3), $\frac{2}{3}(t-a) > t$ is equivalent to $-a > t$, contradicting to $a > 0$ and $t > 0$.

Proof of Lemma 2. For the necessity, we shall prove firstly the claim for the case of LR-type. Suppose by way of contradiction that $\mu_{1R} - \mu_{2R} \leq t - a$. To induce two-stop shopping, the department store needs to set its product prices so that condition (1) holds. The condition, when inducing LR-type two-stop shopping, is written as $\mu_{1R} - \mu_{1L} + \mu_{2L} - \mu_{2R} = (\mu_{1R} - \mu_{2R}) + (\mu_{2L} - \mu_{1L}) > t$. To have such a pair (μ_{1R}, μ_{2R}), the department store must sell good 1 at a lower price than good 2, i.e. $\mu_{1L} + a = p_{1L} < \mu_{2L} = p_{2L}$. This implies that $(\tilde{x}_R - \tilde{x})p_{1L} < (\tilde{x} - \tilde{x}_L)p_{2L}$ because $\tilde{x}_R - \tilde{x} = \tilde{x} - \tilde{x}_L$. Equivalently, $\tilde{x}_R p_{1L} + \tilde{x}_L p_{2L} < \tilde{x}(p_{1L} + p_{2L})$. The department store must prefer to induce the one-stop shopping scenario, and this is a contradiction. Hence, $\mu_{1R} - \mu_{2R} > t - a$ is a necessary condition for the department store to prefer LR-type two-stop shopping scenario. Similarly, for RL-type, suppose that $\mu_{2R} - \mu_{1R} > t - a$. Again, being combined with the condition (1), $(\mu_{2R} - \mu_{1R}) + (\mu_{1L} - \mu_{2L}) > t$, it must be that $\mu_{1L} - \mu_{2L} > a$, or equivalently, $p_{1L} > p_{2L}$. This in turn implies $(\tilde{x} - \tilde{x}_L)p_{1L} > (\tilde{x}_R - \tilde{x})p_{2L}$, thus leading to the same contradiction as in the previous case.

For the other direction, we first show that if $\mu_{1R} - \mu_{2R} > t - a$, the department store prefers to induce the LR-type two-stop shopping scenario rather than to induce the one-stop shopping scenario. Together with the first order conditions $\mu_{1L} = \frac{\mu_{1R}-a}{2}$ and $\mu_{2L} = \frac{t+\mu_{2R}}{2}$, the hypothesis $\mu_{1R} - \mu_{2R} > t - a$ implies the following:

(i) $\tilde{x}_R = \frac{1}{t}(\mu_{1R} - \mu_{1L}) = \frac{1}{2t}(\mu_{1R} + a) \in [0, 1] \iff \mu_{1R} + a \in [0, 2t]$, implying $0 \leq \mu_{1R} = p_{1R} \leq 2t - a$.

(ii) $\tilde{x}_L = 1 - \frac{\mu_{2L}-\mu_{2R}}{t} = \frac{1}{2} + \frac{\mu_{2R}}{2t} \in [0, 1] \iff \mu_{2R} = p_{2R} - a \in [-t, t]$, which implies $p_{2R} \leq t + a$ $(t - a > 0)$.

(iii) $\mu_{1R} - \mu_{1L} + \mu_{2L} - \mu_{2R} > t \iff \mu_{1R} - \mu_{2R} > t - a$.

(iv) $\mu_{1L} < \mu_{1R} \iff \frac{\mu_{1R}-a}{2} < \mu_{1R} \iff \mu_{1R} > -a$.

(v) $\mu_{2L} > \mu_{2R} \iff \frac{t+\mu_{2R}}{2} > \mu_{2R} \iff \mu_{2R} < t$.

(i) and (iii) are true by hypothesis. (iv) trivially hold, because $\mu_{1R} = p_{1R} \geq 0$ (thus larger than $-a$). (ii) and (v) hold true by (i) and (iii). Therefore, the first-order conditions yield the maximum profit of the department store with two-stop shopping. The resulting profit of the department store is thus $\pi_L^{TS}(\mu_{1R}, \mu_{2R}) = \frac{1}{4t}\left[(\mu_{1R} + a)^2 + (\mu_{2R} + t)^2\right]$. On the other hand, in the one-stop (OS) shopping scenario, $\mu_L = \frac{t+\mu_R-a}{2}$ and the resulting profit function is $\pi_L^{OS}(\mu_{1R}, \mu_{2R}) = \frac{1}{8t}(t + \mu_R + a)^2$. By comparing these two, we obtain

$$\pi^{TS} - \pi^{OS} = \frac{1}{8t}(\mu_{1R} - \mu_{2R} + a - t)^2 \geq 0.$$

Similarly, for RL-type scenario, together with the first order conditions, we see that

(i) $\tilde{x}_R \in [0, 1] \iff \mu_{2R} \in [0, 2t]$, implying $p_{2R} \leq 2t + a$.

(ii) $\tilde{x}_L \in [0, 1] \iff \mu_{1R} + t + a \in [-2t, 2t]$, yielding $0 \leq \mu_{1R} = p_{1R} \leq t - a$.

(iii) $\mu_{1L} - \mu_{1R} + \mu_{2R} - \mu_{2L} > t \iff \mu_{2R} - \mu_{1R} > t + a$.

(iv) $\mu_{1R} < \mu_{1L} \iff \frac{\mu_{1R}+t-a}{2} < \mu_{1R} \iff \mu_{1R} < t-a.$

(v) $\mu_{2R} > \mu_{2L} \iff \frac{\mu_{2R}}{2} > \mu_{2R} \iff \mu_{2R} > 0.$

(i) and (iii) are true by hypothesis, thus implying (ii) and thus (iv). Lastly, (v) holds trivially by (i). Then, the first-order conditions yield the maximum profit of the department store with two-stop shopping. By comparing the profits in the two-stop and the one-stop shopping scenarios, we obtain the following: $\pi^{TS} - \pi^{OS} = \frac{1}{8t}(\mu_{2R} - \mu_{1R} + a - t)^2 \geq 0$.

Proof of Lemma 3. By plugging the LR-equilibrium strategy profile of the other players $(\mu_{1L}^*, \mu_{2L}^*, \mu_{2R}^*)$ for $a < 2t$, we obtain the following expression for shop $1R$'s demand function:

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$$q_{1R} = \begin{cases} 1, & \mu_{1R} \in D_1 = \emptyset \\ 1 - \tilde{x}_L = \frac{t-2a-3\mu_{1R}}{3t}, & \mu_{1R} \in D_2 = \emptyset \\ 1 - \tilde{x} = \frac{5t-a-3\mu_{1R}}{6t}, & \mu_{1R} \in D_3 = [0, t-a] \\ 1 - \tilde{x}_R = \frac{4t-2a-3\mu_{1R}}{3t}, & \mu_{1R} \in D_4 = \left(t-a, \frac{2(2t-a)}{3}\right) \\ 0, & \mu_{1R} \in D_5 = \left[\frac{2(2t-a)}{3}, +\infty\right) \end{cases}$$

In order to show that there is no profitable deviation for shop $1R$, we need to check first whether shop $1R$'s LR equilibrium price indeed belongs to D_4. That is, $\mu_{1R}^* = p_{1R}^* = \frac{2t-a}{3} \in D_4$, which leads to the following condition:

$$\frac{a}{2} < t < 2a$$

Next, we need to check whether there is a profitable deviation for shop $1R$. In particular, we first consider shop $1R$'s deviation to D_3, i.e. it decreases its product price in order to induce one-stop. This requires us to compare the corresponding profits in the related demand regions. Recall that in the LR two-stop shopping equilibrium,

$$\mu_{1R}^* = p_{1R}^* = \frac{2t-a}{3}, \quad q_{1R}^* = 1 - \tilde{x}_R = \frac{2t-a}{3t}, \quad \pi_{1R}^* = \mu_{1R}(1-\tilde{x}_R) = \frac{(2t-a)^2}{9t}.$$

Turning to shop $1R$'s deviation to D_3, consider the slope of the corresponding profit maximization problem,

$$\textit{slope}(\mu_{1R}) \equiv \frac{\partial \pi_{1R}}{\partial \mu_{1R}} = \frac{1}{6t}(5t - a - 6\mu_{1R}),$$

where $\pi_{1R} = \pi_{1R}(\mu_{1R}; \mu^*_{1L}(t,a), \mu^*_{2L}(t,a), \mu^*_{2R}(t,a))$. This expression is decreasing in μ_{1R}, thus achieving its minimum at the right end of the interval D_3. Notice that this expression is positive at its minimum within D_3:

$$\min_{\mu_{1R}\in D_3} \mathit{slope}(\mu_{1R}) = \frac{1}{6t}(-t+5a) > \frac{1}{6t}(-t+a+2t) = \frac{1}{6t}(t+a) > 0.$$

The inequality follows from $2a > t$. As the slope is positive on the interval D_3, we may see that the profit maximization problem of shop $1R$ when deviating to D_3 does not have an interior solution. Moreover, shop $1R$'s profit-maximizing price under deviation to one-stop is the right endpoint of D_3, i.e. $\mu^d_{1R} = \frac{3t-3a}{3} = t - a$. The resulting profit is thus

$$\pi^d_{1R} = \frac{1}{3t}(t-a)(t+a).$$

However, this deviation is not profitable:

$$9t(\pi^*_{1R} - \pi^d_{1R}) = (2t-a)^2 - 3(t-a)(t+a) = (2a-t)^2 \geq 0.$$

It is trivial to see that there is no profitable deviation to D_5 in which shop $1R$ earns zero profit.

Proof of Lemma 4. As in the case of shop $1R$, we plug the LR-equilibrium strategy profile of the other players $(\mu_{1L}, \mu_{2L}, \mu_{1R})$ to obtain the following expression for shop $2R$'s demand function:

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$$q_{2R} = \begin{cases} 1, & \mu_{2R} \in D_1 = [-a, -a + \frac{-t+2a}{3}] \\ 1 - \tilde{x}_L = \frac{2t-a-3\mu_{2R}}{3t}, & \mu_{2R} \in D_2 = (-a + \frac{-t+2a}{3}, -a+a) \\ 1 - \tilde{x} = \frac{4t-2a-3\mu_{2R}}{6t}, & \mu_{2R} \in D_3 = [-a+a, -a + \frac{4t+a}{3}] \\ 1 - \tilde{x}_R = \frac{5t-3\mu_{2R}-a}{3t}, & \mu_{2R} \in D_4 = \emptyset \\ 0, & \mu_{2R} \in D_5 = [-a + \frac{4t+a}{3}, +\infty) \end{cases}$$

where $\mu_{2R} = p_{2R} - a$, $1 - \tilde{x}_L = \frac{1}{t}(\mu_{2L} - \mu_{2R}) = \frac{2t-a-3\mu_{2R}}{3t}$, $1 - \tilde{x} = \frac{1}{2} + \frac{1}{2t}(\mu_L - \mu_R) = \frac{4t-2a-3\mu_{2R}}{6t}$, and $1 - \tilde{x}_R = 1 - \frac{1}{t}(\mu_{2R} - \mu_{2L}) = \frac{5t-3\mu_{2R}-a}{3t}$.

We first check whether shop $2R$'s LR equilibrium price indeed belongs to D_2. That is,

$\mu^*_{2R} = \frac{t-2a}{3} \in D_2$, which leads to the following condition:

$$\frac{a}{2} < t < 2a$$

Now turning to check if there is a profitable deviation, we consider shop $2R$'s deviation to D_3. Recall again that in the LR two-stop shopping equilibrium,

$$\mu^*_{2R} = p^*_{2R} - a = \frac{t-2a}{3}, \quad q^*_{2R} = 1 - \tilde{x}_L = \frac{t + a_{2R}}{3t}, \quad \pi^*_{2R} = (\mu_{2R} + a)(1 - \tilde{x}_L) = \frac{(t+a)^2}{9t}.$$

In D_3, the slope of the profit function with respect to μ_{2R} is

$$slope(\mu_{2R}) \equiv \frac{\partial \pi_{2R}}{\partial \mu_{2R}} = \frac{1}{6t}(4t - 5a - 6\mu_{2R}).$$

At the right endpoint of D_3, when $\mu_{2R} = -a + \frac{4t+a}{3}$, the slope is $\frac{1}{6t}(-4t-a) \leq 0$ because the price p_{2R} at the left endpoint of D_5 cannot be negative. On the other hand, at the left endpoint of D_3, the slope is $\frac{1}{6t}(4t-5a)$ and the deviation price to D_3 depends on its sign: (i) if it is non-positive, the deviation price occurs at the left endpoint of D_3; (ii) if the sign is positive, the deviation price would be an interior solution to the profit maximization based on D_3, i.e. the FOC holds (slope = 0).

Case (i) $5a \geq 4t$: The deviation price is $\mu^{d(1)}_{2R} = 0$. The resulting profit is $\pi^{d(1)}_{2R} = \frac{a}{3t}(2t-a)$. Comparing this with the equilibrium profit of shop $2R$, we may see that there is no profitable deviation:

$$9t(\pi^*_{2R} - \pi^{d(1)}_{2R}) = (t+a)^2 - 3a(2t-a) = (2a-t)^2 \geq 0.$$

Case (ii) $5a < 4t$: The interior solution $\mu^{d(2)}_{2R} = \frac{4t-5a}{6}$ would be the deviation price and the profit is thus $\pi^{d(2)}_{2R} = \frac{1}{2t}\left(\frac{4t+a}{6}\right)^2$. Then, the deviation is not profitable if $\pi^*_{2R} \geq \pi^{d(2)}_{2R}$, equivalently,

$$[2(\triangle) + 1]^2 > 2, \text{ where } \triangle = \frac{2a-t}{2t-a} > 0 \tag{4}$$

Notice that $5a < 4t$ can be rewritten equivalently as $\triangle < 1/2$. Therefore, there is no profitable deviation if

$$\frac{\sqrt{2}-1}{2} < \triangle < \frac{1}{2}.$$

Lastly, we consider a deviation to D_1. The slope of the profit function on D_1 is constant as 1, thus being always positive. Hence, the deviation price occurs at the right endpoint of

D_1, thus the profit under this deviation is $\pi^d_{2R} = \frac{-t+2a}{3}$. This deviation is not profitable:

$$\pi^*_{2R} - \pi^d_{2R} = \frac{(t+a)^2}{9t} - \frac{-t+2a}{3} = \frac{(2t-a)^2}{9t} \geq 0$$

Recall that $\frac{\sqrt{2}-1}{2} < \triangle < \frac{1}{2}$ can be explicitly solved for a/t. Note that $\frac{5}{4} < \frac{1}{4}(2+3\sqrt{2}) \approx 1.56$. Therefore, shop $2R$ may benefit from its deviation to D_3 when $\frac{1}{4}(2+3\sqrt{2})a \approx 1.56a < t < 2a$. In other words, no-deviation condition for shop $2R$ is thus $\frac{1}{2}a < t < \frac{1}{4}(2+3\sqrt{2})a \approx 1.56a$. Equivalently,

$$\frac{6\sqrt{2}-4}{7} \approx 0.64 < \frac{a}{t} < 2 \tag{5}$$