RETURN PREDICTABILITY TEST USING THE ENDOGENOUS REGIME SWITCHING MODEL

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Abstract

This paper examines whether stock excess return predictability is different when the stock market is relatively stable and more volatile. The paper introduces a two-state endogenous regime switching model (ERS) for the stock return predictability test. To model regime switching, this paper adopted a new approach proposed by Chang et al. (2017), allowing an endogenous feedback effect channel through which the underlying time series affect the next period volatility regime. Monte Carlo simulation results demonstrated that additional power gain and bias improvement could be achieved in the ERS model, compared to the conventional Markov switching model. Moreover, by jointly estimating the process of return and predictor series with structure given to their innovation correlations, the ERS model could alleviate the distortion in the least squares estimation caused by well-known data characteristics of return and predictor series: persistence in predictors, and correlations between returns and predictors. The empirical test results using the ERS model indicate that none of the tested predictors have significant predictive power when stock returns are highly volatile. However, the dividend-price ratio and macro variables such as T-bill rate and term spread had significant predictability, at least in the low volatility regime.

Keywords: Predictive regression, Regime switching model, Endogenous feedback effect, Persistence, Correlated innovations, Time-varying volatility

1 Introduction

Is stock excess return predictable? As noted by Phillips and Lee (2013), the efficient market hypothesis argues that stock prices have a martingale property, making stock returns unpredictable. However, return predictability is a controversial issue, since there is plenty of empirical evidence claiming that prediction is possible. Campbell and Shiller (1988a) argued that the relationship between fundamental value and asset price might allow the fundamental to price ratio to predict stock returns. Much subsequent literature has considered price ratios, such as dividend-price ratio and earning-price ratio, as predictors for stock returns. Using a newly-developed conditional test, Lewellen (2004) showed that the dividend-price ratio and the earning-price ratio can be used to predict stock returns. Cochrane (2008) clarified that the dividend-price ratio can be used to predict stock returns, emphasizing the variation in dividend yield and the absence of dividend growth predictability. Campbell and Yogo (2006) showed that both price ratios have predictability though it is weak for the dividend-price ratio. Moreover, Chen (1991) contended that consumption smoothing motives make macro variables such as T-bill rate and term spread effective predictors for stock returns. Macro variables can depict the prospects for the future of the economy, along with demand to smooth consumption, affecting the asset market and asset returns. Campbell and Yogo (2006) and Welch and Goyal (2008) showed that T-bill rate and term spread can significantly predict stock return.

However, many literature reports that return predictability is not stable. According to Ang and Bekaert (2007) and Lettau and Ludvigson (2005), the return predictability seems unstable to the inclusion of the period from the mid-1900s. Welch and Goyal (2008) demonstrated that in-sample fitting is consistently better than the out-of-sample forecasting performance. As noted by Paye and Timmermann (2006), time-varying predictive relations might contribute to the discrepancy between the apparent strong in-sample predictability and the weak out-of-sample predictability. Many papers highlighted the importance of time-varying return generating process and its predictability: Viceira (1997), Schaller and Norden (1997), Paye and Timmermann (2006), Lettau and Van Nieuwerburgh (2008), Pettenuzzo and Timmermann (2011) and Hammerschmid and Lohre (2017). Without proper consideration for time-varying relationship between return and forecasting variable, full-sample inference on predictability might yield biased forecasts.

In this paper, return predictability was separately examined for different volatility regimes. It is reasonable to expect that stock return predictability might be different when the stock market is relatively stable and more volatile. The stock price might exhibit abnormal behavior in the highly volatile period, making return predictability vary with its volatility regime. It became a natural starting point for this paper to adopt a two-state regime switching model for the return predictability test.

To model regime switching, a new approach using an autoregressive latent factor proposed by Chang et al. (2017) was implemented. Unlike the conventional Markov switching model, which assumes that the current state is determined only by the past states, this new approach also allows underlying time series to be correlated with the next period latent factor, allowing return series to have an endogenous feedback effect on the next period volatility regime. Such a channel, through which underlying time series affect the next period volatility state, needs to be considered, especially in the return predictability context, due to the leverage effect. It is realistic that a current shock on a return series would affect the next period volatility regime. Monte Carlo simulation shows that additional information from the underlying time series allows a sharper inference of state process, resulting in power gain and bias improvement compared to the conventional Markov switching model.

Additional to its natural anticipation for switching predictability along with volatility regimes, a endogenous regime switching model (ERS) offers benefit in the estimation and hypothesis testing, compared to the least square estimation. There are a few data characteristics of return and predictor series that might distort standard estimation and hypothesis testing. It is well-known that many predictors are highly persistent and that their innovations are correlated with those of returns. If so, as Stambaugh (1999) noted, the limit distribution of test statistics might deviate from the standard normal, resulting in over-rejection of true null when the ordinary least squares regression is conducted. Moreover, least square estimates would suffer severe finite sample bias when innovations of returns and predictors are strongly correlated. By jointly estimating the process of return and predictor series with structure given to their innovation correlations, the ERS model could alleviate the over-rejection problem and finite sample bias.

In Section 4, an empirical test was conducted using the ERS model. Empirical test results indicated that none of the tested variables offer significant predictability in the high volatility regime. However, at least in the low volatility regime, the dividend-price ratio and macro variables such as T-bill rate and term spread did offer significant predictability for stock returns. It is reasonable that the predictability of three variables are restricted to the period when the stock market is less volatile. When exogenous shock strikes the stock market, stock prices would change rapidly, making the stock market enter the high volatility regime. Under such a circumstance, it would be extremely hard to predict stock returns using any past information, due to unpredictable exogenous shock. Unlike the aforementioned variables, the earning-price ratio did not offer significant predictability, even in the low volatility regime. As noted by Fama and French (1988), this might be the result of noisy earnings' data.

According to the empirical test results in Section 4, stock excess return seems to have switching volatilities. The stock excess return volatility of the high volatility regime was more than three times that of the low volatility regime. Moreover, endogeneity parameters were also estimated to have significantly large negative value. This goes well with the leverage effect; a negative shock on the current return increases the next period volatility. Therefore, it is important to consider the volatility regime and the endogenous feedback effect to properly model return series.

In Section 5, Monte Carlo simulation was conducted to examine what happens when either volatility regimes or the endogenous feedback effect is ignored. Section 5.2. investigated the effect of ignoring volatility regimes. The traditional model, estimated using least squares, ignores volatility regimes. According to simulation results, the size distortion and finite sample bias in β was bigger when volatility regimes were ignored. In Section 5.3, the effect of ignoring the endogenous feedback effect was illustrated by comparing the conventional Markov switching model and the ERS model. It was shown that there is a loss of power when a conventional Markov switching model is used without considering an endogenous feedback effect channel. In particular, an interesting pattern of bias improvement was found by simulation results. The bias improvement of the ERS model tends to be more substantial when the regime switching parameter has large switching values.

The remainder of this paper is as follows. Section 2 illustrates the models used for empirical analysis. Section 3 describes how to estimate the ERS model in detail. In Section 4, an empirical analysis using real data is presented. Section 5 provides Monte Carlo evidence that the ERS model outperforms the conventional Markov switching (CRS, i.e., the conventional regime switching) model and the traditional model (OLS) regarding size, power, and bias. Section 6 presents the conclusion.

2 The Model

2.1 Endogenous Regime Switching Model (ERS)

The two-state regime switching model was adopted to capture switching stock return predictability with its volatility regimes. Therefore, a state characterizing rule was set as $\sigma_u(s_t = 0) < \sigma_u(s_t = 1)$, allowing state 1 to be a high volatility regime and state 0 to be a low volatility regime. The predictive regression parameters, α and β, were also modeled to have switching values given volatility regimes.

$$y_t = \alpha(s_t) + \beta(s_t)x_{t-1} + \sigma_u(s_t)u_t \tag{1}$$

When β is not equal to zero, the univariate predictor x_{t-1} has some predictive power on future stock excess returns y_t. Therefore, $\beta(s_t)$ is a parameter of interest indicating whether the predictor has any predictive power over stock excess returns. By doing so, this model provides a way to compare the predictive power of a predictor in high $(\overline{\beta})$ and low $(\underline{\beta})$ volatility regimes.[1] The predictor was also assumed to have regime switching volatilities.[2]

$$x_t = \mu + \phi x_{t-1} + \sigma_\nu(s_t)\nu_t \tag{2}$$

where u_t and ν_t are normalized errors with a variance of one.

$$u_t = \pi\nu_t + \sqrt{1-\pi^2}\varepsilon_t \tag{3}$$

[1]The underline notation is for the parameter value in the low volatility regime while the overline notation is for the parameter value in the high volatility regime.

[2]Among predictors considered in this paper, price ratios (dividend-price ratio, earning-price ratio and smoothed real earnings-real price ratio) were assumed to have regime switching volatility while macro variables such as treasury bill rate and term spread were assumed not. The grounds for such setting will be elaborated on Section 4.2.

where ν_t and ε_t are independent with a normalized variance of one. Therefore, π denotes contemporaneous correlation between the innovation of y_t and that of x_t. As can be shown in (2) and (3), this model give room for persistence in (x_t) and innovation correlation π to be structured within the model.

This paper implemented an innovative approach developed by Chang et al. (2017) to model regime switching. According to their new approach, regimes are determined by whether the autoregressive latent factor (f_t) exceeds the threshold level or not.

$$s_t = 1\{f_t \geq \tau\} \tag{4}$$

$$f_{t+1} = \lambda f_t + \eta_{t+1} \tag{5}$$

where $s_t = \{0, 1\}$ which is determined by latent factor f_t and threshold level τ. The strength of their new method comes from the correlation between the next period innovation of latent factor η_{t+1} and underlying time series innovations u_t and ν_t. With an latent factor that is correlated with previous underlying time series, regime could be determined endogenously in this new approach.

$$\begin{pmatrix} \nu_t \\ \varepsilon_t \\ \eta_{t+1} \end{pmatrix} =_d N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} \right) \tag{6}$$

The current shock on return series may affect next period latent factor and volatility regime since u_t and η_{t+1} are correlated by $\pi\rho_1 + \sqrt{1-\pi^2}\rho_2$. By allowing underlying time series to affect next period latent factor and volatility regime determination, new approach relieves the assumption of a conventional Markov switching (CRS, i.e., conventional regime switching) model that state is determined independently from the underlying time series. Such an endogenous feedback effect of the underlying time series is an important extension the ERS model has made compared to the CRS model.

2.2 Conventional Markov Switching Model (CRS)

The ERS model is a natural extension of the conventional Markov switching (CRS i.e., conventional regime switching) model. Chang et al. (2017) showed their new approach to regime switching reduces to the conventional Markov switching model when the autoregressive latent factor is exogenous and stationary. Therefore, the likelihood ratio test (LR test) could be conducted to check whether the likelihood of the unrestricted model (ERS) is significantly higher than that of the restricted model (CRS). The empirical results in Section 4 shows that the maximum log likelihood significantly increases at the 5% level once the endogenous feedback effect is modeled.

The CRS model can be easily estimated if we impose restriction in equation (5) and (6) as follows:

$$|\lambda| < 1$$

$$\begin{pmatrix} \nu_t \\ \varepsilon_t \\ \eta_{t+1} \end{pmatrix} =_d N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

Compared to the CRS model, the ERS model is more realistic particularly in the context of return predictability. It is realistic to expect stock return innovation to affect next period volatility regime through the latent factor. Negative ρ_1 or ρ_2 implies that a negative shock on stock returns is likely to increase volatility in the next period. Such a phenomenon is well-known as a leverage effect. In the empirical analysis section, it is shown that ρ_2 is estimated to have a significantly negative value which is consistent with the leverage effect.

2.3 Traditional Predictive Regression Model (OLS)

The traditional way to test stock return predictability is using a simple regression model as follows:

$$y_t = \alpha + \beta x_{t-1} + u_t$$
$$x_t = \mu + \phi x_{t-1} + \nu_t$$

where u_t and ν_t are i.i.d. innovations with variance σ_u^2 and σ_ν^2, respectively. The traditional model ignores the existence of volatility regimes. It does not consider the possibility that stock return predictability might change with volatility regimes.

A typical way to analyze the traditional model is using a least squares estimate $\widehat{\beta}$ and its t-ratio. However, some characteristics of series (y_t) and (x_t) might cause size distortion and finite sample bias in $\widehat{\beta}$. Stambaugh (1999) argued that the OLS estimate of β is biased, and the null of no predictability $(\beta = 0)$ is over-rejected when stock return innovation (u_t) is strongly correlated with that of predictor (ν_t) and the predictor is highly persistent. It is important to properly manage such problems, because most predictor series are highly persistent and some of them have innovations strongly correlated with that of stock returns. The ERS model jointly estimate two equations with structure given to innovations u_t and ν_t. As will be shown in Monte Carlo simulation results in Section 5, the ERS model could alleviate size distortion and finite sample bias.

3 Estimation

The endogenous regime switching model (ERS) can be estimated by the maximum likelihood method. For the maximum likelihood estimation of the model, the log-likelihood function can be written as

$$\ell(y_1, \cdots, y_n, x_1, \cdots, x_n) = \log p(y_1, x_1) + \sum_{t=2}^{n} \log p(y_t, x_t | \mathcal{F}_{t-1}) \tag{7}$$

where $\mathcal{F}_t = \sigma((x_s)_{s \le t}, (y_s)_{s \le t})$ is a set of information given for $t = 1, \cdots, n$. There is a set of unknown parameters $\theta \in \Theta$. In this case, θ is a 15-by-1 vector containing $\alpha(s_t)$, $\beta(s_t)$, $\sigma_u(s_t)$, μ, ϕ, $\sigma_\nu(s_t)$, π, λ, τ, ρ_1 and ρ_2.

By allowing an endogenous feedback effect as in (6), the state process is not only influenced by the previous states but also by the underlying time series. Therefore, state process (s_t) alone is not a first-order Markov process; rather, (s_t, y_t, x_t) on $\{0, 1\} \times \mathbb{R} \times \mathbb{R}$ is a first-order Markov process. This makes conventional Markov switching filter not applicable, resulting in a need for a modified Markov switching filter.

$$p(s_t, y_t, x_t | s_{t-1}, y_{t-1}, x_{t-1}) = p(y_t, x_t | s_t, s_{t-1}, y_{t-1}, x_{t-1}) \times p(s_t | s_{t-1}, y_{t-1}, x_{t-1})$$

where

$$\begin{aligned} & p(y_t, x_t | s_t, s_{t-1}, y_{t-1}, x_{t-1}) \\ & = N\left(\begin{pmatrix} \alpha(s_t) + \beta(s_t) x_{t-1} \\ \mu + \phi x_{t-1} \end{pmatrix}, \begin{pmatrix} \sigma_u(s_t)^2 & \sigma_u(s_t)\sigma_\nu(s_t)\pi \\ \sigma_u(s_t)\sigma_\nu(s_t)\pi & \sigma_\nu(s_t)^2 \end{pmatrix} \right) \end{aligned} \tag{8}$$

and

$$p(s_t | s_{t-1}, y_{t-1}, x_{t-1}) = (1 - s_t)\omega_\rho + s_t(1 - \omega_\rho) \tag{9}$$

where $\omega_\rho = \omega_\rho(s_{t-1}, y_{t-1}, x_{t-1})$ is transition probability of (s_t) to the low state conditional on the previous state and the past values of the time series. Given ρ_1 and ρ_2 in (6), the state process, not alone but jointly with the underlying time series, follows Markov process. Therefore, transition probabilities do not stay constant but vary over time, affected by the underlying time series in the previous period. This makes the ERS model more realistic compared to the conventional Markov switching model which assumes constant transition probabilities.

When the endogeneity channel, through which the underlying time series affect the next period latent factor, is modeled, a modified Markov switching filter should be developed as in Chang et al. (2017). To develop a modified Markov switching filter applicable to this model, we let

$$\Phi_\rho(x) = \Phi\left(\frac{x}{\sqrt{1-(\rho_1^2+\rho_2^2)}}\right) \tag{10}$$

For the covariance matrix in (6) to be a non-negative definite matrix, it must be the case that $\rho_1^2 + \rho_2^2 \leq 1$. When $\rho_1^2 + \rho_2^2 = 1$, $\Phi_\rho(x)$ in (10) cannot be defined. For special cases in which $\rho_1^2 + \rho_2^2 = 1$, please refer to Chang et al. (2017). From now on, this paper will focus only on the case where $\rho_1^2 + \rho_2^2 < 1$ and latent factor is stationary($|\lambda| < 1$). If so, transition probability to the low volatility regime is as follows. The exact derivation of (11) will be given in Appendix A.1.

$$\begin{aligned}\omega_\rho &= \omega_\rho(s_{t-1}, y_{t-1}, x_{t-1})\\ &= \frac{\left((1-s_{t-1})\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} + s_{t-1}\int_{\tau\sqrt{1-\lambda^2}}^{\infty}\right)\Phi_\rho\left(\tau - \frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} - \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1} - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right)\rho(x)dx}{(1-s_{t-1})\Phi(\tau\sqrt{1-\lambda^2}) + s_{t-1}[1-\Phi(\tau\sqrt{1-\lambda^2})}\end{aligned} \tag{11}$$

On the other hand, when latent variable is independent with the underlying time series ($\rho_1 = \rho_2 = 0$), transition probabilities are not influenced by u_{t-1} nor ν_{t-1}. The transition probability to the low volatility regime is as follows:

$$\omega_\rho = \omega_\rho(s_{t-1}) = \frac{\left((1-s_{t-1})\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} + s_{t-1}\int_{\tau\sqrt{1-\lambda^2}}^{\infty}\right)\Phi\left(\tau - \frac{\lambda x}{\sqrt{1-\lambda^2}}\right)\rho(x)dx}{(1-s_{t-1})\Phi(\tau\sqrt{1-\lambda^2}) + s_{t-1}[1-\Phi(\tau\sqrt{1-\lambda^2})]}$$

The state process (s_t) follows a first-order Markov process, independent of the time series. This clearly indicates that the ERS model reduces to the conventional model when the underlying autoregressive latent factor is stationary and independent from the model innovations.

The modified Markov switching filter is composed of two steps: prediction and updating.

$$p(y_t, x_t|\mathcal{F}_{t-1}) = \sum_{s_t} p(y_t, x_t|s_t, \mathcal{F}_{t-1})p(s_t|\mathcal{F}_{t-1}) \tag{12}$$

where $p(y_t, x_t|s_t, \mathcal{F}_{t-1})$ is given in (8). To calculate the log-likelihood function in (7), $p(s_t|\mathcal{F}_{t-1})$ is needed which can be obtained from the prediction step. For the prediction step,

$$p(s_t|\mathcal{F}_{t-1}) = \sum_{s_{t-1}} p(s_t|s_{t-1}, \mathcal{F}_{t-1})p(s_{t-1}|\mathcal{F}_{t-1}) \tag{13}$$

where $p(s_t|s_{t-1}, \mathcal{F}_{t-1})$ is given in (9). This can be easily computed once the updating step is conducted for the previous period. For the updating step,

$$\begin{aligned} p(s_t|\mathcal{F}_t) &= p(s_t|y_t, x_t, \mathcal{F}_{t-1}) \\ &= \frac{p(y_t, x_t|s_t, \mathcal{F}_{t-1})p(s_t|\mathcal{F}_{t-1})}{p(y_t, x_t|\mathcal{F}_{t-1})} = \frac{p(y_t, x_t|s_t, \mathcal{F}_{t-1})p(s_t|\mathcal{F}_{t-1})}{\sum_{s_t} p(y_t, x_t|s_t, \mathcal{F}_{t-1})p(s_t|\mathcal{F}_{t-1})} \end{aligned} \tag{14}$$

where $p(y_t, x_t|s_t, \mathcal{F}_{t-1})$ is given in (8) and $p(s_t|\mathcal{F}_{t-1})$ is obtainable from the prediction step. Therefore, iterative computation will allow us to continue the prediction and updating steps.

As Chang et al. (2017) have noted, latent factor f_t can be extracted using a modified Markov switching filter through the prediction and updating steps. The latent factor can be extracted using its conditional density as follows:

$$\mathbb{E}(f_t|\mathcal{F}_t) = \int f_t p(f_t|\mathcal{F}_t) df_t$$

where the conditional density of the latent factor is

$$p(f_t|\mathcal{F}_t) = \frac{p(y_t, x_t|f_t, \mathcal{F}_{t-1}) p(f_t|\mathcal{F}_{t-1})}{p(y_t, x_t|\mathcal{F}_{t-1})} \tag{15}$$

Since $p(y_t, x_t|f_t, \mathcal{F}_{t-1}) = p(y_t, x_t|s_t, \mathcal{F}_{t-1})$ is in (8), allowing us to calculate $p(y_t, x_t|\mathcal{F}_{t-1})$, the updated conditional density of the latent factor (15) is obtainable once $p(f_t|\mathcal{F}_{t-1})$ is obtained from the prediction step. For the prediction step,

$$p(f_t|\mathcal{F}_{t-1}) = \sum_{s_{t-1}} p(f_t|s_{t-1}, \mathcal{F}_{t-1}) p(s_{t-1}|\mathcal{F}_{t-1}) \tag{16}$$

where $p(s_{t-1}|\mathcal{F}_{t-1})$ is obtained from the previous updating step. Therefore, the remaining task is to find $p(f_t|s_{t-1}, \mathcal{F}_{t-1})$, the conditional density of the latent factor on previous state and past information set.

When $|\lambda| < 1$ and $\rho_1^2 + \rho_2^2 < 1$, $p(f_t|s_{t-1}, \mathcal{F}_{t-1})$ is as follows. The relevant proof is in Appendix A.2.

$$
\begin{aligned}
p(f_t|s_{t-1}=1,\mathcal{F}_{t-1}) \quad &= \frac{1-\Phi(X)}{1-\Phi(\tau\sqrt{1-\lambda^2})} \\
&\times N\left(\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1}+\left(\rho_1-\frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1},\frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-\lambda^2}\right)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
p(f_t|s_{t-1}=0,\mathcal{F}_{t-1}) \quad &= \frac{\Phi(X)}{\Phi(\tau\sqrt{1-\lambda^2})} \\
&\times N\left(\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1}+\left(\rho_1-\frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1},\frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-\lambda^2}\right)
\end{aligned} \tag{18}
$$

where

$$
X=\sqrt{\frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-(\rho_1^2+\rho_2^2)}}\left(\tau-\frac{\lambda\left(f_t-\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1}-\left(\rho_1-\frac{\pi\rho_2}{\sqrt{1-\pi^2}}\right)\nu_{t-1}\right)}{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}\right)
$$

4 Empirical Analysis

This section tests whether each predictor has predictive power for stock excess returns. Three models are used to test predictability of each predictor: The traditional model (OLS), the conventional Markov switching (CRS, i.e., conventional regime switching) model and the endogenous regime switching model (ERS). As potential predictors, price ratios such as the dividend-price ratio and earning-price ratio and macro variables such as T-bill rate and term spread were considered.

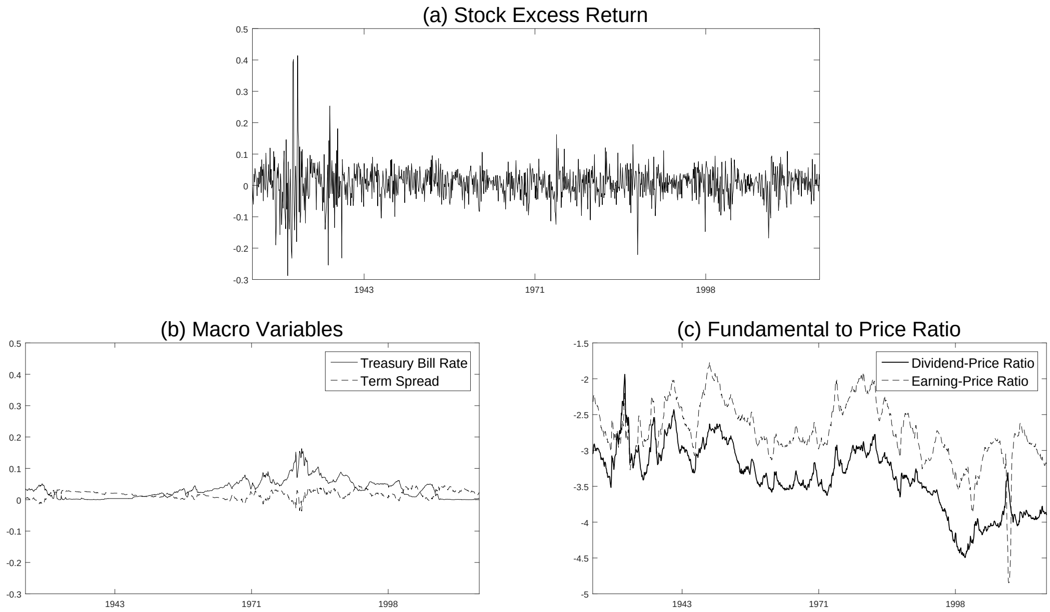
4.1 Description of Data

The full sample period is from January 1926 to December 2016. For stock market returns, monthly *value-weighted return including distributions* (VWRETD) from the Center for Research in Security Prices (CRSP) was used. The monthly excess return was computed by subtracting risk-free rates from stock market returns. The 3-month T-bill rate was used as a risk-free rate.

There are two macro variables considered as potential predictors for stock excess returns: the 3-month T-bill rate and the term spread. As the 3-month T-bill rate, *3-Month Treasury Bill: Secondary Market Rate* from FRED was used from 1934. But for 1926-1934, *U.S. Yields On Short-Term United States Securities, Three-Six Month Treasury Notes and Certificates, Three Month Treasury* from NBER Macrohistory database was used. The term spread is a difference between the long term yield and the T-bill rate. As a long term yield, *Long-Term Government Bond Yield* from Ibbotson's *Stocks, Bonds, Bill and Inflation Yearbook* was used. This dataset was obtained from the 2016 updated version of one used in Welch and Goyal (2008), uploaded to the webpage of Amit Goyal.[3]

[3] http://www.hec.unil.ch/agoyal/

Figure 1: Time Series Plot



Notes. A time series plot of the variables used in this analysis for the sample period 1926/01-2016/12. Figure 1-(a) plots stock excess returns calculated using VWRETD from CRSP and a 3-Month T-bill rate. Figure 1-(b) plots macro variables: 3-Month T-bill rate and term spread, which is the difference between long term yield on government bond and the 3-Month T-bill rate. Figure 1-(c) plots price ratios: log dividend-price ratio and log earning-price ratio.

As indicators for the fundamental to price ratio, two price ratios were considered: the dividend-price ratio and earning-price ratio. According to Campbell and Shiller (1988a), the dividend-price ratio(or earning-price ratio) was calculated as a ratio of dividends(or earnings) over the past year relative to current price. This data was offered by *U.S. Stock Markets 1871-Present and CAPE Ratio*, uploaded in the online data of Robert Shiller.[4] For the actual predictive regression, the natural logarithm was taken on price ratios.

Chang et al. (2017) showed that the ERS model reduces to the CRS model when the autoregressive latent factor is exogenous($\rho_1 = \rho_2 = 0$). Therefore, an LR test could be conducted to check whether the log likelihood of unrestricted model (ERS) is significantly higher than that of

[4]http://www.econ.yale.edu/$\sim$shiller/data.htm

the restricted model (CRS). The following sections will show that the maximum log likelihood significantly increases at the 5% level once the endogenous feedback effect is modeled.

4.2 Estimation Results

The estimation results for the predictability test are presented in this section. Four predictors were tested, respectively not jointly. For each predictor, estimation results using three models, OLS, CRS and ERS, are compared.

As can be seen from Figure 1, stock excess returns seem to have time-varying volatility. To model the switching volatility of stock returns, a two-state regime switching model has been introduced. The state characterizing rule for the model, $\sigma_u(s_t = 0) < \sigma_u(s_t = 1)$, implies that stock return volatility is higher in state 1 than in state 0. Among tested predictors for stock returns, price ratios are highly likely to have a similar volatility pattern with stock returns since all of them were divided by stock price. Therefore, it can be presumed that price ratios also have high volatility in state 1, when stock return volatility is high. To reflect these presumptions, when price ratios are used as an individual predictor for stock returns, a regime was also given for predictor volatility; $\sigma_\nu(s_t = 0) < \sigma_\nu(s_t = 1)$. On the other hand, macro variables are less likely to have higher volatility whenever stock return is in the high volatility regime. Therefore, macro variable volatilities were not assumed as regime switching parameters.

4.2.1 Results with Macro Variables

According to Chen (1991), macro variables such as T-bill rate or term spread can be used to predict asset returns since they can provide prospects for the future economy, which affects the asset market and thereby asset returns. There are many subsequent literatures which supported predictability of macro variables for stock returns: Chen (1991), Fama and French (1989), Campbell and Yogo (2006) and Welch and Goyal (2008). Therefore, in this section, macro variables such as T-bill rate and term spread are considered as potential predictors for stock excess returns.

Estimation results are shown in Table 1. Panel A reports estimation results when the T-bill rate was used as a predictor for stock excess returns, while Panel B reports results when term spread was used as a predictor. In each panel, estimation results from the traditional model which is estimated using least squares are reported in the second column. The next two columns show estimation results from the CRS model, while the last two columns present results from the ERS model. For two-state regime switching models, estimates for the regime switching parameters are presented side by side. The left $(s_t = 0)$ shows estimates in a low volatility regime, while the right $(s_t = 1)$ shows those in a high volatility regime.

The estimated volatility of stock excess returns in the state 1 $(\overline{\sigma_u})$ is almost three times bigger than that in the state 0 $(\underline{\sigma_u})$. When traditional model is used, none of macro variables seem to have any predictive power for stock excess returns. However, two-state regime switching models demonstrated predictability at least in the low volatility regime. It is noticeable that the predictive power was significantly observed at least when stock market is less volatile using two-state volatility regime switching models, though none of them seemed to have significant predictability using OLS.

Significantly detected predictability in the low volatility regime is consistent with many previous papers which support predictability of macro variables. Fama and French (1989) and Chen (1991) emphasized a consumption smoothing motive to explain why economic growth forecasting variables could also play important role in predicting asset returns. A forecasting variable for macroeconomy can also indirectly forecast asset returns because people will reduce their savings when future economic growth is expected, increasing returns on asset. Chen (1991) empirically

Table 1: Estimation Results with Macro Variables

Panel A. T-bill Rate

	OLS	CRS		ERS	
		$s_t = 0$	$s_t = 1$	$s_t = 0$	$s_t = 1$
α	0.0098	0.0141	-0.0061	0.0136	-0.0029
	(0.002)**	(0.002)**	(0.013)	(0.002)**	(0.013)
β	-0.0975	-0.1304	-0.6415	-0.1284	-0.8085
	(0.054)	(0.041)**	(0.520)	(0.041)**	(0.539)
σ_u		0.0386	0.1173	0.0384	0.1191
		(0.001)**	(0.009)**	(0.001)**	(0.009)**
μ	0.0002	0.0002		0.0002	
	(1.65e-4)	(1.65e-4)		(1.65e-4)	
ϕ	0.9934	0.9934		0.9935	
	(0.004)	(0.004)**		(0.004)**	
σ_ν		0.0036		0.0036	
		(7.82e-5)**		(7.82e-5)**	
π		-0.1328		-0.1348	
		(0.030)**		(0.031)**	
λ		0.9947		0.9923	
		(0.004)**		(0.006)**	
τ		11.5103		9.7264	
		(4.555)*		(3.729)**	
ρ_1				0.2436	
				(0.216)	
ρ_2				-0.9602	
				(0.058)**	
log likelihood		6396.8352		6400.6760	
p-value for LR test		0.0215			

Panel B. Term Spread

	OLS	CRS		ERS	
		$s_t = 0$	$s_t = 1$	$s_t = 0$	$s_t = 1$
α	0.0035	0.0052	-0.0202	0.0048	-0.0150
	(0.003)	(0.002)*	(0.023)	(0.002)*	(0.022)
β	0.1772	0.2296	0.3605	0.2326	0.1744
	(0.126)	(0.102)*	(0.935)	(0.099)*	(0.879)
σ_u		0.0388	0.1175	0.0385	0.1187
		(0.001)**	(0.009)**	(0.001)**	(0.009)**
μ	0.0007	0.0007		0.0007	
	(1.79e-4)**	(1.79e-4)**		(1.79e-4)**	
ϕ	0.9610	0.9612		0.9612	
	(0.008)**	(0.008)**		(0.008)**	
σ_ν		0.0036		0.0036	
		(7.71e-5)**		(7.71e-5)**	
π		0.0371		0.0335	
		(0.031)		(0.031)	
λ		0.9951		0.9926	
		(0.004)**		(0.005)**	
τ		12.1415		9.9296	
		(4.788)*		(3.716)**	
ρ_1				-0.2069	
				(0.294)	
ρ_2				-0.8851	
				(0.193)**	
log likelihood		6400.9465		6404.4672	
p-value for LR test		0.0296			

Notes. Standard errors are in parenthesis. The test values that are significant at 95% and 99% level are presented respectively with * and **. The LR test was conducted with the null of no endogeneity ($\rho_1 = \rho_2 = 0$).

showed that T-bill and term spread are valid indicator for future economic growth, implying that they might also have predictive power for stock returns. Campbell and Yogo (2006) and Welch and Goyal (2008) also indicated that T-bill rate and term spread have predictability for stock excess returns.

However, Table 1 indicates that such predictability is only restricted to the period when the stock market is less volatile. When stock return is in high volatility regime, Table 1 suggests that stock excess return is hardly predictable. This might be a result of exogenous shock striking stock market which increases market volatility with rapidly changing asset prices. In such a case, the exogenous shock would be a main driving force in the asset pricing, making stock returns extremely hard to be predicted using any past information. Therefore, it might be hard to predict stock excess returns in the high volatility regime using T-bill or term spread, unlike in the low volatility regime.

Among the endogeneity parameters, ρ_2 was estimated to have quite substantial value while ρ_1 was not. This implies that the innovation of stock excess returns, especially the part that is uncorrelated to the predictor, affects the latent factor in the next period, determining the volatility regime of the following period. Since ρ_2 was significantly negative with a large magnitude, a negative shock on stock excess return seems to make high volatility regime more probable in the next period. This is consistent with the strong leverage effect observed in Chang et al. (2017). As reported at the bottom of Table 1, the null of no endogeneity $(\rho_1 = \rho_2 = 0)$ was rejected at a 5% significance level for both cases. The LR test results imply that the endogenous feedback effect is worth considering, significantly increasing the explanatory power of the model.

4.2.2 Results with Fundamental to Price Ratio

The fundamental to price ratio has been considered as a potential predictor for stock returns in many literatures: Campbell and Shiller (1988a), Lewellen (2004), Campbell and Yogo (2006), Welch and Goyal (2008), Cochrane (2008), and Choi et al. (2016). According to Campbell and Shiller (1988a), asset returns rise when assets are underpriced relative to their fundamental values. The relationship between the fundamental value and price of an asset might allow fundamental to price ratio to predict stock returns.

As indicators for fundamental to price ratio, two variables were considered: the dividend-price ratio and the earning-price ratio. The estimation results are reported in Table 2. Panel A reports estimation results when the dividend-price ratio was used as a predictor for stock excess returns, while Panel B reports those when the earning-price ratio was used as a predictor. In both panels of Table 2, it is noticeable that estimated volatility of stock excess return in the state 1 ($\overline{\sigma_u}$) is almost three times bigger than that in the state 0 ($\underline{\sigma_u}$), as in Table 1. Therefore, stock excess return certainly seems to have switching volatilities. The traditional model, simply using least squares without considering regime switching property in stock return volatilities, implies that both price ratios have significant predictability for stock returns. However, once predictability is modeled separately for each volatility regime, predictability disappears in the high volatility regime; even in the low volatility regime for some case.

When the dividend-price ratio was used as a predictor, Panel A shows that the predictability is significantly observed only in the low volatility regime. It seems that dividends over past year act as a good proxy for the fundamental value of stock, making the dividend-price ratio as a valid predictor for stock returns, at least in the low volatility regime. However, once the market becomes highly volatile, the predictability disappears. This might be due to the exogenous and unpredictable

Table 2: Estimation Results with Fundamental to Price Ratios

Panel A. Dividend-Price Ratio

	OLS	CRS		ERS	
		$s_t = 0$	$s_t = 1$	$s_t = 0$	$s_t = 1$
α	0.0318	0.0298	0.0384	0.0293	0.0900
	(0.012)**	(0.010)**	(0.043)	(0.010)**	(0.043)*
β	0.0075	0.0061	0.0110	0.0061	0.0266
	(0.004)*	(0.003)*	(0.013)	(0.003)*	(0.014)
σ_u		0.0366	0.1093	0.0366	0.1135
		(0.001)**	(0.006)**	(0.001)**	(0.007)**
μ	-0.0162	-0.0113		-0.0130	
	(0.010)	(0.008)		(0.008)	
ϕ	0.9954	0.9976		0.9972	
	(0.003)**	(0.002)**		(0.002)**	
σ_ν		0.0306	0.0930	0.0304	0.0939
		(0.001)**	(0.005)**	(0.001)**	(0.005)**
π		-0.6408		-0.6398	
		(0.019)**		(0.019)**	
λ		0.9774		0.9620	
		(0.010)**		(0.014)**	
τ		4.8608		3.8651	
		(1.150)**		(0.815)**	
ρ_1				0.0654	
				(0.122)	
ρ_2				-0.9697	
				(0.111)**	
log likelihood		4133.2318		4150.0105	
p-value for LR test		5.17e-08			

Panel B. Earning-Price Ratio

	OLS	CRS		ERS	
		$s_t = 0$	$s_t = 1$	$s_t = 0$	$s_t = 1$
α	0.0285	0.0201	0.0003	0.0200	-0.0192
	(0.011)**	(0.009)*	(0.042)	(0.009)*	(0.038)
β	0.0081	0.0041	0.0013	0.0042	-0.0060
	(0.004)*	(0.003)	(0.014)	(0.003)	(0.013)
σ_u		0.0370	0.1147	0.0368	0.1154
		(0.001)**	(0.007)**	(0.001)**	(0.007)**
μ	-0.0293	-0.0141		-0.0136	
	(0.012)*	(0.008)		(0.008)	
ϕ	0.9896	0.9958		0.9961	
	(0.004)**	(0.003)**		(0.003)**	
σ_ν		0.0333	0.1415	0.0329	0.1397
		(0.001)**	(0.009)**	(0.001)**	(0.008)**
π		-0.5419		-0.5433	
		(0.023)**		(0.022)**	
λ		0.9789		0.9771	
		(0.010)**		(0.009)**	
τ		5.1318		5.0000	
		(1.292)**		(1.109)**	
ρ_1				0.2072	
				(0.167)	
ρ_2				-0.7655	
				(0.114)**	
log likelihood		3879.8833		3890.3029	
p-value for LR test		2.98e-05			

Notes. Standard errors are in parenthesis. The test values that are significant at 95% and 99% level are presented respectively with * and **. The LR test was conducted with the null of no endogeneity ($\rho_1 = \rho_2 = 0$).

shocks on the stock market, as mentioned in the previous section. It is deducible that predicting stock excess returns using the dividend-price ratio, as well as T-bill rate and term spread, is extremely difficult in the high volatility regime because stock prices might move in unpredictable ways when the market is in the high volatility regime.

Panel B in Table 2 reports estimation results when the earning-price ratio was used as a predictor for stock excess returns. Unlike the estimation results from the traditional model, the predictability of earning-price ratio was not significantly detected in any of volatility regimes when two-state volatility regime switching model was used. In addition to the earning-price ratio, smoothed real earnings-real price ratio (i.e., Shiller P/E ratio) was tested to deal with a point made by Campbell and Shiller (1988a). They pointed out that yearly earnings might not be a proper measure for the fundamental value since they can be negative during a recession, while the fundamental value can never be negative. Therefore, smoothed real earnings-real price ratio, which compares the moving average of real earnings over the past 10 years and current real price, was also tested as a predictor. This results are reported in Appendix C. However, neither of them had significant predictability for stock returns, regardless of volatility regimes.

It is interesting that neither the earning-price ratio nor the smoothed real earnings-real price ratio (i.e., Shiller P/E ratio) have predictability for stock excess returns, even in the low volatility regime. This might be the result of noisy earnings data. According to Fama and French (1988), the dividend-price ratio predicts returns better than the earning-price ratio since the latter is a noisier measure. They noted that higher variability of earnings makes the earning-price ratio a noisier measure than the dividend-price ratio, if it is unrelated to variations in expected returns.

One of the most interesting point is that the earning-price ratio seems to have valid predictability when OLS is used, while it loses its predictive power when volatility regime switching model is used. It is doubtful whether significantly observed predictability using least squares was real; otherwise, it might have been an illusionary phenomenon caused by specific data characteristics that might distort standard estimation, such as persistence in predictor, correlations between returns and predictors and time-varying volatility in returns. For price ratios, π was significantly different from zero and ϕ was closely estimated to one. Moreover, the estimated values for regime switching parameter $\sigma_u(s_t)$ were highly different in each regime, implying that stock excess returns have switching volatilities. Under these conditions, as many papers have indicated, size distortion might occur, severely damaging the standard estimation using least squares. Therefore, the traditional model using least squares might have overestimated the predictive power of the earning-price ratio and Shiller P/E ratio. By reflecting switching volatilities of stock returns, such problems might have been attenuated in the ERS model.

In Tables 2, ρ_2 was estimated to have a significantly large negative value, while ρ_1 was insignificant. As in Table 1, these results imply that the innovation of stock excess returns, particularly the part that is uncorrelated to the predictor series, affects the latent factor in the next period, influencing the volatility regime of the following period. Moreover, the null of no endogeneity $(\rho_1 = \rho_2 = 0)$ was strongly rejected as reported in the bottom of Table 2. The LR test results indicate that an endogenous feedback effect significantly increases the maximum log likelihood value, allowing the leverage effect to be considered within the model. While the predictability inference using the ERS model seems rarely different from that using the CRS model, the simulation results in Section 5 show that the regime process can be more sharply inferred via the ERS model yielding a gain in test power and bias.

4.2.3 Comparing the Estimated Predictability of Each Predictor

In this part, test results for stock return predictability using the traditional model (OLS) and the ERS model are compared. When test results are reported, β is scaled by the estimated volatility ratio between the predictor and stock excess return(σ_ν/σ_u) as in Campbell and Yogo (2006). Therefore, values of β as reported in Table 3 are $\beta \cdot (\sigma_\nu/\sigma_u)$ in an actual sense. This standardization enables us to compare predictability of different predictors easily. If β is estimated to be significant, we can say that an increase of one standard deviation of predictor(σ_ν) would predict a $\beta \cdot (\sigma_\nu/\sigma_u)$ standard deviation change in expected stock excess returns.

When the T-bill rate was used as a predictor for stock excess returns, we failed to reject the null of no predictability using least squares. However, when the volatility regime was separately modeled in the ERS model, the joint null of no predictability under both regimes $(i.e., \underline{\beta} = \overline{\beta} = 0)$ was rejected significantly. For the implication from the traditional model to be consistent with that from the ERS model, the joint null should not have been rejected. Therefore, test results using two models are not consistent with each other. When predictability in each volatility regime is tested respectively, test results using the ERS model imply that T-bill rate significantly predict stock excess returns, at least in the low volatility regime. The term spread also seems to be an invalid predictor using least squares. However, it turned out that term spread also had significant predictability for stock excess returns, at least in the low volatility regime. Though the joint null was not rejected at the 5% level, the p-value was reduced by a great amount compared to the value from OLS estimation. When the volatility regime is considered, macro variables seem to have valid predictive power, at least in the low volatility regime.

When the dividend-price ratio was used as a predictor for stock excess returns, the null of no predictability was rejected at 5% level using least squares. The joint null of no predictability using

Table 3: Return Predictability Test Results

Predictor	OLS		ERS				
			Jointly tested	Tested under each regime			
				$s_t = 0$		$s_t = 1$	
	β	p-value	p-value for $H_0 : \underline{\beta} = \overline{\beta} = 0$	$\underline{\beta}$	p-value	$\overline{\beta}$	p-value
T-Bill Rate	-0.0065	(0.068)	(0.002)**	-0.0122	(0.002)**	-0.0248	(0.134)
Term Spread	0.0117	(0.160)	(0.059)	0.0217	(0.019)*	0.0053	(0.843)
Dividend -Price Ratio	0.0063	(0.037)*	(0.018)*	0.0051	(0.030)*	0.022	(0.052)
Earning -Price Ratio	0.0089	(0.042)*	(0.412)	0.0038	(0.214)	-0.0073	(0.636)

Notes. β coefficients obtained from the traditional model are scaled by $(\sigma_\nu)/(\sigma_u)$ while those obtained from the ERS model are scaled by $\widehat{\sigma_\nu}(s_t)/\widehat{\sigma_u}(s_t)$ in each volatility regime. The test values that are significant at 95% and 99% level are presented respectively with * and **.

the ERS model $(i.e., H_0 : \underline{\beta} = \overline{\beta} = 0)$ was also rejected. However, the ERS model additionally indicated that predictability was restricted only to the low volatility regime. More interestingly, the earning-price ratio could not reject the joint null of $\underline{\beta} = \overline{\beta} = 0$ at 5% level, though the null of $\beta = 0$ was rejected when using OLS. Even when predictability under each volatility regime was tested respectively, both nulls $(i.e., H_0 : \underline{\beta} = 0$ and $H_0 : \overline{\beta} = 0)$ were failed to be rejected, implying that the earning-price ratio might not be able to predict stock excess return under any volatility regime.

By comparing estimation results from the traditional model and the ERS model, it was shown that ignoring volatility regimes might give significantly different inferences on stock return predictability. When predictability exists only in the low volatility regime, it might be possible that predictability does not appear visibly when volatility regimes are not separately considered, due to the influence of high volatility regimes. The switching predictability with volatility regime might give explanation for the well-known instability in return predictability.

On the other hand, significantly detected predictive power under OLS might become insignificant after volatility regimes are separately considered. This might be the result of over-rejection problem caused by innovation correlation π that is significantly different from zero and highly persistent predictor series. In Section 5, it will be shown that over-rejection problem could be alleviated when volatility regimes are properly considered.

5 Simulation

In this section, Monte Carlo simulation was conducted to examine the effect of stock excess return volatility regimes, the persistence of predictor and innovation correlations between stock returns and predictors on three models: The traditional model (OLS), the CRS model and the ERS model.

The first simulation was conducted to check if regime-switching model is needed for β inference even when predictive power does not exist in both volatility regimes. According to the simulation results in Section 5.2, it is necessary as long as volatility is switching. If volatility regimes are ignored as in OLS, a test would yield more problems in terms of test size and bias, compared to the ERS model.

The second simulation assumed situation where predictability varies with the volatility regime. The traditional model using OLS cannot provide any valid inferences when predictive power exists only in the low volatility regime. If so, a two-state volatility regime switching model is necessary to validly examine predictability under each volatility regime. In addition, the importance of the endogenous feedback effect is emphasized in Section 5.3, indicating that the ERS model is superior to the CRS model in terms of test power and finite sample bias. It seems that the ERS model can infer state process more sharply compared to the CRS model, allowing underlying time series to be reflected upon transition probability which yields power gain and bias improvement.

5.1 Simulation Model

$$y_t = \beta(s_t)x_{t-1} + \sigma_u(s_t)u_t$$

$$x_t = \phi x_{t-1} + \sigma_\nu(s_t)\nu_t$$

$$u_t = \pi\nu_t + \sqrt{1-\pi^2}\varepsilon_t$$

where $\beta(s_t) = \underline{\beta}(1-s_t) + \overline{\beta}s_t$ and $\sigma_u(s_t) = \sigma_\nu(s_t) = 0.03(1-s_t) + 0.10s_t$. The volatility parameters are to have realistic values that was estimated in Section 4. The predictor was assumed to follow a near unit root process.

$$\phi = 1 - \frac{c}{n}$$

In our simulations, the sample size n was 250 and 500. For each n, we considered c=0,2,10.

$$f_{t+1} = \lambda f_t + \eta_{t+1}$$

$$s_t = 1\{f_t \geq \tau\}$$

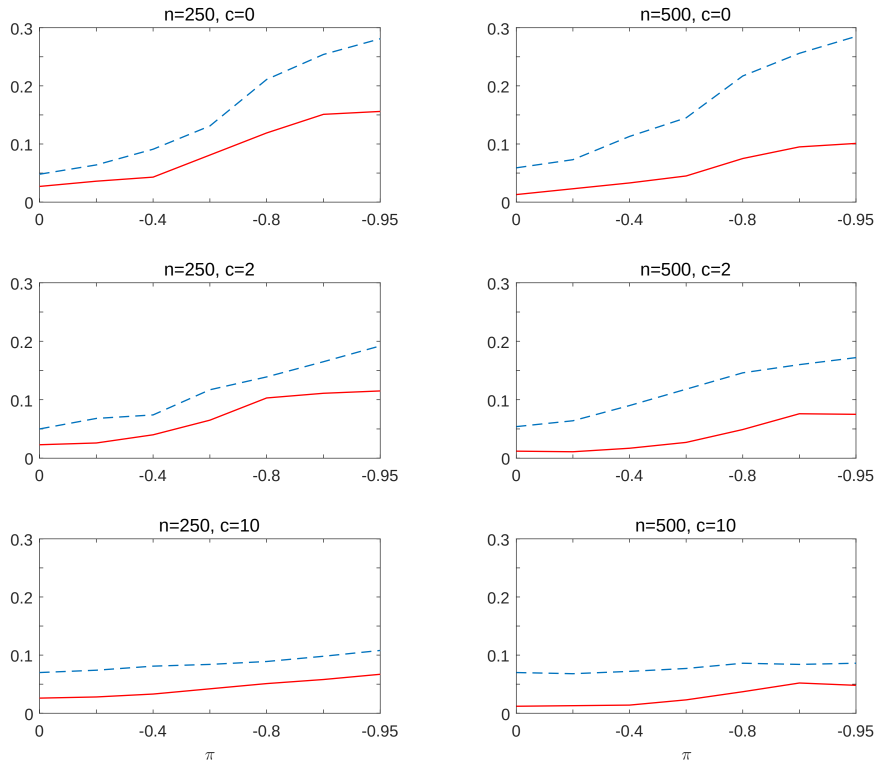
$$\begin{pmatrix} \nu_t \\ \varepsilon_t \\ \eta_{t+1} \end{pmatrix} =_d N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}\right)$$

The autoregressive coefficient of the latent factor $\lambda = 0.9$ and threshold for regime $\tau = 0$. Two endogeneity parameters ρ_1 and ρ_2 were set as $\rho_1 = 0$ and $\rho_2 = -0.9$. All parameters were set similar to estimates from the previous estimation results. As previous literatures have been pointed out, persistence in predictor, correlations between returns and predictors might cause problems in test size and finite sample bias. Therefore, key parameters in this simulation settings are ϕ and π.

5.2 Ignoring Volatility Regimes

In this section, predictability is assumed not to exist under any volatility regime $(i.e., \underline{\beta} = \overline{\beta} = 0)$. If this is so, one might argue that separately considering volatility regimes does nothing

Figure 2: Finite Rejection Rate using OLS and ERS Model (jointly tested)



Notes. Figure 2 presents finite rejection rates for the null of no predictability using the OLS and the ERS model with 5% significance level. The dashed blue line is for the OLS with the null of $H_0 : \beta = 0$ and the solid red line is for the ERS model with the null of $H_0 : \underline{\beta} = \overline{\beta} = 0$.

more than harm the parsimoniousness of the model. However, it was shown by simulation results that the ERS model, two-state volatility regime switching model with endogenous channel, could alleviate many problems in hypothesis testing and estimation, compared to the traditional model. When the predictor is highly persistent and stock return innovation is correlated with that of the predictor, standard OLS might yield size distortion with significant finite sample bias in β. The following part will show that the ERS model can reduce size distortion and finite sample bias by separately modeling volatility regimes.

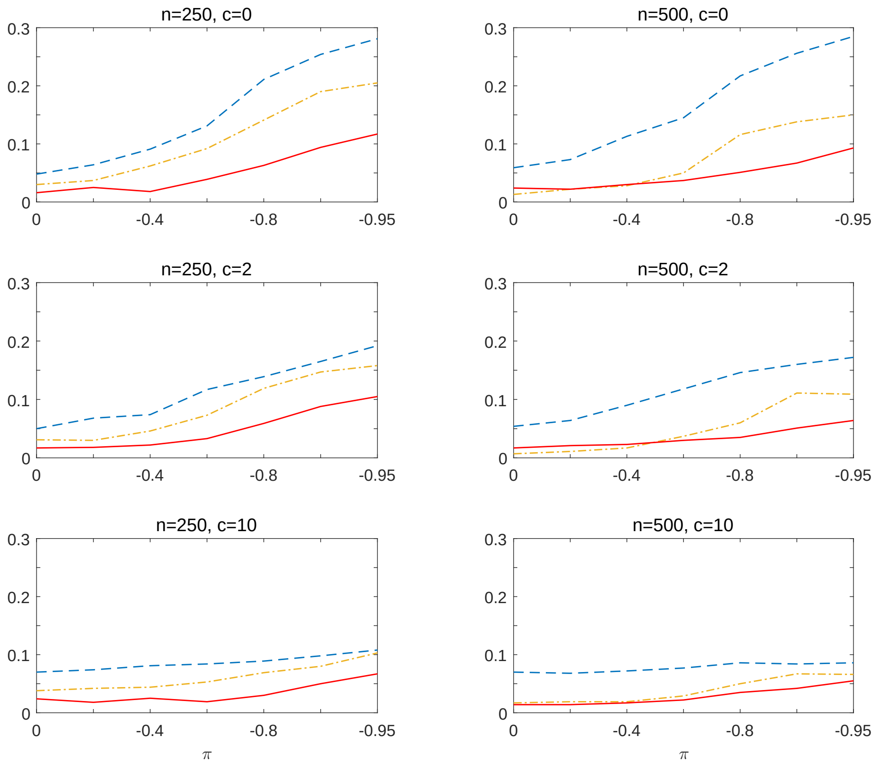
Figure 2 presents the finite rejection rate of the true null that there is no predictability. For the traditional model estimated using least squares, the null $H_0 : \beta = 0$ was tested against the alternative of $H_1 : \beta \neq 0$ with 5% significance level. For the ERS model, which considers volatility regimes with an endogenous feedback effect, the joint null $H_0 : \underline{\beta} = \overline{\beta} = 0$ was tested against the alternative that $H_1 : \underline{\beta} \neq 0$ or $\overline{\beta} \neq 0$ with 5% significance level. In Figure 2, simulation results are presented as figures for each (n, c) with π varying from 0 to -0.95 gradually. The finite rejection could be controlled by no more than 10% for all π values, especially when sample size is as large as 500. When the predictor was not highly persistent $(c = 10)$, least squares estimation did not fail miserably, but still, the ERS model showed a smaller size distortion.

However, the test tended to be undersized when π was around zero. The predictability test using the ERS model seems conservative in a sense that it rejects too little under the true null when innovation correlation between stock returns and the predictor was not strong enough. Still, it is noticeable that test size using the ERS model was quite accurate when π was far from zero, which is important considering that the innovation correlation between stock excess returns and price ratios were estimated to be around -0.6 in Section 4.

In Figure 3, a predictability test using the ERS model was conducted separately for each volatility regime. To test the predictability in the low volatility regime, the null $H_0 : \underline{\beta} = 0$ was tested against the alternative $H_0 : \underline{\beta} \neq 0$. Likewise, the null $H_0 : \overline{\beta} = 0$ was tested against the alternative $H_1 : \overline{\beta} \neq 0$ to test the predictive power in the high volatility regime. As in Figure 2, size distortion from the traditional model was mitigated after volatility regimes were considered separately using the ERS model. The finite rejection rate of $\underline{\beta}$ and $\overline{\beta}$ was close to 5% when π was around -0.6, which is a realistic value for price ratios.

Moreover, once volatility regimes are considered in the model, the bias in β can be reduced by almost half compared to OLS. Stambaugh (1999) argued that the bias in β is proportional to the

Figure 3: Finite Rejection Rate using OLS and ERS Model



Notes. Figure 3 presents the finite rejection rate for null of no predictability using OLS and the ERS model with 5% significance level. The dashed blue line is for OLS with the null of $H_0 : \beta = 0$. The dash-single dotted yellow line is for the ERS model in low volatility regime(i.e., finite rejection rate for the null of $H_0 : \underline{\beta} = 0$) and the solid red line is for the ERS model in high volatility regime(i.e. finite rejection rate for the null of $H_0 : \overline{\beta} = 0$)

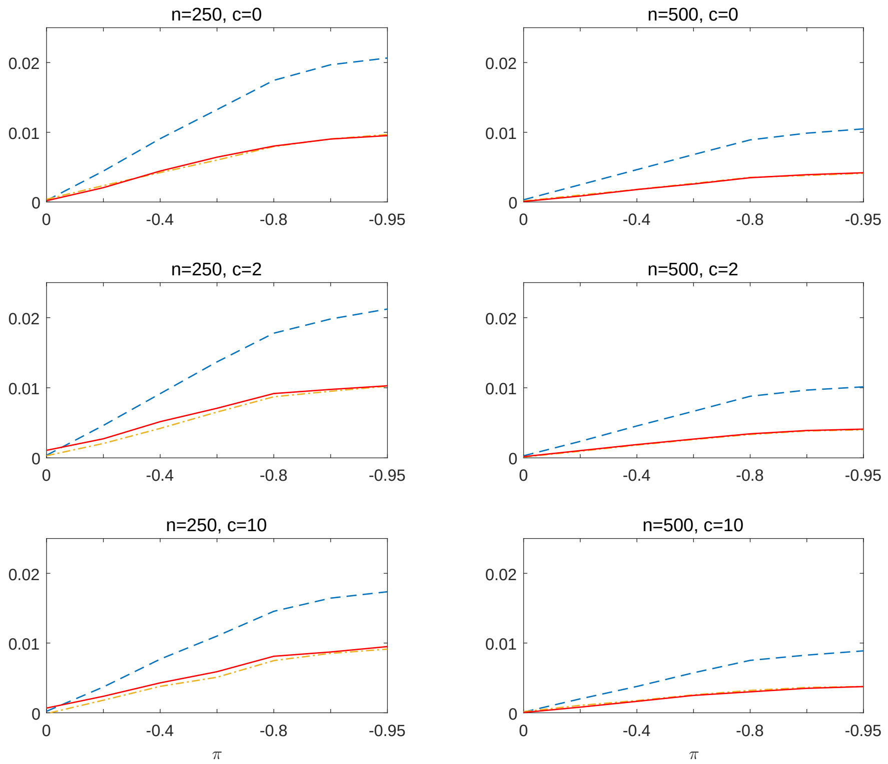
bias in ϕ and π. A slight modification of Stambaugh (1999) yields

$$E[\widehat{\beta}(s_t) - \beta(s_t)] = \left(\frac{\sigma_u(s_t)}{\sigma_\nu(s_t)} \pi \right) E[\widehat{\phi} - \phi] \tag{19}$$

The derivation of (19) is included in Appendix B. When the volatility regimes in stock excess returns and predictors are properly modeled, the bias in the autoregressive coefficient of the predictor (ϕ) can be reduced, making $\beta(s_t)$ less biased.

Figure 4 shows that the finite sample bias in β is negligible when π is zero. However, it

Figure 4: Finite Sample Bias using OLS and ERS Model



Notes. Figure 4 presents finite sample bias in β using OLS and those in $(\underline{\beta}, \overline{\beta})$ using the ERS model. The dashed blue line is bias in β using OLS. The dash-single dotted yellow line is bias in $\underline{\beta}$ while the solid red line is bias in $\overline{\beta}$ using the ERS model.

increases as π increases in an absolute sense. For all n, c and π, the ERS model yields a smaller bias in β for each state $(\underline{\beta}, \overline{\beta})$ compared to the traditional model estimated using least squares. Bias is reduced almost by half under each volatility regime using the ERS model. Table 4 shows that the downward bias in ϕ is smaller in the ERS model for all cases. A smaller bias in ϕ might contribute to a smaller bias in $\beta(s_t)$ in the ERS model.

To sum up Section 5.2, the ERS model can reduce size distortion and finite sample bias compared to the traditional model when stock excess return and predictors have switching volatilities. Even when predictive power does not differ with volatility regime, if the volatility regime switches over time, predictability should be modeled separately for each volatility regimes to relieve problems in hypothesis testing and estimation.

Table 4: Finite Sample Bias in β and ϕ

				Bias in β			Bias in ϕ						Bias in β			Bias in ϕ	
				OLS	ERS		OLS	ERS					OLS	ERS		OLS	ERS
n	c	ϕ	π		$s_t = 0$	$s_t = 1$			n	c	ϕ	π		$s_t = 0$	$s_t = 1$		
250	0	1	0	0.0002	0.0004	0.0002	-0.0215	-0.0100	500	0	1	0	0.0003	0.0001	0.0001	-0.0110	-0.0045
			-0.2	0.0044	0.0024	0.0021	-0.0220	-0.0100				-0.2	0.0025	0.0010	0.0009	-0.0111	-0.0044
			-0.4	0.0091	0.0042	0.0044	-0.0218	-0.0100				-0.4	0.0046	0.0018	0.0018	-0.0108	-0.0043
			-0.6	0.0133	0.0060	0.0064	-0.0217	-0.0099				-0.6	0.0068	0.0027	0.0026	-0.0109	-0.0043
			-0.8	0.0175	0.0080	0.0080	-0.0217	-0.0099				-0.8	0.0089	0.0035	0.0035	-0.0109	-0.0044
			-0.9	0.0197	0.0090	0.0090	-0.0218	-0.0101				-0.9	0.0099	0.0038	0.0039	-0.0108	-0.0042
			-0.95	0.0207	0.0097	0.0095	-0.0218	-0.0101				-0.95	0.0105	0.0041	0.0042	-0.0109	-0.0043
	2	0.992	0	0.0004	0.0003	0.0011	-0.0220	-0.0105		2	0.996	0	0.0003	0.0001	0.0002	-0.0108	-0.0044
			-0.2	0.0046	0.0021	0.0027	-0.0219	-0.0104				-0.2	0.0024	0.0010	0.0010	-0.0108	-0.0044
			-0.4	0.0092	0.0042	0.0052	-0.0221	-0.0106				-0.4	0.0046	0.0018	0.0019	-0.0107	-0.0044
			-0.6	0.0137	0.0065	0.0071	-0.0224	-0.0107				-0.6	0.0066	0.0026	0.0027	-0.0107	-0.0043
			-0.8	0.0178	0.0087	0.0092	-0.0220	-0.0107				-0.8	0.0088	0.0033	0.0034	-0.0107	-0.0042
			-0.9	0.0198	0.0095	0.0098	-0.0220	-0.0105				-0.9	0.0097	0.0038	0.0039	-0.0105	-0.0042
			-0.95	0.0212	0.0102	0.0103	-0.0225	-0.0108				-0.95	0.0101	0.0040	0.0041	-0.0105	-0.0042
	10	0.96	0	0.0002	-0.0001	0.0007	-0.0188	-0.0094		10	0.98	0	0.0001	0.0002	0.0000	-0.0094	-0.0040
			-0.2	0.0037	0.0018	0.0024	-0.0192	-0.0096				-0.2	0.0020	0.0011	0.0008	-0.0095	-0.0041
			-0.4	0.0077	0.0038	0.0043	-0.0190	-0.0097				-0.4	0.0038	0.0017	0.0016	-0.0092	-0.0040
			-0.6	0.0110	0.0051	0.0059	-0.0183	-0.0090				-0.6	0.0057	0.0026	0.0025	-0.0092	-0.0040
			-0.8	0.0146	0.0075	0.0081	-0.0183	-0.0096				-0.8	0.0075	0.0032	0.0030	-0.0092	-0.0038
			-0.9	0.0164	0.0085	0.0087	-0.0184	-0.0094				-0.9	0.0083	0.0036	0.0035	-0.0091	-0.0039
			-0.95	0.0174	0.0091	0.0095	-0.0182	-0.0096				-0.95	0.0089	0.0038	0.0038	-0.0092	-0.0039

Notes. For each n, c and π, this table compares finite sample bias in β and ϕ from the traditional model and the ERS model.

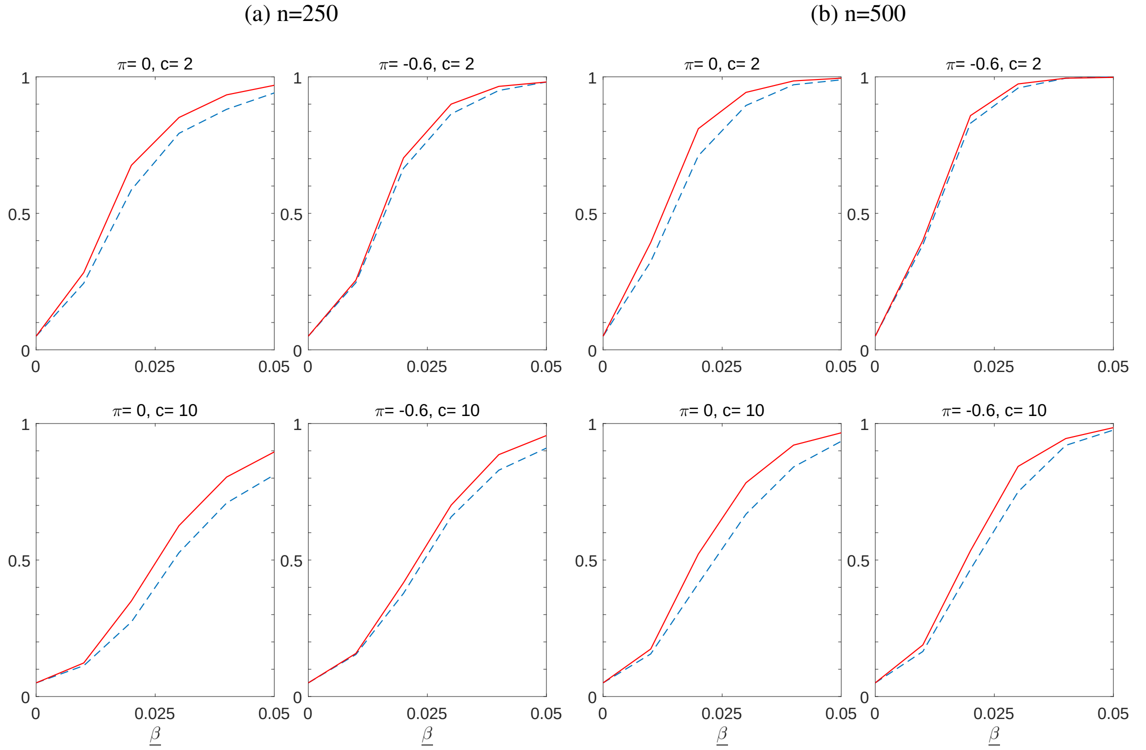
5.3 Ignoring the Endogenous Feedback Effect

In Section 5.3, predictability was assumed to exist only in the low volatility regime $(i.e., \overline{\beta} = 0)$. If so, the traditional model, which ignores volatility regimes, cannot produce any valid inferences. Among two-state regime switching models, the ERS model which considers the endogenous feedback effect of time series on the next period volatility regime, performs better than the CRS model in terms of test power and bias.

In this section, the predictability parameter $\beta(s_t)$ is $\underline{\beta}(1 - s_t) + 0 \cdot s_t$ with $\underline{\beta}$ ranging from zero to a non-zero value in fixed increments; from 0 to 0.05 with an increment of 0.1 when n=250 and from 0 to 0.025 with an increment of 0.005 when n=500. Since we focused on varying $\underline{\beta}$, we only considered $\pi = \{0, -0.6\}$ in Section 5.3. Such π values are realistic values, according to the estimation results in Section 4. Simulation results in this part will show that there is a gain in simulated power in the low volatility regime when the endogenous feedback effect of time series is considered.

Figure 5 compares simulated powers of the CRS model and the ERS model in the low volatility regime. The test powers were all adjusted to have exact 5% size using the simulated critical values. Figure 5-(a) plots power functions when n=250 while 5-(b) plots those when n=500. For each panel, figures in the left column are for $\pi = 0$ while those in the right column are for $\pi = -0.6$. Compared to $\pi = -0.6$, power gain was generally bigger when $\pi = 0$. This is reasonable considering the correlation between u_t and η_{t+1} is $\pi\rho_1 + \sqrt{1 - \pi^2}\rho_2$ and between ν_t and η_{t+1} is ρ_1. In this simulation, ρ_1 was set to zero since it was estimated to have an insignificant value in the estimation section. This implies that only the part of stock excess return innovation, that is uncorrelated to the predictor, affects the next period volatility regime. With $\rho_1 = 0$, innovation of underlying time series y_t affect volatility regime of the next period due to its correlation with

Figure 5: Simulated Power Functions



Notes. Figure 5 presents simulated powers of the CRS model and the ERS model in low volatility regime. The dashed blue line is for the CRS model and the solid red line is for the ERS model. The test powers were all adjusted to have exact 5% size using the simulated critical values. Figure 5-(a) shows simulated power for n=250 and Figure 5-(b) shows simulated power for n=500.

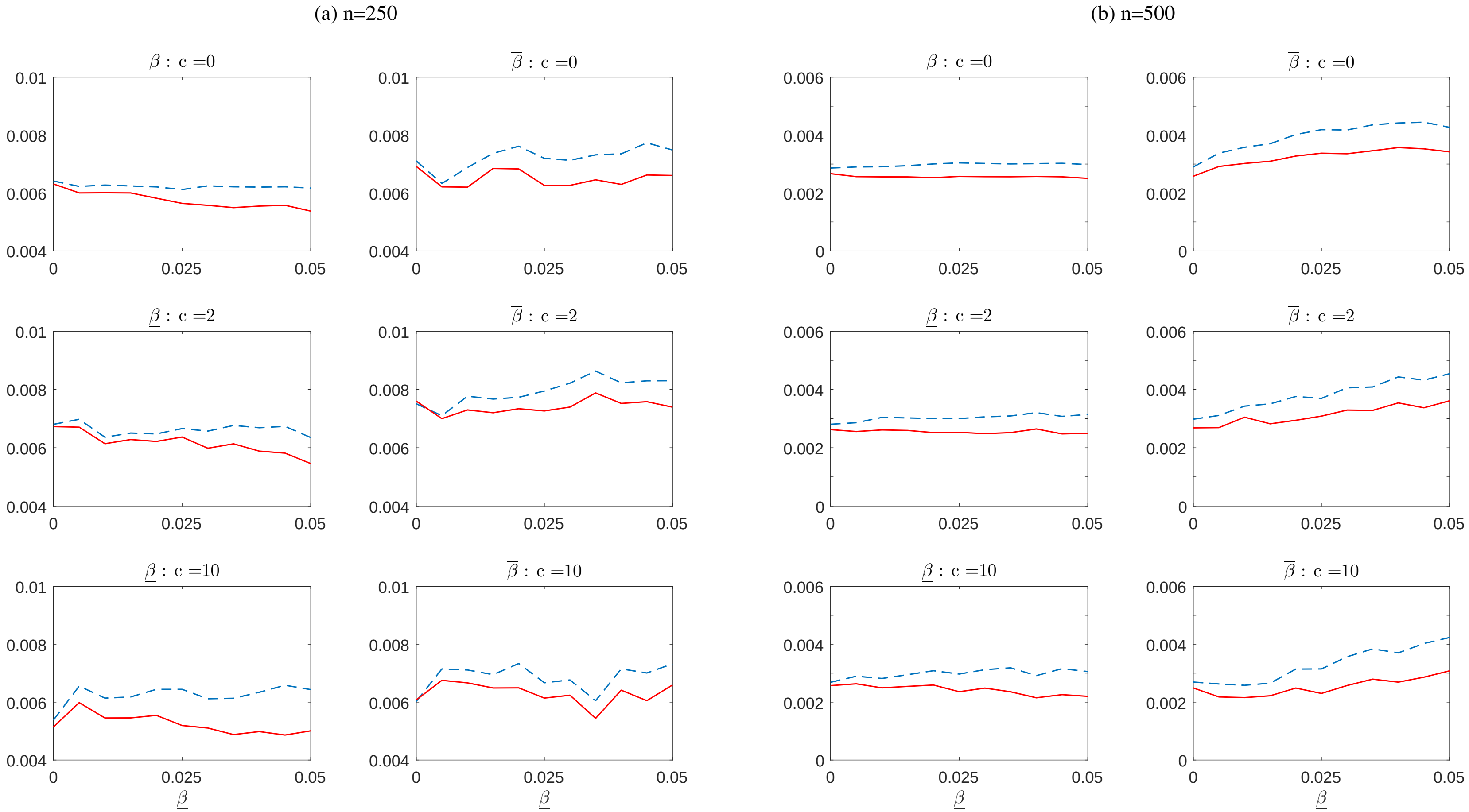
η_{t+1} by $\sqrt{1-\pi^2}\rho_2$. Therefore, compared to $\pi = -0.6$, $\pi = 0$ resulted in a stronger endogeneity effect on the state process. This might be the reason why power gain obtained from allowing an endogeneity effect is bigger when $\pi = 0$. All plots in Figure 5 indicate that the ERS model has power gain compared to the CRS model, for all cases of n, π and c.

In addition to power gain, the ERS model can also reduce finite sample bias in $\beta(s_t)$ compared to the conventional model. As already mentioned in (19), there exists bias in the predictability parameter when the innovation of stock excess return and predictor have contemporaneous correlation. Figure 4 showed that the bias in the predictability parameter $(\underline{\beta}, \overline{\beta})$ can be reduced almost by half when volatility regimes are separately modeled as in the ERS model. In this section, with fixed π at -0.6, which is a realistic value for price ratios, bias improvement of the ERS model will be compared to that of the CRS model. Another realistic case, $\pi = 0$, was not considered because there is no bias problem in β when innovation correlation is zero.

Figure 6 plots the finite sample bias in $(\underline{\beta}, \overline{\beta})$ from the ERS and CRS model, for c=0,2,10 and n=250,500. The finite sample bias in $(\underline{\beta}, \overline{\beta})$ is smaller with the ERS model for all c, n and $\underline{\beta}$. Additionally, an interesting pattern is observable in Figure 6. Since predictability was assumed not to exist in the high volatility regime $(i.e., \overline{\beta} = 0)$, volatility regime-specific difference in predictability grows as $\underline{\beta}$ increases. As $\underline{\beta}$ increases, the gap between $\underline{\beta}$ and $\overline{\beta}$ increases and the bias improvement of the ERS model gets stronger. It has been consistently observed across all plots in Figure 6 that bias improvement becomes more significant as $\underline{\beta}$ deviates from $\overline{\beta} = 0$, while the bias reduction of the ERS model is not noticeable when $\underline{\beta} = \overline{\beta} = 0$. It is interesting to note that the bias improvement obtainable by allowing endogeneity in the state transition process becomes more significant as regime-specific difference in switching parameters gets larger. It seems clear that the benefit from underlying time series information reflected on the state process becomes larger as the difference in parameter values under each regime becomes more dramatic. Therefore, especially for the case when predictability varies with volatility regimes, the ERS model is superior to the CRS model, reducing finite sample bias.

The results in Section 5.3, power gain and bias reduction, would have ensued from the fact that the ERS model enables sharper inference on the state process, especially at the transition period. The ERS model is an extended version of the CRS model, relieving the assumption of the conventional Markov switching model that state is determined independently from the underlying time series. By allowing current innovations of the underlying time series to be correlated with the

Figure 6: Finite Sample Bias using CRS Model and ERS Model



Notes. Figure 6 plots the finite sample bias in $(\underline{\beta}, \overline{\beta})$ using the ERS model (solid red line) and the CRS model (dashed blue line). For n=250, 500 and c=0,2,10, finite sample bias was plotted as predictability in low volatility regime$(\underline{\beta})$ grows. For panel (a) and (b), bias in $\underline{\beta}$ is reported in the left column and bias in $\overline{\beta}$ is reported in the right column. The first row shows the case for c=0, the second row for c=2 and the last row for c=10.

next period latent factor, as in (6), transition probability becomes a function not only of the previous state but also of previous innovations of the underlying time series as in (11). Such time varying transition probability is the result of additional information from the underlying time series being reflected, allowing shaper inferences on the state process. This might have played a key role in improving test power and finite sample bias.

To verify whether the ERS model actually enables better inference on the state process thanks to additional information from the underlying time series, this part compares the state inferences using the ERS and CRS model. Among many available cases, we only considered the case in which n=500 and $\beta(s_t) = 0.05(1 - s_t) + 0 \cdot s_t$. Therefore, six cases were considered for $c = 0, 2, 10$ and $\pi = 0, -0.6$.

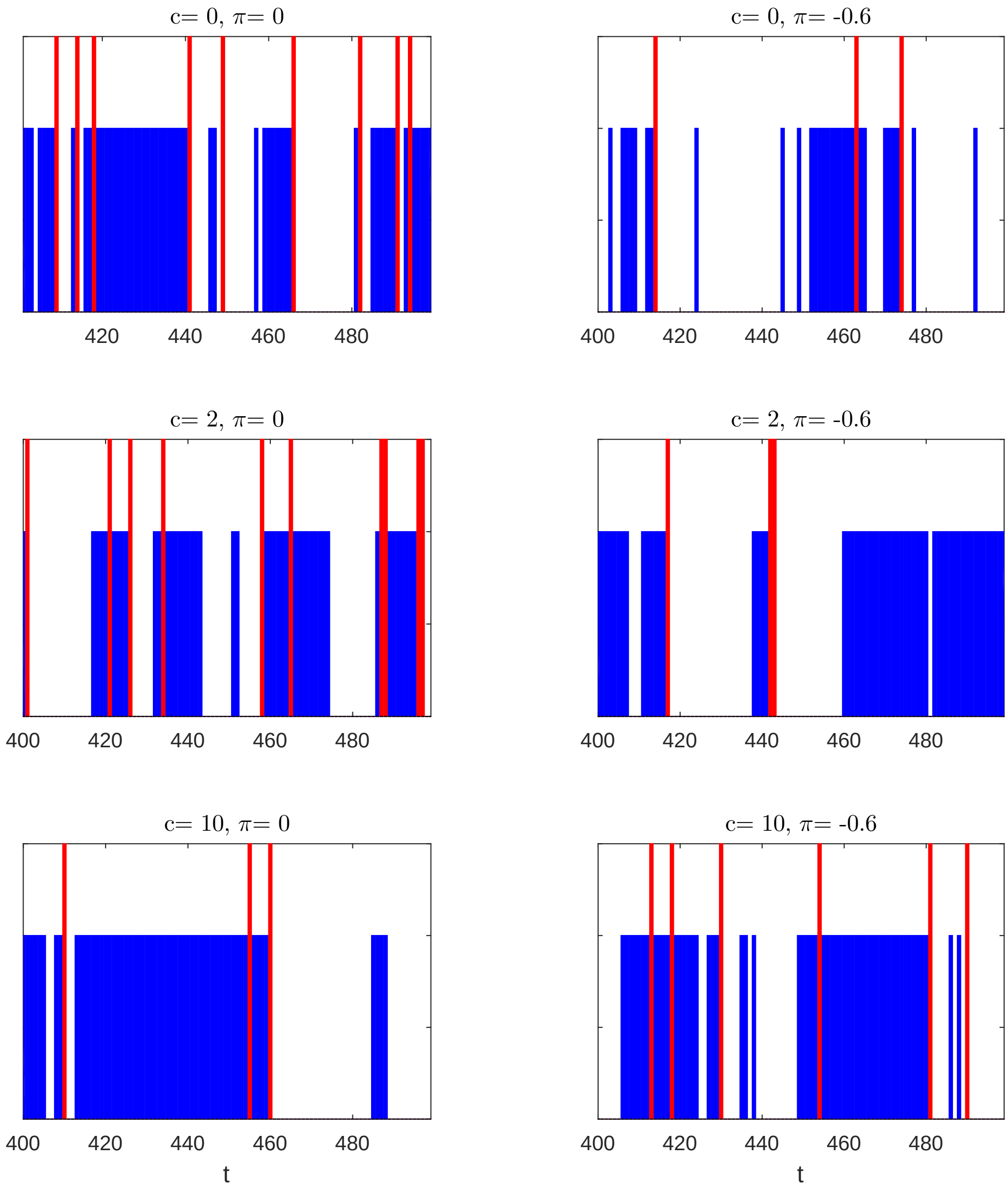
Table 5: The Ratio of Correctly Inferred States

	π=0		$\pi = -0.6$	
	ERS	CRS	ERS	CRS
c=0	0.9306	0.8945	0.9334	0.8988
c=2	0.9275	0.8887	0.9286	0.8916
c=10	0.9262	0.8859	0.9265	0.8867

Notes. The table reports ratios for correctly inferred states using the ERS and CRS model. For each case, the average was taken over 1000 iterations to calculate ratios of correctly inferred states.

Since the data generating process is fully known, we can determine which volatility regime each period belongs to. Using this dataset, the ERS and CRS model could be used to estimate τ and extract latent factor. With $\widehat{\tau}$ and extracted latent factor, an inference could be made on whether each period belongs to the high volatility regime. Table 5 compares the ratio of correctly inferred states based on the inference from the ERS and CRS model. For all cases, the ERS model tends to make correct inferences on the state process more frequently compared to the conventional model. The average ratio using the ERS model was 0.9288 while that using the CRS model was 0.8910.

Figure 7: Inference on State Process



Notes. The shaded areas in Figure 7 are separated into two parts: the one with short blue bars denotes high volatility regimes and the other with long red bars denotes the period when the ERS model correctly inferred which volatility regime it belongs to while the CRS model failed to do so. To present enlarged plots with improved clarity, the results only for the last 100 periods are presented in Figure 7. Results for the whole sample period(n=500) are in Appendix D.

Such pattern is significant especially in the transition period. Among 1000 iterations for each case, the results from one iteration were picked randomly.[5] Figure 7 marked high volatility regimes with short blue bars. The periods when the ERS model correctly inferred which volatility regime it belongs to while the CRS model failed to do so were marked with long red bars in Figure 7. To present enlarged plots with improved clarity, the results only for the last 100 periods were presented in Figure 7. Results for the whole period (n=500) are in Appendix D which show no different patterns from the enlarged version in Figure 7. It is significant that the ERS model outperforms the conventional model especially when the volatility regime changes. This implies that the ERS model can distinguish regimes more sharply during transition periods, using additional information obtainable from the previous period underlying time series.

[5] Among 1000 iterations, one sample was randomly drawn using randomly generated number from uniform distribution. Given seed is 20170816. Other seed were tried too but all gave consistent implication that the ERS model performs better than the CRS model especially in transition period.

6 Conclusion

The stock return might behave differently when the stock market is relatively stable and highly volatile, making return predictability vary with its volatility regimes. This paper introduces a two-state endogenous regime switching model (ERS), which allows one to separately test stock return predictability under low and high volatility regimes. According to empirical analysis using the ERS model, it was clearly shown that none of the tested predictors can significantly predict stock excess returns under the high volatility regime. Only in the low volatility regime did the dividend-price ratio and macro variables such as T-bill rate and term spread show significant predictive power for stock excess returns. The earning-price ratio turned out to be insignificant as predictor even when the return is in the low volatility regime.

The ERS model was more realistic in the context of the return predictability test, compared to the conventional Markov switching (CRS i.e., conventional regime switching) model. It was expected that the stock return innovation would affect the next period volatility regime. A negatively estimated endogeneity parameter ρ_2 was consistent with the leverage effect, indicating that a negative shock on current returns tends to increase the next period volatility. On the other hand, ρ_1 was insignificant no matter what predictor for stock return prediction was used. These results implied the existence of the endogenous feedback effect of the underlying time series on regime process; but it is not past value of predictor series what affect the next period volatility regime. It is past value of return innovation especially the part that is uncorrelated with predictor series.

The ERS model could also alleviate the problems in the estimation and hypothesis testing. Compared to the traditional model estimated using least squares, the ERS model could relieve finite sample bias and the over-rejection problem. The endogenous feedback effect channel modeling, as proposed by Chang et al. (2017), enabled a gain in test power and bias improvement compared to the conventional model. Through endogenous feedback effect channel, the additional information from the underlying time series could be reflected in the latent factor and state process, resulting in sharper inference of the state process particularly during the transition periods. This might have contributed to benefits in hypothesis testing and estimation.

As can be seen from Section 4, the dividend-price ratio and macro variables have valid predictability only in the low volatility regime; and no predictability significantly observed in the high volatility regime. Such switching predictability might be the plausible explanation for widely-observed instability of return prediction in the related literatures. Though return predictability does exist when the stock market is less volatile as the ERS model demonstrated, there remains a question whether it can provide a practical help for real investors, due to the limitation that the state prediction can hardly be perfect. Although the ERS model achieved the improved state inference compared to the conventional model, it has around 93 percent accuracy (not a hundred percent). The future research might examine how useful it is to implement the return predictability limited to the low volatility regime, using the prediction on future volatility regime on which return predictability depends on.

To sum up, the contribution of the ERS model is to suggest a method to respectively test return predictability for different volatility regimes with improved state inference. An endogenous feedback effect channel allowed the additional information from the underlying time series to be reflected in the transition probability, resulting in much sharper state inference and, thereby, better estimation results. Nevertheless, it is the limitation of the ERS model that the problems caused by prediction persistence and innovation correlation have only been attenuated, not solved. It would be of great use if the data characteristics mentioned above could be comprehensively managed within a volatility regime switching model, resulting in a complete solution for over-rejection and the bias problem without the model losing its ability to examine return predictability separately for different volatility regimes.

Appendix

Appendix A.1. Time-varying transition probability derivation

Note that

1. $f_t = \eta_t + \lambda\eta_{t-1} + \lambda^2\eta_{t-2} + \cdots =_d N\left(0, \frac{1}{1-\lambda^2}\right)$ for $|\lambda| < 1$

   $\sqrt{1-\lambda^2} f_t =_d N(0,1)$

2. $$\eta_t|\mathcal{F}_{t-1} =_d N\left(\left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1} + \frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1}, 1-(\rho_1^2+\rho_2^2)\right)$$

   since

   $$\begin{pmatrix} \eta_t \\ \nu_{t-1} \\ u_{t-1} \end{pmatrix} =_d N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 & \pi\rho_1 + \sqrt{1-\pi^2}\rho_2 \\ \rho_1 & 1 & \pi \\ \pi\rho_1 + \sqrt{1-\pi^2}\rho_2 & \pi & 1 \end{pmatrix}\right)$$

   Define z_t as follows.

   $$\begin{aligned} z_t &\equiv \frac{\eta_t - \left[\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}\right]}{\sqrt{1-(\rho_1^2+\rho_2^2)}} \\ &= \frac{f_t - \lambda f_{t-1} - \left[\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}\right]}{\sqrt{1-(\rho_1^2+\rho_2^2)}} \end{aligned}$$

   If so,

   $$z_t|\mathcal{F}_{t-1} = z_t|f_{t-1}, y_{t-1}, x_{t-1} =_d N(0,1)$$

For $|\lambda| < 1$ and $\rho_1^2 + \rho_2^2 < 1$, transition probability is as follows.

$$p(s_t = 0 | s_{t-1} = 0, \mathcal{F}_{t-1}) = p(f_t < \tau \mid f_{t-1} < \tau, \mathcal{F}_{t-1})$$

$$= p(f_t < \tau \mid \sqrt{1-\lambda^2} f_{t-1} < \sqrt{1-\lambda^2}\tau, \mathcal{F}_{t-1})$$

$$= p(\eta_t < \tau - \lambda f_{t-1} | \sqrt{1-\lambda^2} f_{t-1} < \sqrt{1-\lambda^2}\tau, \mathcal{F}_{t-1})$$

$$= p\left(z_t < \frac{\tau - E(\eta_t|\mathcal{F}_{t-1})}{\sqrt{Var(\eta_t|\mathcal{F}_{t-1})}} - \frac{\lambda f_{t-1}}{\sqrt{Var(\eta_t|\mathcal{F}_{t-1})}} | \sqrt{1-\lambda^2} f_{t-1} < \sqrt{1-\lambda^2}\tau, \mathcal{F}_{t-1}\right)$$

$$= \frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\lambda^2}} \int_{-\infty}^{\frac{\tau - \left[\frac{\rho_2}{\sqrt{1-\pi^2}} u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}\right]}{\sqrt{1-(\rho_1^2+\rho_2^2)}}} p(x,y)dydx}{\Phi(\tau\sqrt{1-\lambda^2})}$$

where

$$p(x,y) =_d N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -c_0 \\ -c_0 & 1+c_0^2 \end{pmatrix}\right)$$

and

$$c_0 = \frac{-\lambda}{\sqrt{1-\lambda^2}\sqrt{1-(\rho_1^2+\rho_2^2)}}$$

Likewise,

$$p(s_t = 0|s_{t-1} = 1, \mathcal{F}_{t-1}) = \frac{\displaystyle\int_{-\infty}^{-\tau\sqrt{1-\lambda^2}} \int_{-\infty}^{\frac{\tau - \left[\frac{\rho_2}{\sqrt{1-\pi^2}} u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}\right]}{\sqrt{1-(\rho_1^2+\rho_2^2)}}} p(x,y)dydx}{1 - \Phi(\tau\sqrt{1-\lambda^2})}$$

where

$$p(x,y) =_d N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -c_1 \\ -c_1 & 1+c_1^2 \end{pmatrix}\right)$$

and

$$c_1 = \frac{\lambda}{\sqrt{1-\lambda^2}\sqrt{1-(\rho_1^2+\rho_2^2)}}$$

Appendix A.2. Conditional density of latent factor

Proof is as follows.

$$\begin{aligned}
p(f_t|s_{t-1} = 1, \mathcal{F}_{t-1}) &= p(f_t|f_{t-1} > \tau, u_{t-1}, \nu_{t-1}) \\
&= \frac{\int_\tau^\infty p(f_t, f_{t-1}, u_{t-1}, \nu_{t-1})df_{t-1}}{\int_\tau^\infty p(f_{t-1}, u_{t-1}, \nu_{t-1})df_{t-1}} \\
&= \frac{\int_\tau^\infty p(f_{t-1}|f_t, u_{t-1}, \nu_{t-1})p(f_t|u_{t-1}, \nu_{t-1})p(u_{t-1}, \nu_{t-1})df_{t-1}}{\int_\tau^\infty p(f_{t-1})p(u_{t-1}, \nu_{t-1})df_{t-1}} \\
&= \frac{\int_\tau^\infty p(f_{t-1}|f_t, u_{t-1}, \nu_{t-1})p(f_t|u_{t-1}, \nu_{t-1})df_{t-1}}{\int_\tau^\infty p(f_{t-1})df_{t-1}}
\end{aligned}$$

The last equality holds since f_{t-1} is independent with u_{t-1} and ν_{t-1}. Note followings hold when $|\lambda| < 1$ and $\rho_1^2 + \rho_2^2 < 1$.

1. $f_t = \eta_t + \lambda\eta_{t-1} + \lambda^2\eta_{t-2} + \cdots =_d N\left(0, \frac{1}{1-\lambda^2}\right)$ for $|\lambda| < 1$

   $\sqrt{1-\lambda^2} f_t =_d N(0,1)$

2. $f_{t-1}|f_t, u_{t-1}, \nu_{t-1}$

$$=_d N\left(\frac{\lambda}{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}\left(f_t - \frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} - \left(\rho_1 - \frac{\pi\rho_2}{\sqrt{1-\pi^2}}\right)\nu_{t-1}\right), \frac{1-(\rho_1^2+\rho_2^2)}{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}\right)$$

$$\text{since} \begin{pmatrix} f_{t-1} \\ f_t \\ u_{t-1} \\ \nu_{t-1} \end{pmatrix} =_d N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{1-\lambda^2} & \frac{\lambda}{1-\lambda^2} & 0 & 0 \\ \frac{\lambda}{1-\lambda^2} & \frac{1}{1-\lambda^2} & \pi\rho_1 + \sqrt{1-\pi^2}\rho_2 & \rho_1 \\ 0 & \pi\rho_1 + \sqrt{1-\pi^2}\rho_2 & 1 & \pi \\ 0 & \rho_1 & \pi & 1 \end{pmatrix}\right)$$

3. $f_t|u_{t-1}, \nu_{t-1} \ =_d N\left(\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}, \frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-\lambda^2}\right)$

$$\text{since} \begin{pmatrix} f_t \\ u_{t-1} \\ \nu_{t-1} \end{pmatrix} =_d N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{1-\lambda^2} & \pi\rho_1 + \sqrt{1-\pi^2}\rho_2 & \rho_1 \\ \pi\rho_1 + \sqrt{1-\pi^2}\rho_2 & 1 & \pi \\ \rho_1 & \pi & 1 \end{pmatrix}\right)$$

If so, it is easily deducible that

$$\begin{aligned} p(f_t|s_{t-1}=1, \mathcal{F}_{t-1}) &= \frac{\int_\tau^\infty p(f_{t-1}|f_t, u_{t-1}, \nu_{t-1})p(f_t|u_{t-1}, \nu_{t-1})df_{t-1}}{\int_\tau^\infty p(f_{t-1})df_{t-1}} \\ &= \frac{1-\Phi(X)}{1-\Phi(\tau\sqrt{1-\lambda^2})} \\ &\times \mathcal{N}\left(\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}, \frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-\lambda^2}\right) \end{aligned}$$

Likewise,

$$
\begin{aligned}
p(f_t|s_{t-1}=0,\mathcal{F}_{t-1}) &= \frac{\int_{-\infty}^{\tau} p(f_{t-1}|f_t,u_{t-1},\nu_{t-1})p(f_t|u_{t-1},\nu_{t-1})df_{t-1}}{\int_{-\infty}^{\tau} p(f_{t-1})df_{t-1}} \\
&= \frac{\Phi(X)}{\Phi(\tau\sqrt{1-\lambda^2})} \\
&\times \mathcal{N}\left(\frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} + \left(\rho_1 - \frac{\pi}{\sqrt{1-\pi^2}}\rho_2\right)\nu_{t-1}, \frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-\lambda^2}\right)
\end{aligned}
$$

where

$$
X = \sqrt{\frac{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}{1-(\rho_1^2+\rho_2^2)}}\left(\tau - \frac{\lambda\left(f_t - \frac{\rho_2}{\sqrt{1-\pi^2}}u_{t-1} - \left(\rho_1 - \frac{\pi\rho_2}{\sqrt{1-\pi^2}}\right)\nu_{t-1}\right)}{1-(1-\lambda^2)(\rho_1^2+\rho_2^2)}\right)
$$

Appendix B. Bias in $\beta(s_t)$: A Simple modification of Stambaugh bias

$$
y_t = \alpha(s_t) + \beta(s_t)x_{t-1} + \sigma_u(s_t)u_t
$$

$$
x_t = \mu + \phi x_{t-1} + \sigma_\nu(s_t)\nu_t
$$

$$
u_t = \pi\nu_t + \sqrt{1-\pi^2}\varepsilon_t
$$

where

$$
\begin{pmatrix}\nu_t \\ \varepsilon_t\end{pmatrix} =_d N\left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\right)
$$

and thereby

$$
\begin{pmatrix}u_t \\ \nu_t\end{pmatrix} =_d N\left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}1 & \pi \\ \pi & 1\end{pmatrix}\right)
$$

Let $b_1(s_t) = [\alpha(s_t) \quad \beta(s_t)]'$ and $b_2 = [\mu \quad \phi]'$. Construct $X = [ones(1,n) \quad x]$ for $x = [x_0 \cdots x_{n-1}]'$.

$$\widehat{b_1}(s_t) - b_1(s_t) = \sigma_u(s_t)(X'X)^{-1}X'u$$

$$\widehat{b_2} - b_2 = \sigma_\nu(s_t)(X'X)^{-1}X'\nu$$

Therefore,

$$\begin{aligned}
\widehat{b_1}(s_t) - b_1(s_t) &= \sigma_u(s_t)(X'X)^{-1}X'u \\
&= \sigma_u(s_t)(X'X)^{-1}X'(\pi\nu + \sqrt{1-\pi^2}\varepsilon) \\
&= \frac{\sigma_u(s_t)}{\sigma_\nu(s_t)}\pi[\sigma_\nu(s_t)(X'X)^{-1}X'\nu] + \sigma_u(s_t)\sqrt{1-\pi^2}(X'X)^{-1}X'\varepsilon \\
&= \frac{\sigma_u(s_t)}{\sigma_\nu(s_t)}\pi[\widehat{b_2} - b_2] + \sigma_u(s_t)\sqrt{1-\pi^2}(X'X)^{-1}X'\varepsilon
\end{aligned}$$

Taking conditional expectation on ν and unconditional expectation for both sides gives

$$E[\widehat{b_1}(s_t) - b_1(s_t)] = \left(\frac{\sigma_u(s_t)}{\sigma_\nu(s_t)}\pi\right) E[\widehat{b_2} - b_2]$$

Therefore,

$$E[\widehat{\beta}(s_t) - \beta(s_t)] = \left(\frac{\sigma_u(s_t)}{\sigma_\nu(s_t)}\pi\right) E[\widehat{\phi} - \phi]$$

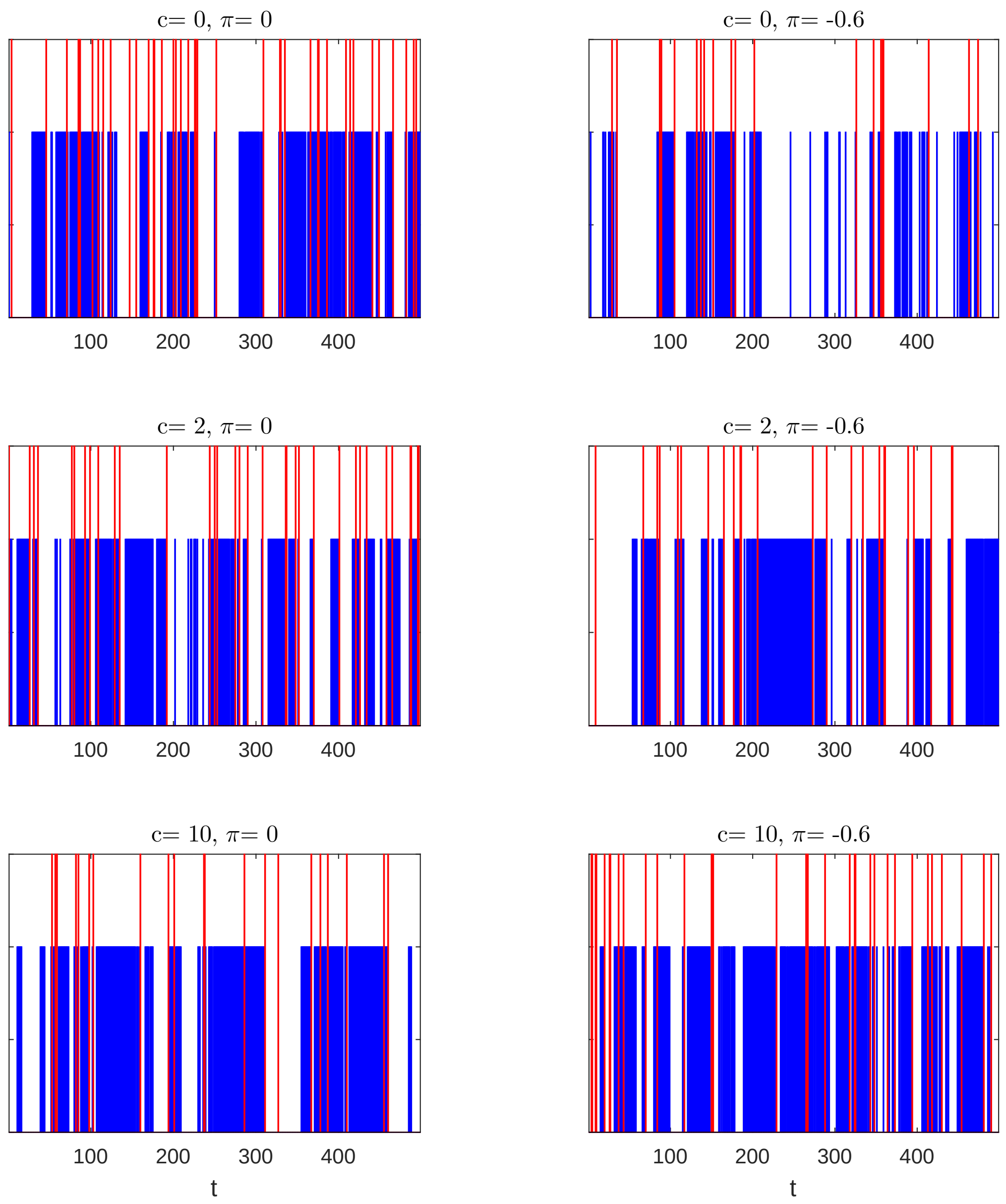
Appendix C. Shiller P/E Ratio

Estimation Results with Smoothed Real earnings-Real price Ratio

	OLS	CRS		ERS	
		$s_t = 0$	$s_t = 1$	$s_t = 0$	$s_t = 1$
α	0.0400	0.0238	0.0803	0.0241	0.0750
	(0.012)**	(0.009)*	(0.042)	(0.009)**	(0.041)
β	0.0120	0.0051	0.0258	0.0055	0.0227
	(0.004)**	(0.003)	(0.015)	(0.003)	(0.015)
σ_u		0.0363	0.1083	0.0364	0.1144
		(0.001)**	(0.006)**	(0.001)**	(0.007)**
μ	-0.0180	-0.0103		-0.0122	
	(0.010)	(0.008)		(0.008)	
ϕ	0.9939	0.9979		0.9974	
	(0.003)**	(0.003)**		(0.003)**	
σ_ν		0.0302	0.0908	0.0300	0.0929
		(0.001)**	(0.005)**	(0.001)**	(0.005)**
π		-0.6625		-0.6642	
		(0.018)**		(0.018)**	
λ		0.9738		0.9489	
		(0.011)**		(0.019)**	
τ		4.4432		3.4269	
		(1.006)**		(0.673)**	
ρ_1				0.1165	
				(0.124)	
ρ_2				-0.8907	
				(0.102)**	
log likelihood		4168.7333		4185.7345	
p-value for LR test		4.13e-08			

Notes. Standard errors are in parenthesis. The test values that are significant at 95% and 99% level are presented respectively with * and **.

Appendix D. Full version of Figure 7



Notes. The shaded areas in this figure are separated into two parts: the one with short blue bars denotes high volatility regimes and the other with long red bars denotes the period when the ERS model correctly inferred which volatility regime it belongs to while the CRS model failed to do so.

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