Endogenous Regime Switching Volatility Model with Time-Varying Threshold

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Abstract

This paper extends endogenous regime switching volatility models by incorporating a predetermined and observable factor, which determines the regimes with an endogenous autoregressive latent factor. The bivalued state process is dependent upon whether the latent factor takes a value above or below time-varying threshold which is a linear function of the predetermined and observable factor. By allowing the threshold to vary over time, the state process becomes partially observable to econometrician as it is determined by both latent and observable factors. Using this property, I can extract the information from the observed time series more effectively, and find that the presence of state observability and endogeneity feedback effect should be considered together when estimating volatility models and conducting state inference.

KEYWORDS: regime switching model, time-varying threshold, state observability, endogeneity feedback effect, predetermined and observable factor, latent factor, state inference, leverage effect, maximum likelihood estimation, markov chain

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1 Introduction

A number of studies have researched regime switching models. Most of these models have the state process which has two regimes, high and low, to determine one of the regimes of an economy. The binary state process is typically modeled as a Markov chain. Hamilton (1989) first introduced the autoregressive model with this type of Markov switching in the mean to explain business cycle. Kim (1994) further analyzed this model which has the fixed transition probabilities. Numerous authors analyzed Markov switching to model a broader set of regression models such as the mean and volatility models. Furthermore, Goldfeld and Quandt (1973), Diebold et al. (1994), and Filardo (1994) considered time-varying transition probabilities in regime switching model. Kim (2004, 2009) has introduced Markov-switching models with endogenous explanatory variables. This paper is closely related to Chang et al. (2017). In this paper, they introduced a regime switching model whose regime is determined by an endogenous autoregressive latent factor.

This paper argues that an endogenous regime switching volatility model with time-varying threshold (TVT) can characterize the dynamics of volatility regimes better than the conventional Markov switching model with time-varying transition probabilities (TVTP) (Diebold et al. 1994 and Filardo 1994) and the endogenous regime switching model with constant threshold (ERS) (Chang et al. 2017). This is because these previous models have some shortcomings. In case of TVTP models, the transition of the state process is exogenously determined. In other words, the Markov chain determining the state process is assumed to be entirely independent from all other parts of the model. This sounds particularly unrealistic in some cases. In addition, the state process itself takes only discrete values and cannot be inferred or extracted out from the model as continuous time series. The ERS models, however, have overcome these drawbacks, except that the state process is assumed to be fully unobservable to econometrician so that precise inference of state process is difficult.

In this paper, I present a regime switching volatility model which incorporates an endogenous autoregressive latent factor and a time-varying threshold (TVT). The volatility process is switching between two volatility regimes, high or low, depending upon the value of state process. The state process is determined by both the latent factor and the TVT, taking value of 1 or 0 if the latent factor is greater or less than the TVT. The TVT is a linear function of predetermined and observable factor, which enables me to partially observe the state process since the state process is determined by both latent and observable factors. The presence of state observability through the TVT allows me to more effectively extract the information about the state process from the observed time series compared to conventional Markov switching models with TVTP and endogenous regime switching model with constant threshold. In addition, the autoregressive latent factor is assumed to be endogenously correlated with the observed time series. The regime switching in the next period is affected by a current shock to the observed time series since the innovation of the latent factor is modeled to be correlated with the previous model innovation. This type of endogeneity in regime switching is introduced by Chang et al. (2017) and will be called endogeneity feedback effect throughout this paper to explain the leverage effect in volatility model.

My model encompasses a broad set of regime switching models including conventional Markov switching models and endogenous regime switching models. If the time-varying threshold is replaced with a constant threshold, the regime switching in my approach reduces to the endogenous regime switching with constant threshold considered by Chang et al. (2017). Thus, one can regard my model as a generalized version of endogenous regime switching model. Furthermore, my model turns into conventional Markov switching models with TVTP if the autoregressive latent factor is exogenous and stationary. If these two restrictions are combined in my model, then it becomes the conventional Markov switching model with fixed transition probabilities.

To empirically illustrate my approach, I analyze the NYSE/AMEX index returns for my volatility switching model. Throughout the results the presence of both endogeneity feedback effect and state observability is strong and clear. The estimated coefficient of slope term of time-varying threshold and correlations between the current shock to the observed time series and the latent factor determining the state in the next period are all significantly different from zero. This implies that the threshold varies over time rather than remains constant over time, and therefore, allows the state process to be partially observable to econometrician by the state observability. The correlation is estimated to be very close to minus unity, -0.999, for both models I consider. This implies that there is an inverse relation between current shock to stock return and volatility. For example, if there is a negative shock to stock returns in the current period, then it will cause a rise in volatility in the next period. This provides a strong evidence of strong leverage effects which is one of the stylized facts of financial time series.

I conduct a comprehensive set of simulations to assess the performance of my model compared to conventional Markov switching models with TVTP and endogenous regime switching model with constant threshold. The simulation results suggest the following points. First, if either the endogeneity feedback effect or the state observability is ignored in regime switching, a considerable amount of bias and efficiency loss occurs when estimating the parameters. Second, the likelihood ratio tests for the state observability perform well in every case I consider. Finally, the confusion matrices computed from the simulations show that my model infers the state process more accurately compared to endogenous regime switching model with constant threshold.

The structure of this paper is as follows. In section 2, I introduce my model and compare it with the conventional Markov switching models and endogenous regime switching model. Section 3 explains the estimation procedure of my model. Section 4 illustrates the empirical results of my model. In section 5, simulation results are presented. Section 6 concludes the paper.

2 Regime Switching Models with Time-Varying Threshold

In this section, I introduce an extended version of endogenous regime switching volatility model with time-varying threshold (TVT) and compare it with the previous approaches used in the conventional and endogenous Markov switching models.

2. 1 Models with Endogenous Regime Switching

In my model, as in the conventional two state Markov switching models, I let a binary state process (s_t) to be determined by a latent factor (w_t) and a predetermined and observable factor (x_t)

$$s_t = s(w_t, x_t) = 1\{w_t \geq \tau(x_t)\} \tag{1}$$

for $t = 1,2,...$, with time-varying threshold $\tau(x_t)$, and $1\{\cdot\}$ is the indicator function. The state process (s_t) takes either 0 or 1 referred respectively to as low and high regimes. I let the latent factor (w_t) be generated as an autoregressive process

$$w_t = \alpha w_{t-1} + v_t \tag{2}$$

with parameter $\alpha \in (-1,1]$ and i.i.d. standard normal innovations (v_t). The time-varying threshold $\tau(x_t)$ is assumed to be a linear function of a predetermined and observable factor (x_t)

$$\tau(x_t) = \tau_t = \tau_c - \tau_s x_t \tag{3}$$

with parameters $\tau = (\tau_c, \tau_s)$.

In order to compare my model with the conventional and endogenous Markov switching models, I use (π_t) as a general notation which denotes a state dependent parameter as follows:

$$\pi_t = \pi(w_t, x_t) = \underline{\pi}1\{w_t < \tau(x_t)\} + \overline{\pi}1\{w_t \geq \tau(x_t)\}, \tag{4}$$

or equivalently,

$$\pi_t = \pi(s_t) = \underline{\pi}(1 - s_t) + \overline{\pi}s_t\,, \tag{5}$$

where $(\underline{\pi}, \overline{\pi})$ are parameters and $\overline{\pi} > \underline{\pi}$. The endogenous regime switching model simply assumes that the threshold τ is constant, whereas my model introduces a time-varying threshold (τ_t) to define the state process (s_t). This approach allows the model to be more flexible since it is more realistic to think that the level of threshold varies over time. Moreover, while the state process (s_t) is assumed to be Markov chain in the conventional models, my approach introduces an autoregressive latent factor (w_t) and observable factor (x_t) to define the state process (s_t). This allows my model to encompass the conventional time-varying transition probabilities (TVTP).

The state process given in (1) can also be understood as follows:

$$s_t = 1\{w_t \geq \tau_c - \tau_s x_t\} = 1\{w_t + \tau_s x_t \geq \tau_c\} = 1\{w_t^* \geq \tau_c\},$$

where $w_t^* = w_t + \tau_s x_t$. In this case, one can understand that this model has a partially observable state process (s_t). That is, the partially observable factor (w_t^*) governs the state process (s_t) and is composed of both latent (w_t) and observable (x_t) factors. The state process (s_t) is therefore determined by whether the partially observable factor (w_t^*) is greater than time-invariant (or constant) threshold τ_c or not. This implies that if I ignore the presence of observability of state process (s_t) with predetermined and observable factor (x_t) (i.e. $\tau_s = 0$), this model reduces to endogenous regime switching volatility model discussed in Chang et al. (2017).

The interpretation of the slope parameter of the observable factor τ_s is nothing but the strength of the observable factor (x_t) in determining the state at time t. If τ_s is very close to 0, the time-varying threshold (τ_t) does not vary much over time. In this case, the state process (s_t) is mainly driven by the latent factor (w_t). On the other hand, if τ_s is not close to 0, then the time-varying threshold (τ_t) varies more sensitively as the observable factor (x_t) changes over time.

A regime switching volatility model is given by

$$\begin{aligned} y_t &= \sigma(w_t, x_t)u_t \\ &= \sigma(s_t)u_t \end{aligned} \tag{6}$$

where the state dependent process (y_t) is defined to be determined by state process (s_t) and the state dependent volatility $(\sigma(s_t))$ with parameters $\underline{\sigma}$ and $\overline{\sigma}$. The innovations (u_t) and (v_t) in (2) are jointly i.i.d. and distributed as

$$\begin{pmatrix} u_t \\ v_{t+1} \end{pmatrix} =_d \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix}\right) \tag{7}$$

with unknown parameter ρ. For the brevity of notation, I write

$$\sigma_t = \sigma(s_t) = \underline{\sigma}(1 - s_t) + \overline{\sigma} s_t. \tag{8}$$

The endogeneity parameter ρ plays an important role in my model. First, the sign of ρ explains whether the innovation u_t of y_t at time t is negatively or positively correlated with the state dependent volatility σ_{t+1} of y_{t+1} at time $t+1$. If $\rho < 0$, this implies that there exists the leverage effect between the innovations: a negative shock to (y_t) in the present period causes an increase in volatility in the next period as (y_t) represents returns of a financial asset. If $\rho > 0$, then this means the opposite, anti-leverage effect. In subsequent discussions of my model, this specific role of the parameter ρ will be called the endogeneity feedback effect. Second, therefore, if I ignore the presence of the endogeneity feedback channel, $\rho = 0$, my model reduces to the conventional Markov switching model with time-varying transition probabilities. This will be further discussed in my simulations and empirical illustrations.

2. 2 Relationship with Previous Markov Switching Models

My model (6) encompasses virtually all models discussed in the literature except the fact it only considers a special case, a volatility model, of general regime switching models. Therefore, one can easily see that it is a generalized version of the endogenous Markov switching model considered by Chang et al. (2017), which particularly allows for the endogeneity feedback effect ρ and nonstationarity $(\alpha = 1)$ in regime changes.

Therefore, I pose some restriction to my model to make my models clearly comparable to the previous Markov switching models. First, I assume

$$\rho = 0 \text{ and } |\alpha| < 1, \tag{9}$$

to make my model reduced to the conventional Markov switching model with TVTP. The above assumptions let the model be stationary and have no endogenous feedback effect but the TVT. In my approach, the transition probabilities of the Markovian state process (s_t) defined in (1) are obtained as follows:

$$\mathbb{P}\{s_t = 0 | w_{t-1}\} = \mathbb{P}\{w_t < \tau_t | w_{t-1}\} = \Phi(\tau_t - \alpha w_{t-1}), \tag{10}$$
$$\mathbb{P}\{s_t = 1 | w_{t-1}\} = \mathbb{P}\{w_t \geq \tau_t | w_{t-1}\} = 1 - \Phi(\tau_t - \alpha w_{t-1}), \tag{11}$$

where $\Phi(\cdot)$ denotes the distribution function of standard normal distribution. Note that if the transition probabilities of the state process (s_t) from the low to the low state (a_t) and from the high to the high state (b_t) are defined as

$$a_t = a(x_t; \alpha, \tau) = \mathbb{P}\{s_t = 0 | s_t = 0\}$$
$$= \frac{\int_{-\infty}^{\tau_{t-1}\sqrt{1-\alpha^2}} \Phi\left(\tau_t - \frac{\alpha z}{\sqrt{1-\alpha^2}}\right) \varphi(z) dz}{\Phi\left(\tau_{t-1}\sqrt{1-\alpha^2}\right)}, \tag{12}$$

$$b_t = b(x_t; \alpha, \tau) = \mathbb{P}\{s_t = 1 | s_t = 1\} \tag{13}$$

$$= 1 - \frac{\int_{\tau_{t-1}\sqrt{1-\alpha^2}}^{\infty} \Phi\left(\tau_t - \frac{\alpha z}{\sqrt{1-\alpha^2}}\right)\varphi(z)dz}{1 - \Phi\left(\tau_{t-1}\sqrt{1-\alpha^2}\right)},$$

where $\varphi(\cdot)$ denotes the density function of standard normal distribution, then the time-varying transition matrix P_t is given by

$$P_t = \begin{pmatrix} a_t & 1-a_t \\ 1-b_t & b_t \end{pmatrix}. \tag{14}$$

Since the state process (s_t) is a Markov chain on a binary state space $\{0,1\}$, it has a transition density given by

$$p(s_t|s_{t-1}) = (1-s_t)\omega(s_{t-1}) + s_t[1-\omega(s_{t-1})], \tag{15}$$

where $\omega(s_{t-1})$ denotes transition probability from (s_{t-1}) to the low state $(s_t = 0)$ as

$$\omega(s_{t-1}) = (1-s_{t-1})a_t + s_{t-1}(1-b_t)$$
$$= \frac{[(1-s_{t-1})\int_{-\infty}^{\tau_{t-1}\sqrt{1-\alpha^2}} + s_{t-1}\int_{\tau_{t-1}}^{\infty}]\Phi\left(\tau_t - \frac{\alpha z}{\sqrt{1-\alpha^2}}\right)\varphi(z)dz}{(1-s_{t-1})\Phi\left(\tau_{t-1}\sqrt{1-\alpha^2}\right) + s_{t-1}[1-\Phi\left(\tau_{t-1}\sqrt{1-\alpha^2}\right)]}. \tag{16}$$

By the assumption (9) in my model (6), it is clear to see that the transition of the state process (s_t) evolves with the predetermined and observable factor (x_t) and the pair (α, τ) of parameters. This allows my model to have the time-varying transition probabilities (TVTP) as the one considered in Diebold et al. (1994) and Filardo (1994). The difference between my model and theirs, however, is that while my model assumes the model innovations to be standard normal distribution so that the transition probabilities are calculated from standard normal density function, theirs define the transition probabilities (a_t, b_t) as the logistic functional form.[2]

Furthermore, I also additionally let

$$\tau_s = 0, \tag{17}$$

i.e. ignoring the presence of the observability of the state process (s_t) with predetermined and observable factor (x_t). In this case, the time-varying threshold (τ_t) reduces to the time-invariant threshold τ_c, $\tau_t = \tau_c$. This makes my model (6) to be the conventional Markov switching model with the fixed transition probabilities (FTP)[3] considered in Hamilton (1989). See Chang et al. (2017) for more

[2]In their model, transition probabilities are presented as

$$a_t = a(\mathrm{x_t};\theta_a) = \frac{exp\left(\theta_{a0} + \sum_{j=1}^{J_1}\theta_{aj}x_{t-j}\right)}{1 + exp\left(\theta_{a0} + \sum_{j=1}^{J_1}\theta_{aj}x_{t-j}\right)}, b_t = b(\mathrm{x_t};\theta_b) = \frac{\exp\left(\theta_{b0} + \sum_{j=1}^{J_2}\theta_{bj}x_{t-j}\right)}{1 + \exp\left(\theta_{b0} + \sum_{j=1}^{J_2}\theta_{bj}x_{t-j}\right)},$$

where $\mathrm{x_t} = (x_t, x_{t-1}, \dots)$ is the history of predetermined and observable variables with parameters $\theta_a = (\theta_{a0}, \theta_{a1}, \dots, \theta_{aJ_1})$ and $\theta_b = (\theta_{b0}, \theta_{b1}, \dots, \theta_{aJ_2})$.

[3] $a(\alpha,\tau) = a(\alpha,\tau_c) = \frac{\int_{-\infty}^{\tau_c\sqrt{1-\alpha^2}}\Phi\left(\tau_c - \frac{\alpha z}{\sqrt{1-\alpha^2}}\right)\varphi(z)dz}{\Phi\left(\tau_c\sqrt{1-\alpha^2}\right)}$, $b(\alpha,\tau) = b(\alpha,\tau_c) = 1 - \frac{\int_{\tau_c\sqrt{1-\alpha^2}}^{\infty}\Phi\left(\tau_c - \frac{\alpha z}{\sqrt{1-\alpha^2}}\right)\varphi(z)dz}{1-\Phi\left(\tau_c\sqrt{1-\alpha^2}\right)}$

discussion on observational equivalence between the regime switching model with autoregressive latent factor and any given conventional Markov switching model.

Now if I release the assumption (9) and leave only (17), that is,

$$|\rho| \leq 1,\ \alpha \in (-1,1],\ \text{and}\ \tau_s = 0,$$

then my model (6) becomes the same as the endogenous regime switching volatility model considered in Chang et al. (2017). This is an obvious consequence since my model is a generalized version of their model. Therefore, it is trivial to show that they are equivalent.

3 Estimation

My endogenous regime switching model with time-varying threshold (6) can be estimated by the maximum likelihood method. The log-likelihood function of the maximum likelihood estimation can be written as

$$l(y_1, y_2, \dots, y_n) = \log p(y_1) + \sum_{t=2}^{n} \log p(y_t|\mathcal{F}_{t-1}) \tag{18}$$

where $\mathcal{F}_t = \sigma(x_t, (y_s)_{s\leq t})$, which is the information given by $x_t, y_1, y_2, \dots, y_t$ for each $t = 1, \dots, n$. A vector of the unknown parameters $\theta \in \Theta$ is included in the log-likelihood function, but is omitted for the brevity. The vector of unknown parameters consists of state dependent volatility parameters $(\underline{\sigma}, \overline{\sigma})$, the autoregressive coefficient α of the latent factor, the endogeneity feedback effect coefficient ρ, and the parameters of the TVT function (τ_c, τ_s).

The predetermined and observable factor (x_t) is obtained by the natural logarithm of lagged indicator variable (X_{t-1})

$$x_t = \log X_{t-1}. \tag{19}$$

The maximum likelihood estimator $\hat{\theta}_{ML}$ of θ can be obtained by

$$\hat{\theta}_{ML} = \underset{\theta\,\in\,\Theta}{\operatorname{argmax}}\, l(y_1, \dots, y_n).$$

To estimate my endogenous regime switching model using maximum likelihood method, I adopt the modified Markov switching filter (MMSF) developed by Chang et al. (2017). The process is similar to theirs except that my model has the TVT instead of the constant threshold. To begin with, I let $\Phi_\rho(z) = \Phi\left(z/\sqrt{1-\rho^2}\right)$ for $|\rho| < 1$ and $u_t = y_t/\sigma_t$.

When the Markov switching model allows the endogenous feedback effect $(\rho \neq 0)$, the state process (s_t) itself is no longer a Markov chain. Instead, the bivariate process (s_t, y_t) on $\{0,1\} \times \mathbb{R}$ is a 1st order Markov process with probability density

$$p(s_t, y_t|s_{t-1}, y_{t-1}) = p(y_t|s_t, y_{t-1})p(s_t|s_{t-1}, y_{t-1}) \tag{20}$$

where

$$p(y_t|s_t, y_{t-1}) = \mathbb{N}(0, \sigma_t^2) \tag{21}$$

and

$$p(s_t|s_{t-1}, y_{t-1}) = (1 - s_t)\omega_\rho(x_t, s_{t-1}, y_{t-1}) + s_t(1 - \omega_\rho(x_t, s_{t-1}, y_{t-1})) \tag{22}$$

with the transition probability $\omega_\rho(s_{t-1}, y_{t-1})$ of state process (s_t) to the low state conditional on the previous state and the past value of state dependent variable. For $|\alpha| < 1$,

$$\omega_\rho(x_t, s_{t-1}, y_{t-1}) = \frac{[(1 - s_{t-1}) \int_{-\infty}^{\tau_{t-1}\sqrt{1-\alpha^2}} + s_{t-1} \int_{\tau-1}^{\infty}] \Phi_\rho \left(\tau_t - \rho u_{t-1} - \frac{\alpha z}{\sqrt{1-\alpha^2}}\right) \varphi(z) dz}{(1 - s_{t-1})\Phi(\tau_{t-1}\sqrt{1-\alpha^2}) + s_{t-1}[1 - \Phi(\tau_{t-1}\sqrt{1-\alpha^2})]}. \tag{23}$$

The modified Markov switching filter consists of the prediction and updating steps, which are iterated to compute the log-likelihood function in (18). To obtain the modified Markov switching filter, I have the conditional probability density of (y_t) on past information on observed time series $\mathcal{F}_{t-1}$ by marginalizing the state process s_t from (20)

$$\begin{aligned} p(y_t|\mathcal{F}_{t-1}) &= \sum_{s_t} p(s_t, y_t|\mathcal{F}_{t-1}) \\ &= \sum_{s_t} p(y_t|s_t, \mathcal{F}_{t-1})\, p(s_t|\mathcal{F}_{t-1}). \end{aligned} \tag{24}$$

While $p(y_t|s_t, \mathcal{F}_{t-1}) = p(y_t|s_t, y_{t-1})$ is the state dependent probability density of (y_t) given by (21), $p(s_t|\mathcal{F}_{t-1})$ should be computed to obtain the log-likelihood function in (18). This can be calculated in the prediction step by marginalizing the state process s_{t-1} from the joint density of s_t and s_{t-1} given $\mathcal{F}_{t-1}$. So, I have

$$\begin{aligned} p(s_t|\mathcal{F}_{t-1}) &= \sum_{s_{t-1}} p(s_t, s_{t-1}|\mathcal{F}_{t-1}) \\ &= \sum_{s_{t-1}} p(s_{t-1}|\mathcal{F}_{t-1})\, p(s_t|s_{t-1}, \mathcal{F}_{t-1}), \end{aligned} \tag{25}$$

where $p(s_t|s_{t-1}, \mathcal{F}_{t-1}) = p(s_t|s_{t-1}, y_{t-1})$ is the transition probability given in (22). Therefore, I can calculate $p(s_t|\mathcal{F}_{t-1})$ from (25), once I obtain $p(s_{t-1}|\mathcal{F}_{t-1})$ from the previous updating step. For the updating step, I write

$$\begin{aligned} p(s_{t-1}|\mathcal{F}_{t-1}) &= p(s_{t-1}|y_{t-1}, \mathcal{F}_{t-2}) \\ &= \frac{p(s_{t-1}, y_{t-1}|\mathcal{F}_{t-2})}{p(y_{t-1}|\mathcal{F}_{t-2})} \\ &= \frac{p(y_{t-1}|s_{t-1}, \mathcal{F}_{t-2}) p(s_{t-1}|\mathcal{F}_{t-2})}{p(y_{t-1}|\mathcal{F}_{t-2})}, \end{aligned} \tag{26}$$

where $p(y_{t-1}|s_{t-1},\mathcal{F}_{t-2})$ is given by (21), and $p(s_{t-1}|\mathcal{F}_{t-2})$ and $p(y_{t-1}|\mathcal{F}_{t-2})$ can be obtained in the prediction step (25) given $\mathcal{F}_{t-2}$.

I can also extract the latent factor (w_t) using the modified Markov switching filter. This can be easily done through the steps described in (25) and (26). For the prediction step, I write

$$p(w_t|\mathcal{F}_{t-1}) = \sum_{s_{t-1}} p(w_t|s_{t-1},\mathcal{F}_{t-1})p(s_{t-1}|\mathcal{F}_{t-1}). \tag{27}$$

Having already obtained $p(s_{t-1}|\mathcal{F}_{t-1})$ from the updating step in (26), I may easily obtain $p(w_t|\mathcal{F}_{t-1})$ once I define the conditional probability density of latent factor (w_t) on previous state and past information on the observed time series. To derive this, I assume $|\alpha| < 1$ and $|\rho| < 1$, then I have as follows:

$$p(w_t|s_{t-1} = 0,\mathcal{F}_{t-1}) = \frac{\Phi\left(\sqrt{\frac{1-\rho^2+\alpha^2\rho^2}{1-\rho^2}}\left(\tau_{t-1}-\frac{\alpha(w_t-\rho u_{t-1})}{1-\rho^2+\alpha^2\rho^2}\right)\right)}{\Phi\left(\tau_{t-1}\sqrt{1-\alpha^2}\right)} \times \mathbb{N}\left(\rho u_{t-1},\frac{1-\rho^2+\alpha^2\rho^2}{1-\rho^2}\right), \tag{28}$$

$$p(w_t|s_{t-1} = 1,\mathcal{F}_{t-1}) = \frac{1-\Phi\left(\sqrt{\frac{1-\rho^2+\alpha^2\rho^2}{1-\rho^2}}\left(\tau_{t-1}-\frac{\alpha(w_t-\rho u_{t-1})}{1-\rho^2+\alpha^2\rho^2}\right)\right)}{1-\Phi\left(\tau_{t-1}\sqrt{1-\alpha^2}\right)} \times \mathbb{N}\left(\rho u_{t-1},\frac{1-\rho^2+\alpha^2\rho^2}{1-\rho^2}\right). \tag{29}$$

Now I may obtain

$$p(w_{t-1}|\mathcal{F}_{t-1}) = \frac{p(y_{t-1}|w_{t-1},\mathcal{F}_{t-2})p(w_{t-1}|\mathcal{F}_{t-2})}{p(y_{t-1}|\mathcal{F}_{t-2})}, \tag{30}$$

in the updating step. Once this is done, the inferred factor $(\widehat{w_t})$ is given as

$$\widehat{w_t} = \mathbb{E}(w_t|\mathcal{F}_t) = \int w_t p(w_t|\mathcal{F}_t)dw_t \tag{31}$$

Furthermore, I may easily obtain the inferred states $(\widehat{s_t})$

$$\widehat{s_t} = 1\{\widehat{w_t} \geq \widehat{\tau_t}\} = 1\{\widehat{w_t} \geq \widehat{\tau_c} - \widehat{\tau_s}x_t\}. \tag{32}$$

for all $t = 1,2,\dots$. Therefore, I may extract not only the inferred factor but also the inferred states, once the maximum likelihood estimates of $p(w_t|\mathcal{F}_t), 1 \leq t \leq n$, are obtained.

4 Empirical Illustrations

To empirically illustrate my model, the US excess stock market returns are analyzed using the regime switching volatility model. To define the excess returns, I used the monthly series of value weighted returns including dividends on the NYSE/AMEX index as stock market returns. For risk free

Table 1: Estimation Results

Model	ERS[4]	TVTP		TVT	
Time-Varying Threshold (τ_t)	Ignored	Allowed		Allowed	
Endogeneity (ρ)	Allowed	Ignored		Allowed	
Observable Factor (x_t)	–	VIX	RV	VIX	RV
$\underline{\sigma}$	0.028	0.024	0.025	0.024	0.025
	(0.003)	(0.002)	(0.002)	(0.002)	(0.002)
$\overline{\sigma}$	0.059	0.052	0.055	0.053	0.055
	(0.007)	(0.004)	(0.005)	(0.003)	(0.008)
α	0.950	0.900	0.030	0.861	0.700
	(0.052)	(0.352)	(1.115)	(0.136)	(0.139)
τ_c	1.531	70.75	-16.53	80.49	-19.66
	(0.961)	(39.80)	(5.892)	(31.57	(3.580)
τ_s		24.09	5.10	27.68	6.03
		(14.22)	(1.753)	(10.76)	(1.077)
ρ	-0.999			-0.999	-0.999
	(0.009)			(0.008)	(0.000)
log-likelihood	585.76	594.08	588.21	596.28	591.69
p-value (LR test for τ_s=0)				0.00	0.00
p-value (LR test for ρ=0)				0.04	0.01

rate of return, monthly risk-free rates are imputed from the daily observations of three months T-bill rates. Both NYSE/AMEX index returns and T-bill rates are obtained from the Center for Research in Security Prices (CRSP) for the period of January 1990-December 2015.[5] By subtracting the risk free returns from the NYSE/AMEX index returns, I obtained the demeaned excess market returns (y_t) to fit the volatility model in (6). For the candidates of the indicator variable (X_t), I used the Chicago Board Options Exchange (CBOE) Volatility Index (VIX) and the realized volatility (RV)[6] of daily returns on NYSE/AMEX index from CRSP. Thus, the predetermined and observable factor (x_t) is derived as in (19). Since the VIX is only available from January 1990, I analyze my model with a sample period January 1990-December 2015. Both the VIX and RV are in monthly format as well.

To estimate the volatility switching model by the maximum likelihood method using the MMSF, the numerical optimization method including the generally used BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm is implemented. The estimation results are reported in Table 1.[7] To compare my

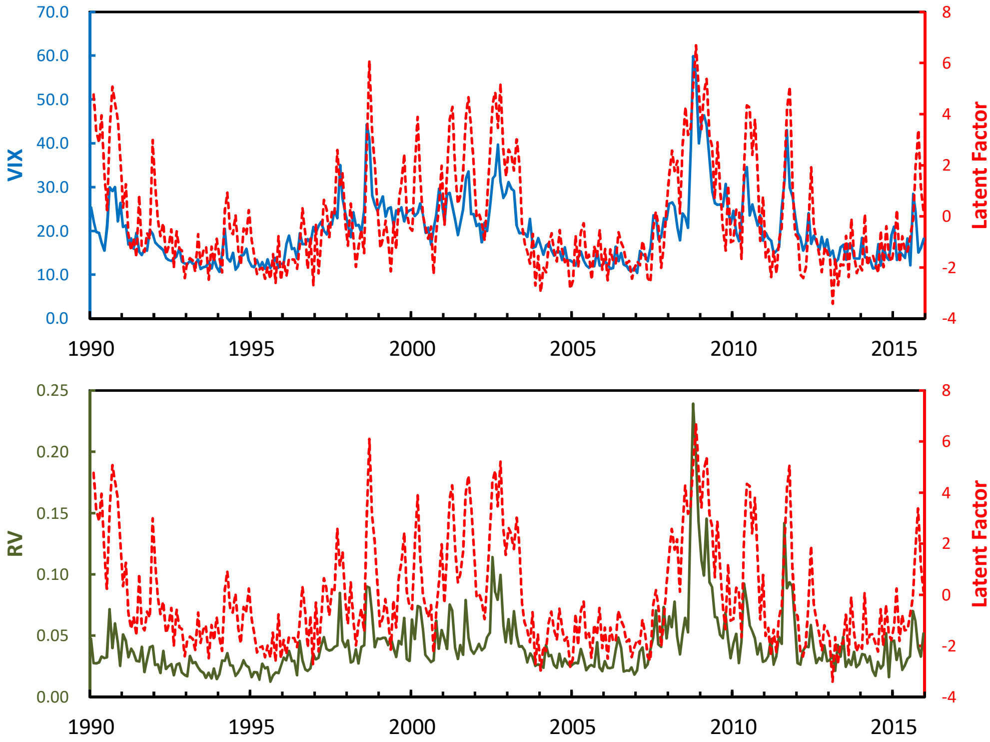
[4] In fact, the endogenous regime switching volatility model (Chang et al. 2017) also has the time-varying transition probability as the transition probabilities vary over time. However, as they do not depend upon the history predetermined and observable factors, but depend upon the lagged value of the excess market return y_{t-1}, the ERS is distinguished from the conventional Markov-switching models with TVTP.

[5] To compute monthly excess return, I followed the steps provided by Chang et al. (2017). First, I obtain monthly risk-free rate of return by continuously compounding daily risk free rate between the quotation dates. The range of the number of days between quotation dates is from 28 to 33. Monthly series of annualized yield to maturity (TMYTM), constructed from nominal price of three-month treasury bill, are provided by CRSP. Hence, by the conversion formula provided by CRSP, $TMYLD_t = \frac{1}{365}\frac{1}{100}TMYTM_t$, I can obtain the yield to maturity at monthly frequency by first converting the annual yield to maturity to daily (TMYLD). To obtain the monthly yield, this daily yield is then continuously compounded as $\exp(TMYLD_{t-1} \times N_t) - 1$, where N_t is the number of days between the quote date for the current month and the quote date, $MCALDT_{t-1}$ for the previous month. Lastly, monthly excess market returns (y_t) are obtained by subtracting the above monthly risk-free rate from the market return.

[6] The realized volatility refers to the annualized monthly volatility calculated as the square root of sum of squared daily returns of NYSE/AMEX index.

[7] The standard errors are presented in parenthesis.

Figure 1: Extracted Latent Factor from ERS (Chang et al. 2017), VIX, and RV



Notes: Figure 1 presents the sample path of the latent factor extracted from the endogenous volatility switching model (red dashed line) presented in Change et al. (2017) along with that of CBOE VIX (solid blue line) and RV (solid green line) for the period of 1990-2015, respectively, on the left and right vertical axis.

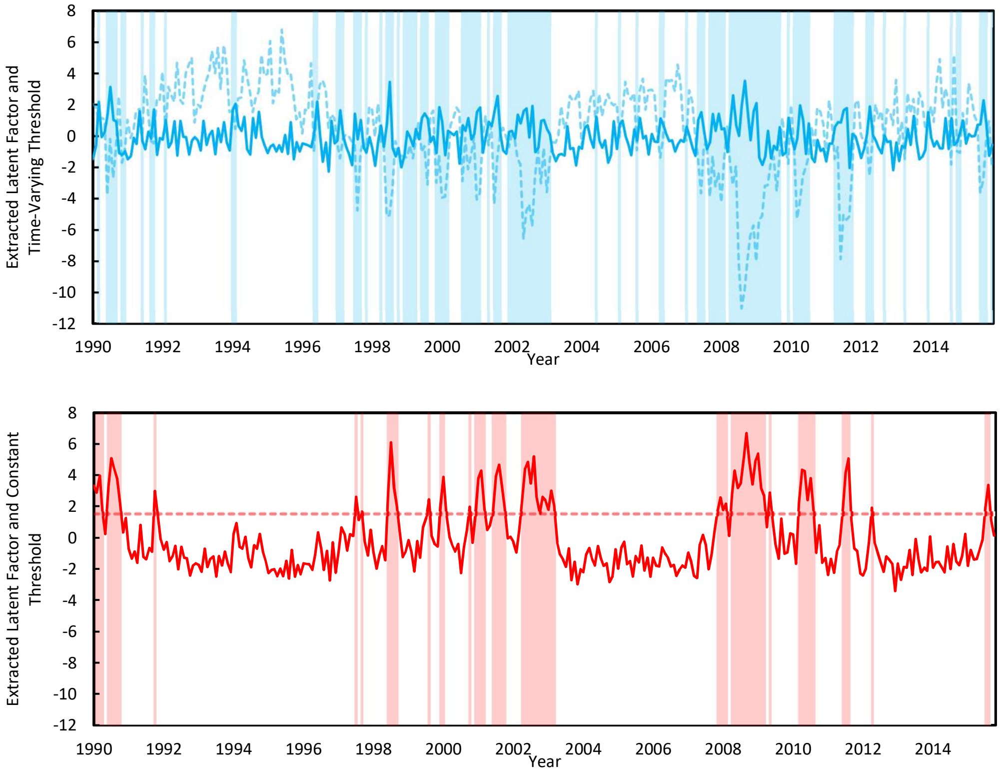
model (6) with the conventional Markov switching model with time-varying transition probabilities (TVTP) and the endogenous regime switching model (ERS), the estimates from these models are also presented in Table 1. For the models with observable factor (x_t), results using the VIX and RV are given, respectively.

The reason why these variables are chosen as an observable factor is clearly shown in Figure 1. I compare the sample paths of the extracted latent factor from the endogenous regime switching volatility model which is presented in Change et al. (2017) and that of VIX and RV. The latent factor stayed relatively high during 1998-2004 and 2008-2012 periods showing that the volatility was high during these periods. The VIX and RV were also relatively high during these periods and seem to move closely together with the extracted latent factor. This is an evidence that these variables can be utilized to partially explain the state process (s_t) which is assumed to be fully unobservable to econometrician in previous models.

An analysis with the simple ordinary least square (OLS) linear regression model using the latent factor and one of indicators as dependent and independent variables, respectively, shows that the coefficient of determination (R^2) is 0.54 and 0.48 for each VIX and RV as dependent variable. Since these factors play a role as a measure of market expectations of volatility, or a gauge for fear factor, it is reasonable to incorporate them into my model (6) to explain volatility regimes.

The estimates for the slope of the observable factor τ_s are quite considerable both in TVTP and TVT models using either of indicators as an observable factor (x_t), 24.09 (14.22) and 5.1 (1.75), and 27.68 (10.76) and 6.03 (1.08) for VIX and RV models, respectively. This provides a significant evidence

Figure 2: Extracted Latent Factor and Threshold



Notes: Top panel plots the extracted latent factor (blue solid line), time-varying threshold (blue dashed line), and inferred high volatility states (shaded blue area) obtained from the TVT model, while the bottom panel plots those from ERS model with red line and shaded area.

that allowing for the presence of the observability of state process (s_t) with observable factor (x_t) is crucial in regime switching volatility models when explaining the state process (s_t). In addition, the null hypothesis of zero constant threshold $(\tau_c = 0)$ is rejected at 1% significance level in my TVT model, whereas the same null cannot be rejected in ERS model. This implies that the state process (s_t) in the ERS model may not be accurately estimated as they were solely determined by imprecise time-invariant threshold τ_c.

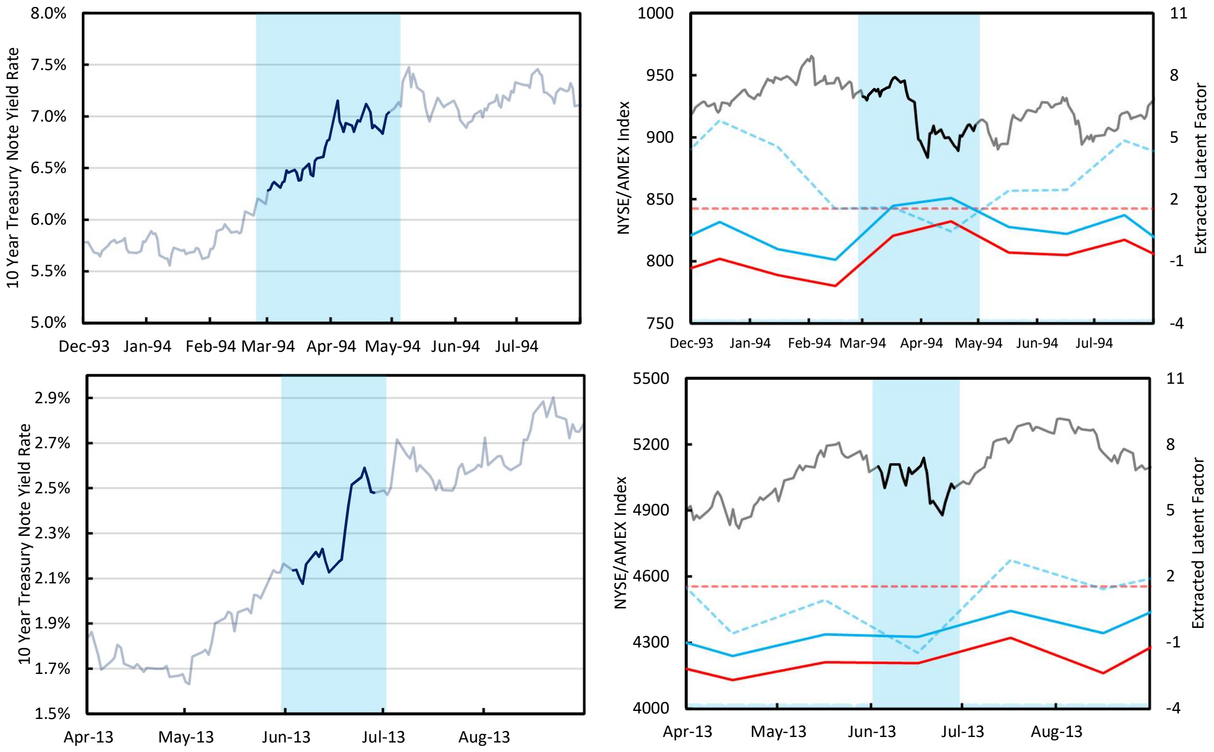
The estimates of the endogeneity feedback effect ρ in both the ERS and TVT are very close to minus unity, which implies that there exists the leverage effect between model innovation (u_t) and innovation of latent factor (v_{t+1}). In addition, the standard errors of the estimates of the endogenous feedback effect ρ in my TVT models are smaller than that of ERS. This shows that the endogeneity feedback effect is more precisely realized when considering the state observability rather than ignoring it.

Furthermore, to test for the presence of observability of the state process with predetermined and observable factor τ_s and endogeneity feedback effect ρ, the likelihood ratio test was conducted, which is given by

$$2(\ell(\hat{\theta}) - \ell(\tilde{\theta})), \tag{33}$$

where ℓ represents the log-likelihood function and the parameter θ with tilde and hat signify their maximum likelihood estimates with and without either one of the restrictions, no observability of state

Figure 3: Bond Market Massacre in 1994 and 2013



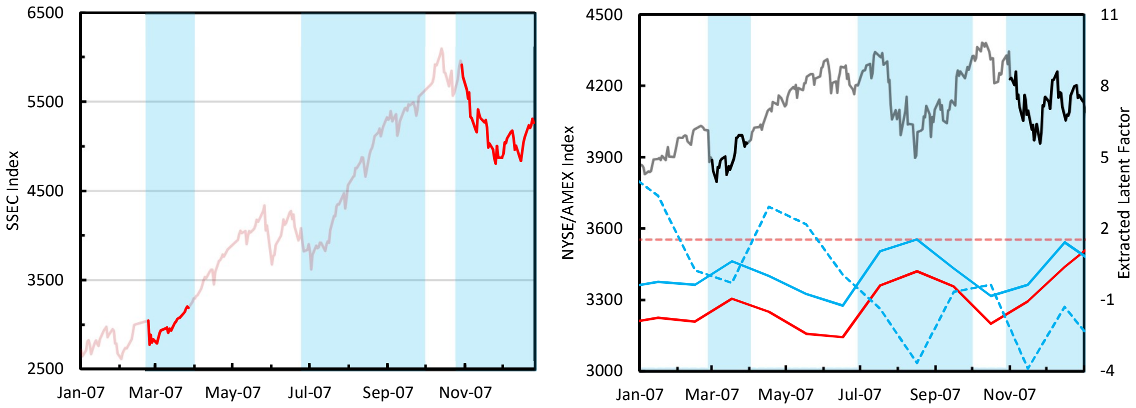
Notes: 10-year treasury note yield rates in 1994 (top right) and 2013 (bottom left) are presented on left hand side column of the panel of Figure 3. On the right-hand side of the panel are the sample paths of extracted latent factor, estimated threshold, and NYSE/AMEX indices presented. The blue solid and dashed line signify the extracted latent factor and time-varying threshold obtained from the TVT, whereas the red solid and dashed line represent those from the ERS. The shaded blue areas refer to the inferred high volatility states from the TVT.

process with observable factor $\tau_s = 0$, or no endogeneity feedback effect $\rho = 0$. The likelihood ratio test has a chi-square distribution with one degree of freedom (χ_1^2). The p-values from each test are also shown in Table 1. For the presence of the observability of the state process, the null of no observability between ERS and TVT model is rejected at 1% significance level in both VIX and RV modes. In case of the presence of the endogeneity feedback effect, the null of no endogeneity between the TVTP and TVT model is rejected at 5% significance level in both VIX and RV models. This result suggests an important result as my TVT model incorporates both effects, the observability and endogeneity, to explain the regime switching volatility.

The effect of the observability of state process with observable factor can be clearly seen from Figure 2, which shows the sample paths of the extracted latent factor $(\widehat{w_t})$, the estimated threshold $(\widehat{\tau_t})$, and inferred states $(\widehat{s_t})$ from the TVT (VIX) and ERS models. The estimated threshold of the state process (s_t) from the ERS is constant over the entire sample period, whereas the corresponding threshold from the TVT varies much over time, and depend upon the predetermined and observable factor (x_t). This results in the difference between the number of inferred high states $(\widehat{s_t} = 1)$ from each model, 70 (22%) and 126 (40%) from the ERS and TVT out of 312 months, respectively.

The observability of the state process in regime switching does not simply increase the number of the inferred high states, but also enables the model to capture historically high volatility regimes more precisely than when ignored. The shaded areas in Figure 3 and 4 indicate some of the examples of high

Figure 4: Chinese Stock Market Bubble



Notes: The red solid line in the left panel signifies the Shanghai Stock Exchange Composite (SSEC) Index in 2007 during which the Chinese stock market bubble occurred. On the right-hand side of the panel, the extracted latent factor, estimated threshold, and NYSE/AMEX index are presented. The blue solid and dashed lines signify the extracted latent factor and estimated time-varying threshold from the TVT, respectively, whereas the red solid and dashed lines represent the extracted latent factor and estimated constant threshold from the ERS. The shaded blue areas in both panels are the inferred high volatility states from the TVT.

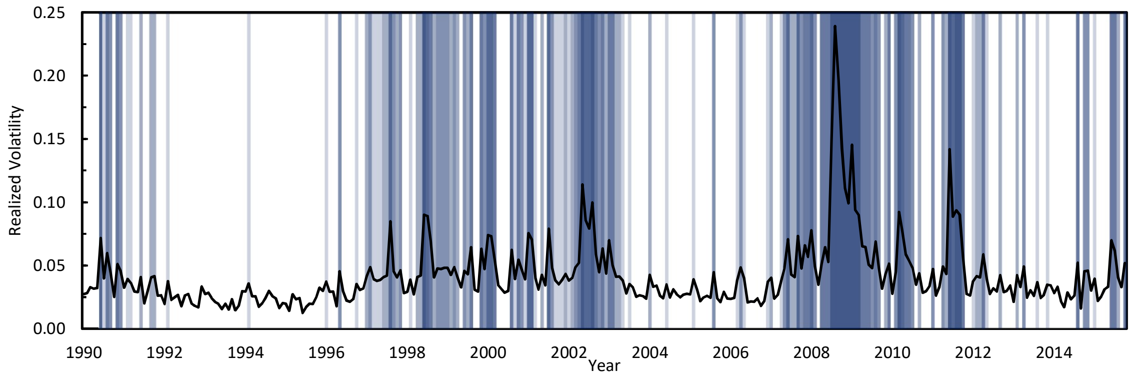
volatility regimes which are inferred to be high volatility states ($\hat{s}_t = 1$) not by the ERS but by the TVT (VIX).

The U.S. 10-year treasury note yield rates are presented on the left-hand side column of Figure 3, and the NYSE/AMEX indices are on the right-hand side along with the extracted latent factor and estimated threshold. The shaded blue area indicates the period which is inferred to be high volatility state only from the TVT model. The top two panels show what happened in March and April 1994 in U.S. stock market as the interest rates spiked. During that period, the interest rates experienced a 31% increase in 98 calendar days while the NYSE/AMEX index dropped more than 4%. The role of the time-varying threshold is clearly seen from top-right of the Figure 3. Whereas the extracted latent factor from the ERS (solid red line) stays below the constant threshold (dashed red line) during the period, that from the TVT (solid blue line) exceeds the time-varying threshold (dashed blue line) as it varies over time with the VIX. That is, the observability of the state process by the VIX pushes the threshold downward so that the state can be transitioned to be high.

A similar situation happened in June 2013 is described in bottom panels, during which the interest rates have increased about 30% in just 39 calendar days. The NYSE/AMEX index declined about 2.7% in one day as the interest rates spiked more than 5% at the same time. Again, the state during this period is inferred to be high from the TVT while it is not from the ERS.

The inferred high volatility states from my TVT model also captures the Chinese stock market bubble happened in 2007, which is presented in Figure 4. It was the global stock market plunge of February 27, and November 2007. After the rumors of government economic authorities introducing varying policies that would restrict foreign investment, the SSEC Index tumbled 9%, the largest drop in 10 years. The U.S. stock market experienced a rapid drop during the same period, the NYSE/AMEX index dropping down more than 3% in one day. This period seems to be unarguably high volatility regime, in which the inferred state from the ERS in Figure 4, however, is still zero referring low volatility state, whereas that from the TVT correctly indicate the high volatility states as the estimated time-varying threshold falls as VIX increases.

Figure 5: Realized Volatility and Volatile Period



Notes: The solid black line presents the time series of annualized monthly realized volatility of NYSE/AMEX index. The shaded areas signify the months which have higher than median of realized volatility from 1990-2015. The darker the color of the shaded areas, the higher the realized volatility during that month.

Table 2: Accuracy of Inferred States

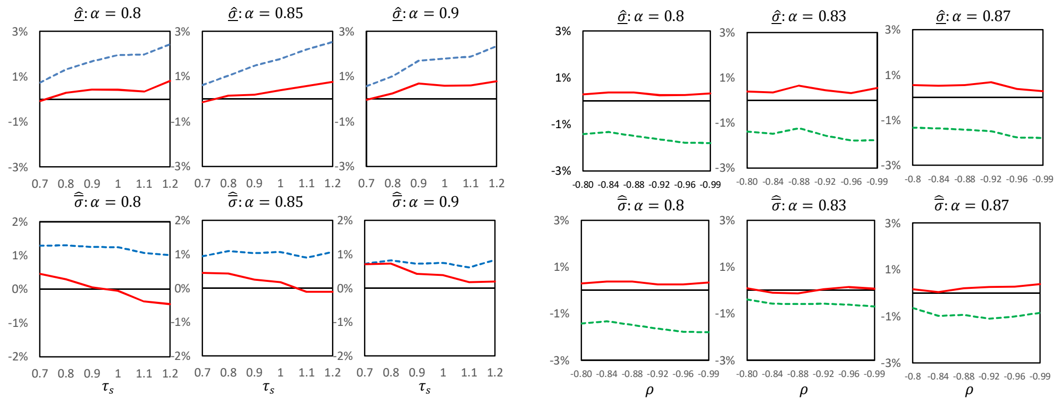
Percentile	Above 5%			Above 10%			Above 20%		
Jan.1990~Dec.2015 (312 Obs.)	Obs.	s_t^{ERS}	s_t^{TVT}	Obs.	s_t^{ERS}	s_t^{TVT}	Obs.	s_t^{ERS}	s_t^{TVT}
Number of High States	16	70	126	31	70	126	62	70	126
% of High States	5%	23%	41%	10%	23%	41%	20%	23%	41%
True Positive Rate	-	94%	100%	-	84%	100%	-	73%	100%

Percentile	Above 30%			Above 40%			Above 50%		
Jan.1990~Dec.2015 (312 Obs.)	Obs.	s_t^{ERS}	s_t^{TVT}	Obs.	s_t^{ERS}	s_t^{TVT}	Obs.	s_t^{ERS}	s_t^{TVT}
Number of High States	93	70	126	124	70	126	155	70	126
% of High States	30%	23%	41%	40%	23%	41%	50%	23%	41%
True Positive Rate	-	56%	98%	-	47%	89%	-	41%	79%

Notes: Table 2 present the accuracy of inferred states from the endogenous regime switching model with constant threshold (ERS) and with time-varying threshold (TVT). True high states are assumed to be determined by the realized volatility above arbitrary percentiles. The number of high states refers to the number of high volatility regimes from each model (ERS, TVT) and underlying assumption. The percentage of high states indicates the rate of high volatility regimes out of 312 months. The true positive rate refers to the rate of inferred high states that match with the true high regimes derived from the periods when the realized volatility was above a certain percentile.

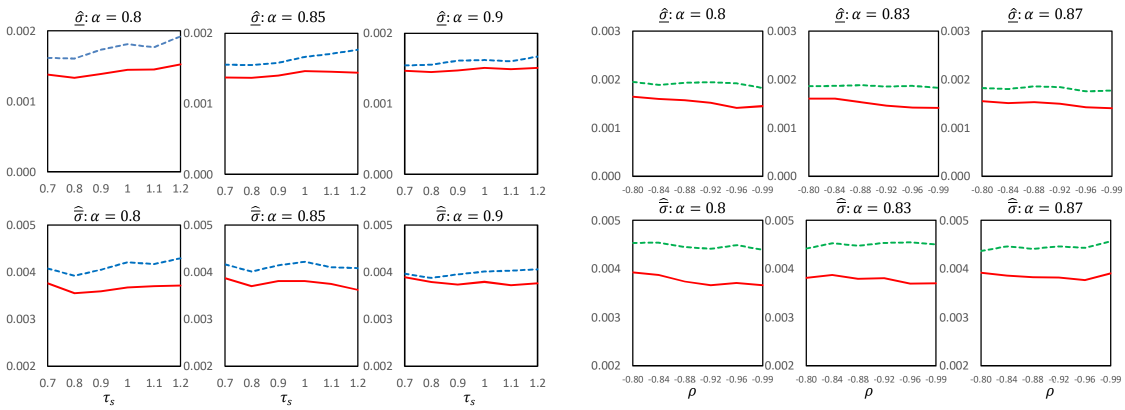
In addition to historical events that caused high volatility regimes, the overall performance of inference of the high volatility states by each model is presented in Figure 5 and Table 2. By comparing the inferred high volatility states ($\widehat{s_t} = 1$) from ERS and TVT(VIX) with periods in which the annualized monthly realized volatility of NYSE/AMEX index was higher than median from 1990-2015, I can clearly measure the true positive rates. In Table 2, the results are presented from above 5 percentiles to above median for inferred high states from ERS and TVT. What can be clearly seen from the results is that the accuracy rate of TVT is higher than that of ERS regardless of the percentile. Moreover, for quite high volatility periods such as above 20 percentiles, the ERS only captures 73% of them, whereas the TVT contain all of them. This can be an obvious result since the TVT infers more high volatility states than the ERS, and the TVT model utilizes the VIX as an observable factor (x_t) which moves very close to the RV. Nevertheless, this result shows the importance of incorporating the observable factor when modelling the regime switching models.

Figure 6: Bias



Notes: On the left panel, the bias in ML estimates of $\underline{\hat{\sigma}}$ and $\hat{\overline{\sigma}}$ of $\underline{\sigma}$ and $\overline{\sigma}$ from the volatility regime switching models from part one is presented respectively in the upper and lower parts, for three persistency level of latent factor $\alpha = 0.8, 0.85, 0.9$, in each of three columns. Each of the six individual graphs plots the bias from TVT models (red solid line) and ERS models (blue dashed line) across different levels of strength of state observability parameter τ_s on the horizontal axis. Presented in the same manner on the right panel are the bias in the ML estimates for part two, for three persistency level of latent factor $\alpha = 0.8, 0.83, 0.87$. Each of the six individual graphs plots the bias from TVT models (red solid line) and TVTP models (green dashed line) across different levels of endogeneity parameter ρ on the horizontal axis.

Figure 7: Efficiency Loss



Notes: Respectively presented in the left and right panel of Figure 7 are the standard errors of the ML estimates of the parameters in my volatility switching models. The graphs on the left and the right panels present the standard error of ML estimates from part one and two in the exactly the same manner as in Figure 7

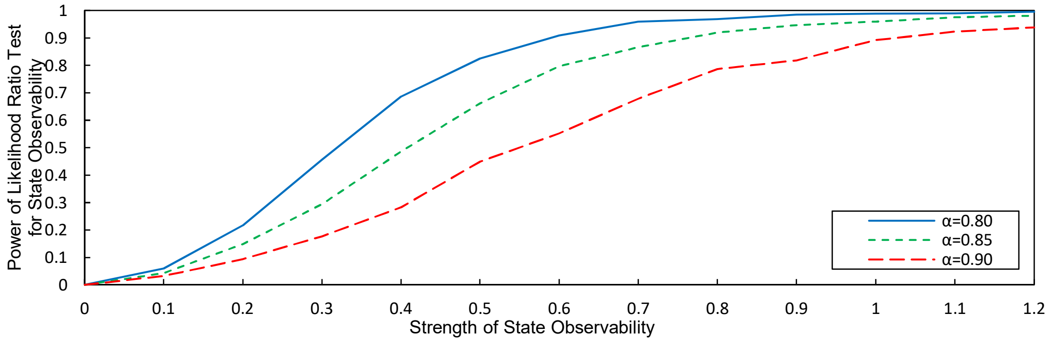
.

5 Simulations

In this section, I conduct a broad set of simulations to evaluate the performance of my model and estimation procedure. In what follows I will present my simulation models and results in two parts. In part one, I compare the endogenous regime switching with time-varying threshold (TVT) with the endogenous regime switching volatility model with constant threshold (ERS). The aim of this part is to see the effect of the state observability in regime switching. In part two, I compare the TVT with conventional Markov switching volatility model with time-varying transition probabilities (TVTP). In this case, the effect of endogenous feedback channel is expected to be observed.

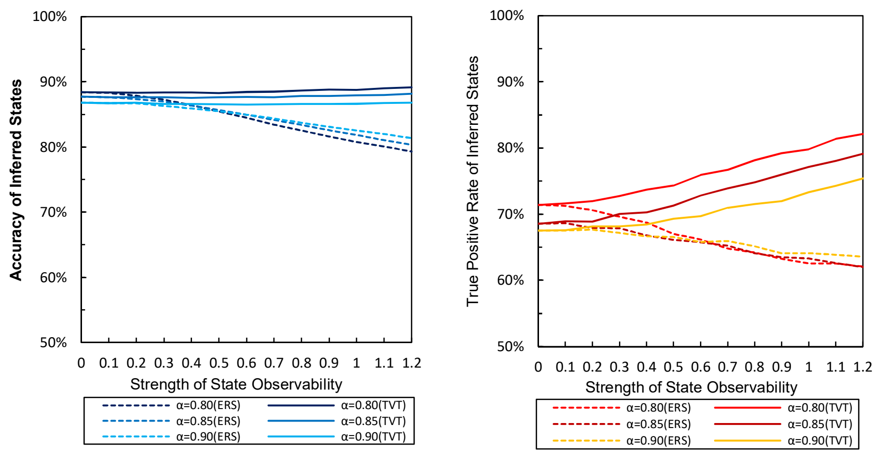
5.1 Simulation Models

Figure 9: Power Function of Likelihood Ratio Test for State Observability



Notes: Figure 8 presents the power function of the likelihood ratio test computed from the volatility models for three different levels of persistency in the latent factor measured by its AR coefficient $\alpha = 0.8, 0.85, 0.9$.

Figure 9: Accuracy and True Positive Rate of Inferred States



Notes: Figure 9 presents the accuracy and true positive rate of inferred states from the volatility models for three different levels of persistency in the latent factor measured by its AR coefficient $\alpha = 0.8, 0.85, 0.9$. For both panels on the left and right, solid and dashed lines refer to the results from TVT and ERS models, respectively.

In my simulations, my volatility model is defined as

$$y_t = \sigma(s_t)u_t, \quad \sigma(s_t) = \underline{\sigma}(1 - s_t) + \bar{\sigma}s_t. \tag{34}$$

I set the parameters $\underline{\sigma}$ and $\bar{\sigma}$ at $\underline{\sigma} = 0.025$ and $\bar{\sigma} = 0.055$, which are approximately the same as my estimates for the endogenous regime switching volatility model with time-varying threshold(TVT) for the endogenous regime switching model with time-varying threshold(TVT) for the stock returns I analyze in the previous section. In addition, I assume that the predetermined and observable factor is generated as autoregressive of order one

$$x_t = c + \gamma x_{t-1} + \varepsilon_t$$

with i.i.d. standard normal innovation (ε_t). I set the parameters at $c = 0$ and $\gamma = 0.85$, which are roughly the same as the estimates from AR(1) model for monthly VIX. For both part one and two, the state process (s_t) and the model innovation (u_t) are generated as specified in (1), (2) and (7) for the samples of size 500, and iterated 700 times.

For part one, the strength of the state observability τ_s is set to be positive, as in most of my empirical results reported in the previous section. I allow τ_s to vary from 0 to 1.2 in increment of 0.1 to more effectively investigate the effect of state observability on the estimation of my model parameters. The correlation coefficient ρ between the current model innovation u_t and the next period innovation v_{t+1} of the latent autoregressive factor is fixed as -0.99, which is also the same as empirical results. On the other hand, I consider three pairs of the autoregressive coefficient α of the latent factor and the constant term τ_c of time-varying threshold given by $(\alpha, \tau_c) = (0.8,0.7), (0.85,0.82), (0.9,0.94)$. In part two, the strength of the state observability τ_s is fixed as 1. Instead, to more thoroughly study the impact of the endogeneity feedback effect, I allow the correlation coefficient ρ to vary from -0.8 to -0.99 in increment of 0.04. In this part, I also consider three pairs of α and τ_c given by $(\alpha, \tau_c) = (0.8,0.7)$, $(0.83,0.75)$, $(0.87,0.91)$.

If $\rho = 0$ and $\tau_s = 0$, my model reduces to conventional Markov switching model with the fixed transition probabilities(FTP), as discussed earlier. Then, as proven by Chang et al. (2017), there exists a one-to-one correspondence between the (α, τ_c) pair and the pair (a, b) of transition probabilities of state process, where a and b denote respectively the transition probabilities from the low state to the low state and from the high state to the high state. All the pairs of (α, τ_c) considered in above have the same equilibrium distribution given by $(a^*, b^*) = (2/3, 1/3)$, which also becomes the common invariant distribution.[8]

5.2 Simulation Results

First, I examine the bias from part one and two. In part one, the estimators of parameters in my models are expected to be biased if the presence of the state observability in the time-varying threshold is ignored. To check the amount of bias caused from ignoring the state observability in TVT, I set $\tau_s = 0$ for the endogenous regime switching model with constant threshold (ERS). In this case, my TVT model reduces to the endogenous regime switching model with constant threshold. In part two, if the presence of the endogeneity feedback effect in regime switching is neglected, the estimators of parameters are also likely to be biased. Once again, to see the magnitude of bias resulting from the ignored endogeneity feedback effect, I let $\rho = 0$. This makes my TVT model become the conventional Markov switching model with time-varying transition probabilities(TVTP).

My simulation results are summarized in Figure 6Figure 7: Efficiency Loss

. On the left panel of Figure 6are the bias in the maximum likelihood estimates of $\hat{\underline{\sigma}}$ and $\hat{\overline{\sigma}}$ of $\underline{\sigma}$ and $\overline{\sigma}$ from part one, whereas the estimates from part two are presented on the right panel of Figure 6Figure 7: Efficiency Loss

. For the upper and the lower part of the panel for both parts present the bias in the estimates of $\hat{\underline{\sigma}}$ and $\hat{\overline{\sigma}}$ for three different levels of α in each column in the panel. For the part one (the left panel), each graph shows the bias of the estimates from the TVT (red solid line) models and the ERS (blue dashed line) models across different levels of the state observability τ_s on the horizontal axis. Likewise, the right panel presents the bias from the TVT (red solid line) models and the TVTP (green dashed line) models in part two across different levels of the endogeneity feedback effect ρ.

Both the state observability and the endogeneity feedback effect in regime switching, even if one of them is ignored, may cause a significant amount of bias in the estimates of model parameters. First,

8 Note that the invariant distribution of the binary state Markov transition given by a 2×2 transition matrix P is defied by $\pi^* = (a^*, b^*)$ such that $\pi^* = \pi^* P$.

the gap of the magnitude of bias between the TVT and the ERS in part one grows larger as the strength of the state observability increases in part one. This implies that there are more chances to improve the bias problem in the estimates of the model parameters when there exists a strong state observability. On average, if one considers the presence of the state observability in the model to estimate the parameters, then the magnitude of bias in $\hat{\underline{\sigma}}$ and $\hat{\overline{\sigma}}$ can improve by 76% and 79% compared to ignoring it. On the other hand, the amount of bias from the TVT and the TVTP models in part two seems to be almost invariant across different values of ρ. Even though the bias does not heavily depend on the level of endogeneity, the amount of bias caused from ignorance of endogeneity feedback effect is substantial. Therefore, if one considers it rather than neglects it, then the magnitude of bias in $\hat{\underline{\sigma}}$ and $\hat{\overline{\sigma}}$ can improve by 73% and 88% on average.

The presence of state observability and endogeneity feedback effect can lead to a severe amount of bias in parameter estimates if they are not properly adopted in regime switching models. In other words, if they are accounted for appropriately as in my TVT models, one can obtain a chance to improve the precision of parameter estimates. Indeed, in the TVT models, the state process (s_t) is determined not by either the autoregressive latent factor (w_t) or the predetermined and observable factor (x_t) but by both of them, and thus I have an additional channel for the information in (y_t) to be reflected in the likelihood function compared to ERS and TVTP models which ignores the state observability $(\tau_s = 0)$ and the endogeneity feedback effect $(\rho = 0)$, respectively. The simulation results in Figure 7 presents the efficiency loss of parameter estimates from ignoring one of the channels of information to determine the state process. The standard errors of maximum likelihood estimates of $\underline{\sigma}$ and $\overline{\sigma}$ are shown in the left and right panel of Figure 7 for part one and two, respectively, in the same manner as in Figure 6.

Figure 7 presents the efficiency loss from neglecting the presence of state observability or endogeneity feedback effect in the analysis of regime switching models. Although there is a little difference in level of the amount of efficiency loss, it is equally substantial for part one and two. In other words, I can improve the efficiency of parameter estimates by correctly accounting for τ_s or ρ. For example, if I set $\alpha = 0.8$, the standard deviations of the estimators $\hat{\underline{\sigma}}$ and $\hat{\overline{\sigma}}$ from my TVT models improve by 16.6% and 9.8%, respectively, if compared with the ERS models with $\tau_s = 0$. On average, the standard deviations can improve by 11.8% and 7.3% for part one, and 18.2% and 14.8% for part two by incorporating τ_s or ρ properly in regime switching models.

Next, I consider testing for the presence of state observability in regime switching models by the likelihood ratio test shown above in (33). In this case, the parameter θ with tilde and hat refers to their maximum likelihood estimates with (ERS) and without (TVT) the no state observability restriction, $\tau_s = 0$, respectively. The power functions of the likelihood ratio test obtained from the simulated volatility switching models for three different levels of autoregressive coefficient of latent factor $\alpha = 0.8, 0.85, 0.9$ are shown in Figure 8. The power of the test gradually increases as the strength of state observability τ_s grows for all levels of α. For instance, $\alpha = 0.85$, which is similar as in my empirical results, the power reaches 90% when the strength of state observability is above 0.7. As the persistency of the autoregressive latent factor gets larger, the overall power of the test grows weaker. This is because the autoregressive latent factor becomes more persistent and nonstationary as α gets close to 1. But the power of the test is powerful enough to reach 90% if there exists a certain amount of state observability which is about 1.

Finally, to see whether my TVT models estimate the state process better than ERS models, I compute the accuracy and the true positive rate of inferred states from simulation part one and the results are shown in Figure 9. To measure these rates, I constructed a confusion matrix for volatility regimes. True conditions are obtained from simulation models as in (1), referring each high $(s_t = 1)$ and low $(s_t = 0)$ volatility regime as positive and negative condition. Predicted conditions are determined in the same manner except that the inferred high $(\hat{s_t} = 1)$ and low $(\hat{s_t} = 0)$ states from the TVT and ERS models are used. As shown in (32), I can obtain the inferred states from regime switching models.

On the right panel of Figure 9 are the true positive rates from simulation part one for three different levels of autoregressive coefficient of latent factor $\alpha = 0.8, 0.85, 0.9$. While the true positive rates from the TVT models (solid lines) increase as the strength of state observability grows, those from the ERS (dashed lines) decrease. This implies that if I ignore the presence of state observability, i.e. with restriction $\tau_s = 0$, it has a deleterious impact on the precision of estimated high volatility regimes. Likewise, the accuracy of inference of volatility states is shown on the left panel of Figure 9. In this case, only the accuracy of ERS models decrease while that of TVT models is stable as the strength of state observability varies. This result implies the same result as in case of true positive rates except that the accuracy measures the overall performance of inferred states from each model.

6 Conclusion

This paper introduces a regime switching model whose state process is determine by two important factors: the autoregressive latent factor and the predetermined and observable factor. My approach has numerous distinct advantages over the conventional Markov switching model and endogenous regime switching model with constant threshold. Above all, I may allow for two channels of information to affect the change in volatility regime. First, the latent factor creates a link between the observed time series and latent part of the state process. This makes the transition of the state process be endogenously determined, and therefore the regime switching becomes endogenous. Second, the predetermined and observable factor affects the value of threshold, making the threshold vary over time. This is acceptable assumption as it is more reasonable and realistic to believe that the level of threshold does not remain constant over time. These two links are referred as the endogeneity feedback effect and the state observability. In my volatility model, the endogeneity feedback effect implies the presence of leverage effect, and the state observability enables the precise inference of state process. Finally, my regime switching volatility model becomes observationally equivalent to the conventional Markov switching model with time-varying transition probabilities or the endogenous regime switching model with constant threshold if there is no endogeneity or no state observability.

The empirical evidence for the presence of both the endogeneity feedback effect and the state observability in volatility regime switching seems to be clear and strong. Especially, my empirical results show that ignoring the presence of the state observability can cause a severe loss in precision of state inference in endogenous regime switching models. This is because the predetermined and observable factor creates a crucial additional link to partially observe the state process to econometrician, which would otherwise be fully unobservable. My simulations clearly show that the presence of endogeneity and state observability in regime switching is so convincing and unmistakable that neglecting even one of them can cause not only a crucial amount of bias, but also a substantial information loss. If I do not consider the time-varying threshold by incorporating the observable factor, then the state observability is ignored, and therefore, a loss in the accuracy and true positive rate of inferred states occurs. Therefore, the additional information that enters through the time-varying threshold is very crucial in regime switching since the state process which plays important roles in the model is otherwise latent to econometrician and cannot be accurately inferred out if the presence of the state observability is ignored.

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