Competitive Advertising and Pricing

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February 2018

Abstract

We consider an oligopoly market in which each firm decides not only its price but also how much information about its product to reveal to consumers. Utilizing a recently developed technique in information design, we fully characterize symmetric pure-strategy market equilibria of this game. We illustrate how a firm's advertising strategy is shaped by its pricing decision and how the equilibrium advertising level depends on the underlying distribution of consumers' true values. A direct but important corollary of our analysis is that more intense competition (more firms in the market) induces each firm to reveal more product information.

JEL Classification Numbers: D43, L11, L13, L15, M37.

Keywords: Informative advertising; information disclosure; Bertrand competition.

1 Introduction

One of the fundamental questions in the economics of advertising is how much, and what, product information a seller should provide for consumers.[1] More product information enables the seller to price-discriminate consumers more effectively or be more aggressive in pricing. However, it comes at the cost of losing some consumers who do not find revealed product characteristics appealing. This essential trade-off has been extensively studied in the monopoly context.[2] A common insight

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[1] In this paper, we restrict attention to this unbiased information transmission role of advertising. However, the literature has identified several other roles of advertising. See Bagwell (2007) and Renault (2015) for comprehensive overviews of the literature.

[2] Lewis and Sappington (1994) is a seminal contribution in this literature. Subsequent important contributions include Che (1996) (return policies as a way to facilitate consumer experimentation), Ottaviani and Prat (2001) (the value of revealing a signal affiliated to the buyer's private signal), Anderson and Renault (2006) (optimal advertising for search goods), Johnson and Myatt (2006) (U-shaped profit function based on the rotation order), and Roesler and Szentes (2017) (buyer-optimal signal for experience goods).

in that literature is that in the absence of varying advertising costs, a monopolist wishes to provide either no information (on values above the marginal cost) or full information: the former enables her to serve all consumers without charging too low a price, while the latter allows her to extract most from high-value consumers.

We study sellers' incentives to provide product information in a competitive oligopoly environment. Specifically, our investigation is built upon the random utility discrete-choice framework of Perloff and Salop (1985). There are $n(\geq 2)$ firms, each selling a horizontally differentiated product. Each consumer's true value for each product is independently and identically drawn from the underlying distribution F. Each firm chooses its price and advertising strategy (i.e., how much information to provide about its own product).[3] Unlike in previous studies, we place no ad hoc restriction on advertising strategy: each firm can reveal, or hide, no matter how much product information it wants, as long as it is consistent with F. In other words, we assume that each firm has access to numerous advertising channels and adjust its advertising content with full flexibility. As in Anderson and Renault (2006) and in the literature on information design, we implement this flexibility by allowing each firm to directly choose any mean-preserving contraction G_i of F. Given G_i, it is as if each consumer's value for product i is independently and identically drawn according to G_i. In this regard, our model can be interpreted as the one in which the distribution of consumer values, which is exogenously given in Perloff and Salop (1985), is endogenized through informative advertising.

Our analysis is enabled by recent technical developments in the literature on information design. As is well-known, the elegant con-cavification method in Aumann and Maschler (1995) and Kamenica and Gentzkow (2011), which is powerful for binary-state problems, is less useful when there are many types (in particular, when the state space is continuous). The fundamental problem is that it is not possible to visualize the sender's (information designer's) value function with respect to the posterior belief she induces, because belief itself is a function over the state space. Recently, several researchers have developed methods that can be used to solve for an optimal signal in such a problem. As anticipated by Kamenica and Gentzkow (2011), the analysis becomes significantly more tractable if the receiver's optimal action depends only on the expected state, and much progress has been made for that class of problems. Assuming risk-neutral consumers, an individual seller's optimal advertising problem in our model belongs to that class, because each consumer's optimal purchase decision depends only on his expected values and prices.

In particular, we make extensive use of a verification technique by Dworczak and Martini (2017) (DM, hereafter).[4] They consider a general programming problem in which the sender's

[3] We do not allow for "comparative advertising", in which a firm not only controls its own product information but also can provide information about the rivals' products. See Anderson and Renault (2009) for an important contribution on comparative advertising.

[4] Other relevant contributions include Gentzkow and Kamenica (2016), Ivanov (2015), and Kolotilin et al. (2017).

(indirect) payoff depends only on the expected value (state) she induces, represented by a function $u(v)$, and show that in order to evaluate the optimality of a signal (equivalently, a mean-preserving contraction G_i of F), it suffices to check whether there exists a convex function $\phi(v)$ which touches $u(v)$ only on the support of G_i. As illustrated by DM with a series of examples and shown by our subsequent analysis, this result permits a simple geometric analysis for a variety of information design problems.

Our analysis differs from Dworczak and Martini (2017) mainly in two ways. First, we look for Nash equilibrium in a strategic environment. As is typically the case, the programming problem each firm faces, which is exogenously given in DM, is endogenously determined in our model, and certain equilibrium restrictions streamline the analysis, rather than complicate it. Second, in our model, an individual firm chooses not only its advertising strategy but also its price. The latter also affects the firm's payoff function $u(v)$ and, therefore, DM's result does not apply directly. We address this problem by finding an optimal signal for each price, to which DM's result applies unchanged, and verifying whether any of those (double) deviations is profitable, for which we develop our own method.

Under some canonical regularity assumptions on F, we provide extensive characterization for symmetric pure-strategy equilibrium. We show that the equilibrium advertising level crucially depends on the shape of F^{n-1}. The firms disclose full product information if and only if F^{n-1} is convex. If F^{n-1} is concave (which is relevant only when $n = 2$), then each firm employs a uniform advertising strategy: each firm's equilibrium choice of G_i is a uniform distribution. If F^{n-1} is neither convex nor concave, then equilibrium exhibits both properties: each firm fully reveals values below a certain threshold v^* and manipulates values above v^*, so that G^{n-1} is uniform above v^*.

For an intuition behind our characterization results, consider the simplest duopoly environment. If F is convex (equivalently, density f is increasing), each firm has to offer expected value close to the maximal value (i.e., the upper bound of the support of F) in order to win a consumer. Therefore, each firm wishes to inflate consumer values as much as possible but is subject to the mean-preserving contraction constraint. In equilibrium, the constraint is binding and each firm reveals all product information. If F is concave (equivalently, f is decreasing) then, opposite to the previous case, firm i wants to provide no information given full disclosure by firm j. Clearly, this cannot be an equilibrium, because firm j would then deviate. They would then engage in a matching-penny type of competition and end up with playing a uniform strategy, not because of its superiority over other advertising strategies but because of the other firm's indifference over various advertising strategies: a uniform strategy is one of many best responses to itself. If F is neither convex nor concave then, naturally, both effects are present and, therefore, equilibrium is a mixture of the two previous equilibrium structures.

An important corollary of our analysis is that more intense competition induces each firm to reveal more product information: this follows directly from our equilibrium characterization and the fact that the convex portion of F^{n-1} continuously expands and eventually covers the entire support of F as n grows unboundedly. This is quite plausible and, therefore, perhaps not surprising at all. However, to our knowledge, this result has not been formally and completely established in the literature.[5] Note that this suggests another beneficial effect of competition on consumer welfare: in the standard random-utility model, an increase in n is beneficial to consumers for two reasons, one that it implies more options for consumers and the other that, under certain regularity, it lowers market prices. In our model, it also makes each firm to reveal more product information and, therefore, enables each consumer to better identify the best product for her.

By now, there are several studies that analyze the effects of competition on information provision (design), including Boleslavsky and Cotton (2015); Gentzkow and Kamenica (2017); Li and Norman (2017); Au and Kawai (2017b,a). Our paper is closest to Au and Kawai (2017b): their model can be interpreted as the one in which the firms compete only through informative advertising (with exogenously given prices). They consider discrete types and apply a different characterization technique. Nevertheless, they have several results in common with us. In particular, they also emphasize the emergence of uniform strategies (i.e., linear equilibrium payoff structure).

The remainder of this paper is organized as follows. We introduce the formal model in Section 2. We characterize full information equilibrium, which turns out to exist only when F^{n-1} is convex, in Section 3. We then analyze the case where F^{n-1} is concave in Section 4 and the case where F^{n-1} is neither convex nor concave in Section 5.

2 The Model

There are $n(\geq 2)$ ex ante homogeneous firms and a unit mass of consumes. Each firm sells a product, which it can product at no fixed cost and zero (normalized) marginal cost. The firms' products are horizontally differentiated: each consumer's true value for each product is drawn according to the distribution function F independently and identically across the products and across consumers. We assume that F has positive, bounded, and continuously differentiable density f over the support $[\underline{v}, \overline{v}]$, where both $\underline{v} = -\infty$ and $\overline{v} = \infty$ are allowed. We also assume that the derivative of f is uniformly bounded.

Each firm chooses its price p_i and advertising content (how much product information to reveal

[5] An exception is Ivanov (2013). He studies the same economic problem but considers only the case where all advertising strategies are ranked in terms of the rotation order defined by Johnson and Myatt (2006). In addition, he characterizes only full information equilibrium and shows that it is likely to exist when n is sufficiently large, without identifying an equilibrium structure when a full information equilibrium does not exist and explaining how the convergence to full information occurs.

to consumers), in order to maximize its profit (the measure of consumers who purchase product i times its price p_i). We represent firm i's advertising strategy by a distribution function $G_i : [\underline{v}, \overline{v}] \rightarrow [0, 1]$, where $G_i(v)$ denotes the measure of consumers whose expected (interim) value for product i is less than or equal to v. For example, if firm i fully reveals its product information, then $G_i = F$. If firm i provides no information about its product, then G_i puts all mass on $\mu_F \equiv \int v dF(v)$. We impose no structure on feasible advertising content. In other words, each firm is allowed to reveal, or hide, whatever product information it wants. The only restriction is that all consumers are rational and, therefore, each firm can provide only "real" information. This implies that each firm can only garble true information and, therefore, G_i must be a mean-preserving contraction of F. Since each firm is unrestricted otherwise, the set of feasible advertising strategies coincides with the set of all mean-preserving contractions of F. In other words, each firm can choose any distribution function G_i, as long as it is a mean-preserving contraction of F.

The market proceeds as follows. The firms simultaneously choose their price p_i and advertising strategy G_i. Then, each consumer draws her interim value for each product according to G_i and purchases a product that maximizes her expected utility. For tractability, we assume that each consumer is risk-neutral and must purchase one of the products, even though her eventual payoff is negative.[6] These imply that each consumer's purchase decision depends only on her interim values and prices: given interim values $(v_1, ..., v_n)$ and prices $(p_1, ..., p_n)$, a consumer purchases product i if and only if $v_i - p_i > v_j - p_j$.[7]

We focus on symmetric pure-strategy equilibria of this market game. Since consumers' optimal purchase decisions are straightforward, we define a market equilibrium in regard to the firms' optimality. Let $D(p_i, G_i, p, G)$ denote firm i's demand (i.e., the measure of consumers who purchase product i) when firm i's strategy is (p_i, G_i), while all the other firms adopt an identical strategy (p, G). Given consumers' optimal choice rule,[8]

$$D(p_i, G_i, p, G) = Pr\{v_i - p_i > v_j - p^*, \forall j \neq i\} = \int G(v_i - p_i + p^*)^{n-1} dG_i(v_i).$$

A tuple (p^*, G) is a symmetric market equilibrium (henceforth, just an equilibrium) if (p^*, G) is a solution to the following programming problem:

$$\max_{(p_i, G_i)} \pi(p_i, G_i, p^*, G) \equiv D(p_i, G_i, p^*, G) p_i,$$

[6] The latter "full market coverage" assumption is common in the literature. See, e.g., Perloff and Salop (1985), Caplin and Nalebuff (1991), and Zhou (2017).

[7] As shown later, in any equilibrium in this paper, the measure of consumers who are indifferent among multiple products is negligible. Therefore, for notational and expositional simplicity, we ignore them throughout the paper.

[8] Note that our derivation of $D(p_i, G_i, p, G)$ does not account for the possibility of atoms in G. This significantly simplifies the notation. In addition, as shown later, equilibrium G never includes any atom and, therefore, there is no loss.

subject to the feasibility constraint that G_i is a mean-preserving contraction of F.

We make use of the following regularity assumption on the underlying distribution F throughout the paper.

Assumption 1 *The density function f is log-concave.*

This assumption has been proved to be useful in various contexts and is satisfied with many commonly used distribution functions (see Bagnoli and Bergstrom, 2005). Among many desirable implications of this assumption, the following will be particularly useful in this paper.

Lemma 1 *Suppose that the density function f is log-concave.*

(i) Both distribution function F and survival function $1 - F$ are log-concave.

(ii) For any $n \geq 2$, F^{n-1} has at most one inflection point, that is, the density function of F^{n-1} is singled-peaked.

(iii) For each v, define $\mu_F(v) \equiv E_F[x|x \geq v]$. Then, the mean residual lifetime function $MRL_F(v) \equiv \mu_F(v) - v$ decreases in v.

Proof. For (i) and (iii), see Bagnoli and Bergstrom (2005). For property (ii), observe that

$$\left(F(v)^{n-1}\right)'' = (n-1)F^{n-2}(v)f(v)\left((n-2)\frac{f(v)}{F(v)} + \frac{f'(v)}{f(v)}\right).$$

The result follows from the fact that both F and f are log-concave, that is, both f/F and f'/f are decreasing in v. ■

3 Full Information Equilibrium

In this section, we study under what conditions there exists an equilibrium in which the firms provide all product information (i.e., $G = F$).

3.1 Competitive Pricing under Full Information

Suppose that all the firms adopt a fully informative advertising strategy. Then, our model reduces to the random-utility discrete choice model of Perloff and Salop (1985). Since the model has been extensively studied in the literature, we present only key findings that are relevant to the subsequent analysis.

Let p^F denote the symmetric equilibrium price under full information. Given that the other firms choose an identical price p^F, firm i's first-order condition can be expressed as

$$\frac{1}{p_i} = -\frac{\partial D(p_i, F, p^F, F)/\partial p_i}{D(p_i, F, p^F, F)}. \tag{1}$$

This is a familiar optimal pricing formula, which states that the optimal price (markup) is inversely related to the proportion of marginal consumers ($\partial D/\partial p_i$) among those who purchase product i ($D(p_i, F, p_F, F)$). Intuitively, when there are more marginal consumers, firm i faces a stronger incentive to capture them by lowering its price. Now imposing the symmetry requirement (that p^F must be firm i's optimal price as well) and using the fact that $D(p_F, F, p_F, F) = 1/n$, we arrive at the following formula for the equilibrium price p^F:

$$\frac{1}{p^F} = -n\frac{\partial D(p_F, F, p_F, F)}{\partial p_i} = n(n-1)\int_{\underline{v}}^{\overline{v}} F(v)^{n-2} f(v) dF(v). \tag{2}$$

Although the candidate equilibrium price is uniquely determined by equation (2), it is non-trivial to establish the existence of symmetric pure-strategy equilibrium, because the profit function $\pi(p_i, F, p^F, F)$ depends on the fine details of the distribution function F and, therefore, may not be well-behaved (i.e., may not satisfy the second-order condition). Assumption 1, however, provides sufficient regularity into the demand function $D(p_i, F, p^F, F)$ and, therefore, ensures the existence of symmetric pure-strategy equilibrium.

Proposition 1 *Under Assumption 1, there exists a unique symmetric (pricing) equilibrium if the firms fully reveal product information.*

Proof. The result follows from Theorem 1 in Quint (2014), which states that if both F and $1-F$ are log-concave, then the demand function $D(p_i, F, p^F, F)$ is also log-concave in p_i, which implies that the right-hand side in equation (1) rises in p_i. ■

3.2 Individual Optimal Advertising under Full Information

Now we examine when each individual firm is willing to fully reveal product information. Unlike in most previous studies that impose a particular structure on advertising technology (e.g., Johnson and Myatt, 2006; Ivanov, 2013), this problem cannot be addressed with classical calculus-based techniques in our model, because each firm can choose any mean-preserving contraction of F. We utilize a technique that has been recently developed by Dworczak and Martini (2017) for a class of Bayesian persuasion problems. We introduce their result and illustrate how to apply it to our strategic problem.

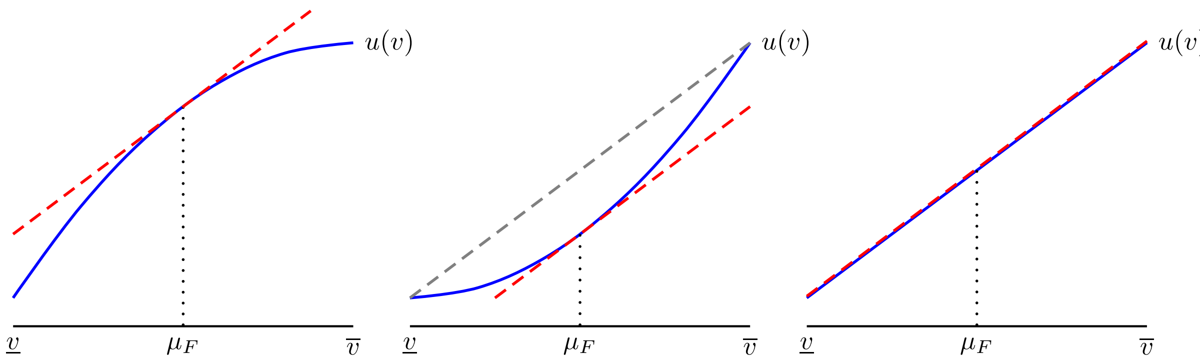

Figure 1: The three representative shapes of $u(v)$ for Theorem 1. In the left panel, it is optimal to put all mass on μ_F. In the middle panel, it is optimal to fully reveal all values. Finally, in the right panel, any feasible distribution is optimal.

Theorem 1 (Dworczak and Martini, 2017) *Let Ω denote the set of all probability distribution functions over the interval $[\underline{v}, \overline{v}]$. Consider the following programming problem:*

$$\max_{G \in \Omega} \int_{\underline{v}}^{\overline{v}} u(x) dG(x)$$

subject to the constraint that $F \in \Omega$ is a mean-preserving spread of G. A distribution function G is a solution to the problem if there exists a convex function $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$, with $\phi(x) \geq u(x)$ for all $x \in [\underline{v}, \overline{v}]$, that satisfies (i) supp$(G) \subset \{x \in [\underline{v}, \overline{v}] : u(x) = \phi(x)\}$, (ii) $\int_{\underline{v}}^{\overline{v}} \phi(x) dG(x) = \int_{\underline{v}}^{\overline{v}} u(x) dF(x)$, and (iii) F is a mean-preserving spread of G.

We illustrate the basic ideas in Theorem 1 with three representative examples in Figure 1. First, consider the case where $u(v)$ is concave (the left panel). In this case, clearly, it is optimal to reduce dispersion as much as possible, which can be implemented by putting all mass on μ_F. Notice that this is equivalent to letting $\phi(x)$ coincide with the straight line that is tangent to $u(v)$ at μ_F. Now consider the case where $u(v)$ is convex (the middle panel). Opposite to the previous case, it is optimal to maximize dispersion. It would be best to put all mass on the two extreme points, $\underline{v}$ and $\overline{v}$. However, such a strategy is not feasible, because the resulting binary distribution would not be a mean-preserving contraction of F. In this case, the mean-preserving contraction constraint binds and, therefore, a constrained optimal solution is to fully reveal all the values (i.e., $G = F$). Notice that it suffices to set $\phi(x) = u(x)$ for any x in this case. Finally, if $u(x)$ is linear (the right panel) then, again, $\phi(x) = u(x)$. However, there are multiple solutions in this case: the value of the objective function is independent of dispersion, which implies that any distribution function G yields the same value. Therefore, any G that is feasible (i.e., satisfies the mean-preserving contraction constraint) is optimal.

In our model, given the symmetric equilibrium price $p^* = p^F$ and fully informative advertising by the other sellers, $u(v) = F(v)^{n-1}p^F$. The following result is then straightforward from Theorem 1 and the discussion above.

Proposition 2 *A full information equilibrium (in which $p^* = p^F$ and $G = F$) can exist only when F^{n-1} is convex.*

Proof. For (p^F, F) to be an equilibrium, it is necessary that $G_i = F$ is an optimal advertising strategy given that $p_i = p^F$ and the other sellers play an identical strategy (p^F, F). Now notice that if $u(v)$ is not convex over a certain interval, then $\phi(v) > u(v)$ on a subset of the interval and, therefore, $G_i = F$ cannot satisfy the second property in Theorem 1. This implies that $u(v) = F(v)^{n-1}$ being convex is necessary for $G_i = F$ to be optimal. ■

Proposition 1 suggests that the optimality of fully informative advertising reduces to whether F^{n-1} is convex or not. In fact, under Assumption 1, it suffices to check the second derivative of F^{n-1} at $\overline{v}$: by straightforward calculus,

$$\left(F(v)^{n-1}\right)'' = (n-1)F^{n-2}(v)f(v)\left((n-2)\frac{f(v)}{F(v)} + \frac{f'(v)}{f(v)}\right).$$

Assumption 1 ensures that the bracket term is decreasing in v. Therefore, F^{n-1} is convex over the entire region if and only if

$$(n-2)\frac{f(\overline{v})}{F(\overline{v})} + \frac{f'(\overline{v})}{f(\overline{v})} \geq 0 \Leftrightarrow (n-2)f(\overline{v})^2 + f'(\overline{v}) \geq 0.$$

Notice that this inequality always holds if $f'(\overline{v}) \geq 0$, which is equivalent to convex F (equivalently, increasing f) under Assumption 1. Even if $f'(\overline{v}) < 0$, the condition holds as long as n is sufficiently large and $f(\overline{v}) > 0$.

3.3 Existence of Full Information Equilibrium

Proposition 1 provides a necessary condition for the existence of full information equilibrium, but does not establish the existence itself. To put it different, in Section 3.1 we consider a firm's price deviation given its advertising strategy, while in Section 3.2 we analyze a firm's deviation in advertising strategy given its price. However, a firm can deviate jointly in both price and advertising strategy. We can conclude that an equilibrium exists if and only if no such double deviation is profitable.

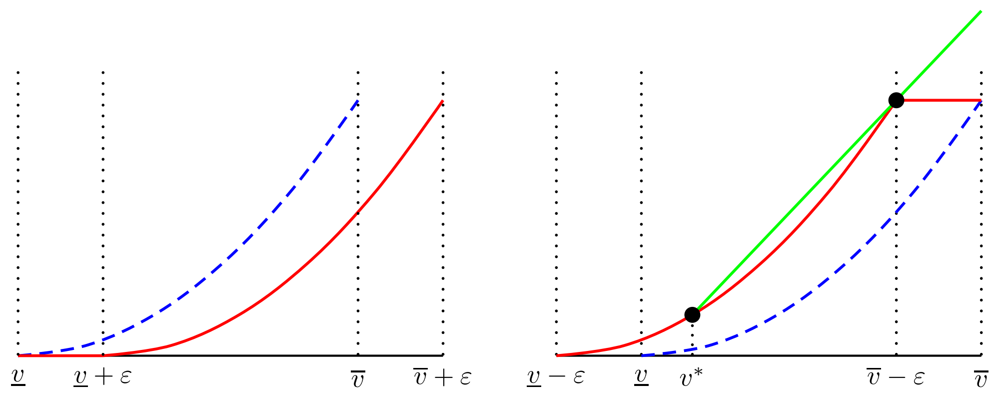

Figure 2: This figure shows how $u(v)/p_i$ (the solid line in each panel) behaves depending on firm i's price deviation when F^{n-1} is convex. The left panel is for an upward deviation where $p_i > p^*$, while the right panel is for a downward deviation where $p_i < p^*$. In both panels, $\varepsilon = |p_i - p^*|$ and the dashed lines are for the benchmark case where $p_i = p^*$ (no deviation). In the right panel, the solid line above the solid curve depicts the convex function $\phi(v)$ in Theorem 1.

In order to analyze the profitability of double deviations, we first fix price p_i. Then, applying Theorem 1, we identify an optimal advertising strategy that corresponds to p_i. Suppose that firm i has chosen price p_i. Since each consumer purchases product i if and only if $v_i - p_i > v_j - p^*$ for all $j \neq i$, firm i's profit conditional on inducing expected value v is given by

$$u(v) = Pr\{v - p_i > v_j - p^*, \forall j \neq i\}p_i = G(v - p_i + p^*)^{n-1}p_i.$$

As always, an increase in p_i lowers firm i's demand (from $G(v)^{n-1}$ to $G(v-p_i+p^*)^{n-1}$) but raises firm i's profit conditional on selling.

For our purpose, it is crucial how the shape of $u(v)$ (equivalently, $u(v)/p_i$ because p_i is fixed) changes as p_i increases. If $p_i > p^*$ then, as shown in the left panel of Figure 2, $u(v)$ stays convex over the support $[\underline{v}, \overline{v}]$: relative to the case where $p_i = p^*$, $u(v)$ shifts to the right, while $u(v) = 0$ if $v \in [\underline{v}, \underline{v}+\varepsilon]$ where $\varepsilon = |p_i - p^*|$. It then follows from Theorem 1 that fully informative advertising (i.e., $G_i = F$) remains optimal. Notice that this implies that this (upward) double deviation shrinks to a deviation only in price, which proved to be unprofitable in Section 3.1 already. We conclude that no firm has an incentive to deviate to $p_i > p^*$ even if it can also adjust its advertising strategy.

Now suppose $p_i < p^*$. In this case, as shown in the right panel of Figure 2, $u(v)$ is no longer convex over the support $[\underline{v}, \overline{v}]$: it remains convex until $\overline{v} - \varepsilon$ but is flat thereafter. It then follows from Theorem 1 that $G_i = F$ is no longer firm i's optimal advertising strategy: for any convex $\phi(v)$ such that $\phi(v) \geq u(v)$ for all $v \in [\underline{v}, \overline{v}]$, $\int \phi(v)dF(v) > \int u(v)dF(v)$, because $\phi(v) > u(v)$ if $v > \overline{v} - \varepsilon$ and $1 - F(\overline{v} - \varepsilon) > 0$.

When $p_i < p^*$, an optimal advertising strategy takes the following structure:

$$G_i(v) = \begin{cases} F(v), & \text{if } v \leq v^*, \\ F(v^*), & \text{if } v \in (v^*, \overline{v} - \varepsilon), \\ 1, & \text{if } v \geq \overline{v} - \varepsilon, \end{cases}$$

where v^* is the value such that

$$\mu_F(v^*) \equiv E[v|v \geq v^*] = \overline{v} - \varepsilon.$$

In other words, it is optimal to reveal values below v^* and put all remaining mass on $\overline{v} - \varepsilon$. In order to show the optimality of this strategy, consider the the function $\phi(v)$ such that $\phi(v) = u(v)$ if $v \leq v^*$ and $\phi(v)$ is a linear function that crosses $(v^*, u(v^*))$ and $(\overline{v} - \varepsilon, u(\overline{v} - \varepsilon))$. As shown in the right panel of Figure 2, this function is convex. In addition, by construction, G_i is a mean-preserving contraction of F (because all mass above v^* is concentrated at $\overline{v} - \varepsilon$) and $\int \phi(v)dG_i(v) = \int u(v)dF(v)$ (by the definition of v^*).

Under the above optimal signal, firm i's demand is given by

$$\begin{aligned} D_i(p_i, G_i, p^F, F) &= \int_{\underline{v}}^{v^*} F(v+\varepsilon)^{n-1} dF(v) + (1 - F(v^*)) \\ &= \int_{\underline{v}}^{\overline{v}} F(v+\varepsilon)^{n-1} dF(v) + \int_{v^*}^{\overline{v}-\varepsilon} (1 - F(v+\varepsilon)^{n-1}) dF(v) \\ &= D_i(p_i, F, p^F, F) + \int_{v^*}^{\overline{v}-\varepsilon} (1 - F(v+\varepsilon)^{n-1}) dF(v). \end{aligned}$$

Since $v^* < \overline{v} - \varepsilon$, this double deviation is strictly more profitable than the corresponding deviation only in price, unlike in the above upward deviations. This implies that a firm has a stronger incentive to deviate to a lower price when it can also adjust its advertising strategy and, therefore, the existence of equilibrium in the benchmark full information environment does not ensure the existence of equilibrium in our model. Nevertheless, the extra benefit due to flexible advertising (the second term in the equation for $D_i(p_i, G_i, p^F, F)$ above) is typically not so large that the full information equilibrium does exist with most well-behaved distribution functions.[9]

[9]It can be easily shown that $\partial D_i(p^F, G_i, p^F, F)/\partial p_i = \partial D_i(p^F, F, p^F, F)/\partial p_i$, that is, the extra term has at best a second-order effect around the equilibrium price p^F.

4 Duopoly with Concave F

We now analyze the case where F^{n-1} is not convex and, therefore, there cannot exist a full information equilibrium. We begin by analyzing the case where F^{n-1} is concave. With bounded density, this case can arise only when there are two firms, because if $n > 2$ then

$$\lim_{x \to \underline{v}} (F(x)^{n-1})' = (n-1)F(\underline{v})^{n-2} f(\underline{v}) = 0 < (F(v)^{n-1})'$$

and, therefore, $F(v)^{n-1}$ cannot be concave around $\underline{v}$. Nevertheless, some common distribution functions, such as exponential and half-normal distributions, are concave. In addition, as shown in Section 5, the analysis of this case is informative about the subsequent and more general analysis, because equilibrium characterization is particularly clean and strong in this case, as formally stated in the following proposition.

Proposition 3 *Suppose $n = 2$ and F is concave (equivalently, f is monotone decreasing). Then, there exists an equilibrium in which $p^* = \mu_F - \underline{v}$ and $G = U[\underline{v}, 2\mu_F - \underline{v}]$, that is,*

$$G(v) = \min\left\{\frac{v - \underline{v}}{2(\mu_F - \underline{v})}, 1\right\} \text{ for any } v \geq \underline{v}.$$

The most striking fact in Proposition 3 is that the equilibrium advertising strategy is to induce a uniform distribution over $[\underline{v}, 2\mu_F - \underline{v}]$: it is straightforward to show that $p^* = \mu_F - \underline{v}$ is the equilibrium price given that $G = U[\underline{v}, 2\mu_F - \underline{v}]$. Importantly, this equilibrium is independent of the underlying distribution F, as long as it is concave.

We prove Proposition 3 in three steps. First, we show that G is feasible and a best response to itself given $p_i = p_j$. Second, we prove that any upward deviation (with $p_i > p^*$ and any G_i that is a mean-preserving contraction of F) is unprofitable. Finally, we show that any downward deviation is also unprofitable.

Sep 1: optimality of G. Given $G_j = G$ and $p_i = p_j = p^*$, firm i's expected payoff by inducing value v is equal to $u(v) = G(v)p^*$, which is linear in v until $2\mu_F - \underline{v}$ and stays flat thereafter. First, G is a mean-preserving spread of F, because, as shown in the left panel of Figure 3, (linear) G necessarily crosses only once (concav) F from below and $\mu_G = \mu_F$. Second, by Theorem 1, G is an optimal advertising strategy:[10] as shown in the right panel of Figure 3, it suffices to linearly

[10] As explained in Section 3.2, there are many optimal solutions if $u(v)$ is linear. It applies here even though $u(v)$ is linear only until $2\mu_F - \underline{v}$. Any distribution G_i that is a mean-preserving contraction of F with the support included in $[\underline{v}, 2\mu_F - \underline{v}]$ (such as a degenerate distribution at μ_F) is optimal.

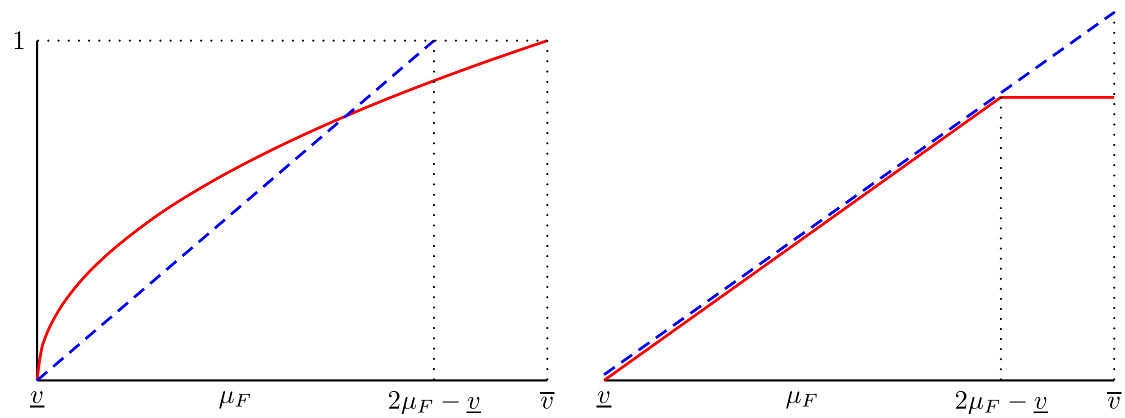

Figure 3: This figure illustrates the equilibrium in Proposition 3. The left panel depicts F (solid) and G (dashed), while the right panel exemplifies $u(v) = G(v)p^*$ (solid) and $\phi(v)$ (dashed).

extend $u(v)$, so that

$$\phi(v) = \frac{v - \underline{v}}{2(\mu_F - \underline{v})} p^* \text{ for all } v \geq \underline{v}.$$

Step 2: downward deviations. Now suppose $p_i < p^*$. In this case, letting $\varepsilon = |p_i - p^*|$, firm i's expected payoff by incuding v is given by

$$u(v) = \min\left\{\frac{v + \varepsilon - \underline{v}}{2(\mu_F - \underline{v})}, 1\right\}(p^* - \varepsilon).$$

As shown in the left panel of Figure 4, this function is concave. By Theorem 1, an optimal advertising strategy is to put all mass on μ_F, which yields the following total profit to firm i:

$$\pi(p^* - \varepsilon, \delta_{\mu_F}, p^*, G) = u(\mu_F) = \begin{cases} \frac{\mu_F + \varepsilon - \underline{v}}{2(\mu_F - \underline{v})}(p^* - \varepsilon) = \frac{\mu_F - \underline{v} + \varepsilon}{2(\mu_F - \underline{v})}(\mu_F - \underline{v} - \varepsilon), & \text{if } \varepsilon \leq \mu_F - \underline{v}, \\ \mu_F - \varepsilon, & \text{if } \varepsilon > \mu_F - \underline{v}. \end{cases}$$

Clearly, π is strictly decreasing in ε, which implies that firm i has no incentive to take a downward deviation.

Step 3: upward deviations Now suppose $p_i \geq p^*$. If $\varepsilon \equiv |p_i - p^*| \geq \overline{v} - \underline{v}$, then no consumer purchases product i and, therefore, we restrict attention to $\varepsilon < \overline{v} - \underline{v}$. Similarly to the above, firm i's expected payoff by inducing value v is given by

$$u(v) = \min\left\{\max\left\{0, \frac{v - \varepsilon - \underline{v}}{2(\mu_F - \underline{v})}\right\}, 1\right\}(p^* + \varepsilon).$$

As shown in the right panel of Figure 4, this function is piecewise linear but neither concave nor convex over the support $[\underline{v}, \overline{v}]$. Nevertheless, we can use the fact that $u(v)$ is convex at least over

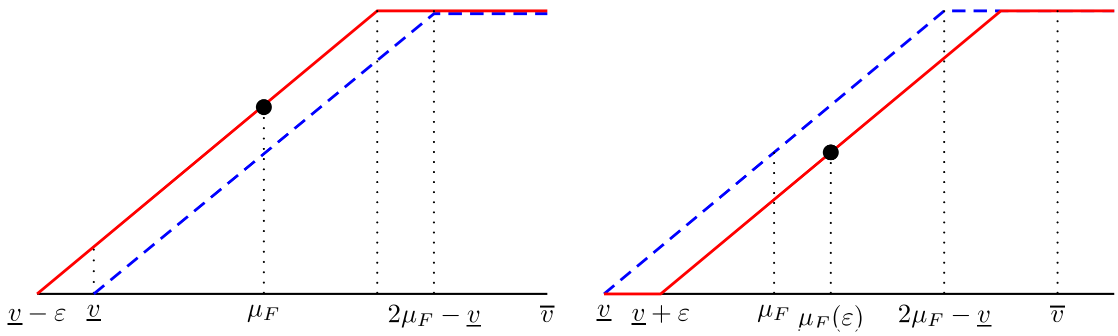

Figure 4: This figure shows how $u(v)/p_i$ (the solid line in each panel) depends on p_i given $G_j = U[\underline{v}, 2\mu_F - \underline{v}]$ and $p_j = p^*$ when $n = 2$ and F is concave. The left panel is when $p_i < p_F$, while the right panel is for the case where $p_i > p_F$. In both panels, $\varepsilon = |p_i - p^*|$.

the interval $[\underline{v}, 2\mu_F - \underline{v} + \varepsilon]$. Consider the following optimal strategy:

$$G_i(v) = \begin{cases} F(v), & \text{if } v \le \underline{v} + \varepsilon, \\ F(\varepsilon), & \text{if } v \in (\underline{v} + \varepsilon, \mu_F(\varepsilon)), \\ 1, & \text{if } v \ge \mu_F(\varepsilon), \end{cases}$$

where $\mu_F(\varepsilon) \equiv E_F[v|v \ge \underline{v} + \varepsilon]$. In other words, this strategy fully reveals all values below $\underline{v} + \varepsilon$ and puts all remaining mass on $\mu_F(\varepsilon)$. By construction (in particular, by the definition of $\mu_F(\varepsilon)$), this strategy is a mean-preserving contraction of F. In addition, under Assumption 1 (because of property (iii) in Lemma 1), $\mu_F(\varepsilon) < 2\mu_F - \underline{v}$ for any $\varepsilon < \overline{v} - \underline{v}$. For the optimality of this strategy, it suffices to define $\phi(v)$ as a piecewise linear and convex function that extends $u(v)$, that is,

$$\phi(v) = \max\left\{0, \frac{v - \varepsilon - \underline{v}}{2(\mu_F - \underline{v})}\right\}(p^* + \varepsilon).$$

The optimal signal above yields the following total profit to firm i:

$$\begin{aligned} \pi(p^* + \varepsilon, G_i, p^*, G) &= F(\underline{v} + \varepsilon)0 + (1 - F(\underline{v} + \varepsilon))\frac{\mu_F(\varepsilon) - \varepsilon - \underline{v}}{2(\mu_F - \underline{v})}(p^* + \varepsilon) \\ &= \frac{1}{2(\mu_F - \underline{v})}\left(\int_{\underline{v}+\varepsilon}^{\overline{v}} v dF(v) - (1 - F(\underline{v} + \varepsilon))(\underline{v} + \varepsilon)\right)(p^* + \varepsilon). \end{aligned}$$

Now notice that

$$\frac{d\pi(p^* + \varepsilon, G_i, p^*, G)}{dp_i} = \frac{1}{2(\mu_F - \underline{v})}(1 - F(\underline{v} + \varepsilon))(\mu_F(\varepsilon) - 2\varepsilon - \mu_F).$$

If $\varepsilon = 0$, then $\mu_F(\varepsilon) = \mu_F$ and, therefore, $d\pi/dp_i = 0$. In addition, under Assumption 1, $\mu_F(\varepsilon) - 2\varepsilon$ necessarily decreases in ε (see property (iii) in Lemma 1) and, therefore, $d\pi/dp_i < 0$ for any $\varepsilon < \underline{v} - \overline{v}$. It follows that no firm has an incentive to deviate upward as well.

5 Partial Information Equilibrium

We now consider the case where F^{n-1} is neither concave nor convex. As explained in Section 4, this case arises whenever F is concave (i.e., f is monotone decreasing) and $n > 2$. It is also relevant whenever F is neither concave nor convex.

5.1 Equilibrium Structure

We begin by identifying the unique equilibrium structure. Assumption 1 implies that there exists a unique value of $\widehat{v} \in (\underline{v}, \overline{v})$ such that $(F(v)^{n-1})'' \geq 0$ if and only if $v \leq \widehat{v}$ (see property (ii) in Lemma 1). This suggests that the equilibrium structure when F^{n-1} is neither concave nor concave is likely to exhibit both the properties of full information equilibrium in Proposition 1 and those of the uniform equilibrium in Proposition 3. The following proposition explains how the two sets of equilibrium properties can be combined for the current case. Note that given G, the equilibrium price p^* can be calculated according to the usual formula (see equation 2).

Proposition 4 *If F^{n-1} is neither convex nor concave, then the equilibrium advertising strategy is given as follows:*

$$G(v) = \begin{cases} F(v), & \textit{if } v < v^*, \\ (F(v^*)^{n-1} + \alpha(v - v^*))^{1/(n-1)}, & \textit{if } v \in [v^*, v^{**}], \\ 1, & \textit{if } v \geq v^*, \end{cases}$$

where

$$\alpha \equiv (F(v^*)^{n-1})' = (n-1)F(v^*)^{n-2} f(v^*)$$

and $v^*(< \widehat{v})$ and $v^{**}(> \widehat{v})$ are the values that satisfy

$$\mu_F(v^*) = \mu_G(v^*) \textit{ and } 1 = G(v^{**}) = (F(v^*)^{n-1} + \alpha(v^{**} - v^*))^{1/(n-1)}.$$

The strategy profile in Proposition 4, whose structure is visualized by Figure 5, is to reveal all values below v^* and manipulate the values above v^* in a specific way. In particular, G is such that G^{n-1} is uniform (linear) above v^*, that is, $G(v)^{n-1} = F(v^*)^{n-1} + \alpha(v - v^*)$. This explains how the equilibrium structure in Proposition 4 connects the previous two equilibrium structures: full

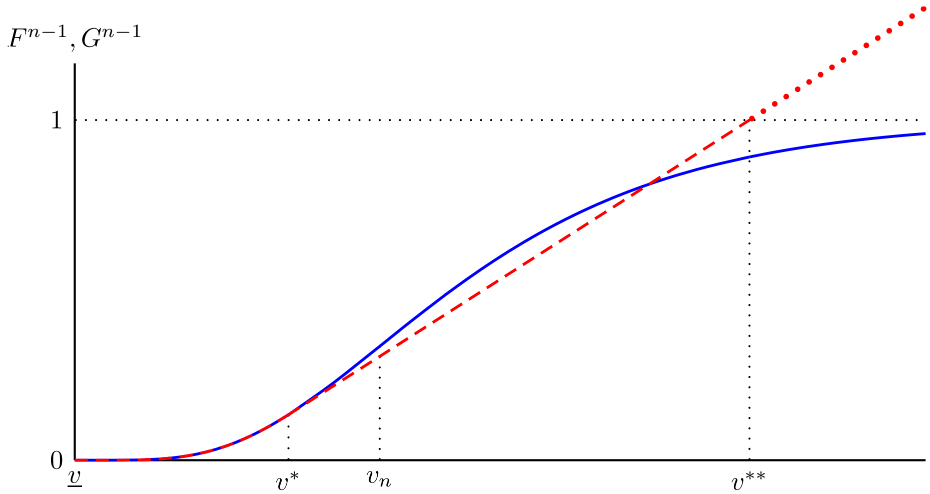

Figure 5: This figure exemplifies the equilibrium structure in Proposition 4. The solid line shows the underlying distribution F^{n-1}, while the dashed line depicts the equilibrium distribution G^{n-1}. The specific parametric assumptions used for this figure are $n = 7$ and $F(v) = 1 - exp(-0.5v)$.

information equilibrium is the case when $v^* = \overline{v}$, while the uniform equilibrium in Proposition 3 is when $v^* = \underline{v}$.

To show that G is feasible, observe that whenever $v \in [v^*, v^{**}]$,

$$G(v) = F(v^*)^{n-2} f(v^*)(F(v^*)^{n-1} + \alpha(v - v^*))^{(2-n)/(n-1)}.$$

Clearly, $G(v^*) = f(v^*)$, and G crosses F only once from below: see the relationship between the solid line (F^{n-1}) and the dashed line (G^{n-2}) in Figure 5. Finally, $\mu_G = \mu_F$ follows from the fact that $G(v) = F(v)$ if $v < v^*$ and $\mu_G(v^*) = \mu_F(v^*)$. We conclude that G is a mean-preserving contraction of F.

For the individual optimality of G, notice that, by construction, G^{n-1} is convex until v^{**}. It follows from Theorem 1 that G is an optimal strategy: as in Section 4 and shown in Figure 5 with the increasing dotted line, it suffices to set $\phi(v) = u(v)$ until $v \leq v^{**}$ and extend the line beyond v^{**}. In other words,

$$\phi(v) = \begin{cases} F^*(v)^{n-1} p^*, & \text{if } v \leq v^*, \\ (F(v^*)^{n-1} + \alpha(v - v^*))^{1/(n-1)} p^*, & \text{if } v > v^*. \end{cases}$$

5.2 Existence of Equilibrium

Proposition 4 provides a candidate equilibrium strategy profile but does not asset the existence of equilibrium. The following proposition provides a tractable necessary and sufficient condition for the strategy profile to be indeed an equilibrium.

Proposition 5 *Let $\tilde{G} : [\underline{v}, \infty) \to \mathcal{R}_+$ be a function such that*

$$\tilde{G}(v) = \begin{cases} F(v), & \text{if } v < v^*, \\ \left(F(v^*)^{n-1} + \alpha(v - v^*)\right)^{1/(n-1)}, & \text{if } v > v^*. \end{cases}$$

The strategy profile in Proposition 4 is an equilibrium if and only if p^* is the solution to the following problem:

$$\max_{p_i \in [0, \overline{v} - \underline{v} + p^*]} \tilde{\pi}(p_i, F, p^*, \tilde{G}) \equiv \int_{\underline{v}}^{\overline{v}} \tilde{G}(v - p_i + p^*)^{n-1} dF(v) p_i.$$

In order to understand Proposition 5, first notice that $\tilde{G}$ is fairly similar, but not identical, to G in Proposition 4. $\tilde{G}$ coincides with G until v^{**} but is strictly larger than G if $v > v^{**}$. In fact, $\tilde{G}$ is not a distribution function, because $\tilde{G}(v) > 1$ whenever $v > v^{**}$. This means that in the programming problem in the proposition, firm i is allowed to obtain a higher payoff than p_i if it induces $v > v^{**}$ (i.e., sell to a consumer with probability larger than 1). On the other hand, in the same programming problem, firm i's advertising strategy is fixed at $G_i = F$, that is, firm i is not allowed to vary its advertising strategy depending on its choice of price. Proposition 5 states that these two effects exactly offset each other and, therefore, no firm has an incentive to take any double deviation if and only if no firm has an incentive to choose a different price in the hypothetical programming problem.

Consider $p_i > p^*$. In the original problem, this upward deviation induces $u(v)/p_i$ to shift to the right (i.e., the probability of selling to a consumer with value v for product i decreases). This changes firm i's optimal advertising strategy: now it is optimal to reveal all values below $v^* + \varepsilon$ and use the remaining mass on the interval $[v^* + \varepsilon, v^{**} + \varepsilon]$ (on which $u(v)/p_i$ is linear). One optimal strategy is to put all mass above $v^* + \varepsilon$ on $\mu_F(v^* + \varepsilon)$, which yields the following profit to firm i:

$$\pi(p_i, G_i, p^*, G) = \left(\int_{\underline{v}}^{v^* + \varepsilon} G(v - \varepsilon)^{n-1} dF(v) + (1 - F(v^* + \varepsilon)) G(\mu_F(v^* + \varepsilon) - \varepsilon)^{n-1} \right) p_i.$$

Applying the definition of G, the second term inside the parenthesis can be rewritten as follows:

$$\begin{aligned} (1 - F(v^* + \varepsilon)) G(\mu_F(v^* + \varepsilon) - \varepsilon)^{n-1} &= \int_{v^* + \varepsilon}^{\overline{v}} \left(F(v^*)^{n-1} + \alpha(\mu_F(v^* + \varepsilon) - \varepsilon - v^*)\right) dF(v) \\ &= \int_{v^* + \varepsilon}^{\overline{v}} F(v^*)^{n-1} dF(v) + \alpha \int_{v^* + \varepsilon}^{\overline{v}} (v - v^* - \varepsilon) dF(v) \\ &= \int_{v^* + \varepsilon}^{\overline{v}} \tilde{G}(v - \varepsilon)^{n-1} dF(v). \end{aligned}$$

Plugging this back into the above equation and arranging the terms, we arrive at

$$\pi(p_i, G_i, p^*, G) = \tilde{\pi}(p_i, F, p^*, \tilde{G}). \tag{3}$$

It then follows that no upward (double) deviation is profitable in the original problem ($\pi(p_i, G_i, p^*, G) \leq \pi(p^*, G, p^*, G)$) if and only if

$$\tilde{\pi}(p_i, F, p^*, \tilde{G}) \leq \tilde{\pi}(p^*, F, p^*, \tilde{G}) = \tilde{\pi}(p^*, G, p^*, \tilde{G}) = \pi(p^*, G, p^*, G).$$

Now consider $p_i < p^*$. Opposite to the previous case, this downward deviation shifts $u(v)/p_i$ to the left: it is convex until $v^{**} - \varepsilon$ and stays flat thereafter. If p_i is so low (equivalently, ε is sufficiently large) that $\mu_F(v^* - \varepsilon) \geq v^{**} - \varepsilon$, then it is clearly optimal to put all mass on $\mu_F(v^* - \varepsilon)$, because firm i can capture the entire market with that advertising strategy. Once it occurs, firm i has no incentive to further lower its price. Therefore, without loss of generality, we restrict attention to the case where $\mu_F(v^* - \varepsilon) \leq v^{**} - \varepsilon$. For such a price deviation, by Theorem 1, an optimal advertising strategy is to fully reveals the values below $v^* - \varepsilon$ and put all remaining mass on $\mu_F(v^* - \varepsilon)$, which yields the following profit to firm i:

$$\pi(p_i, G_i, p^*, G) = \left(\int_{\underline{v}}^{v^* - \varepsilon} G(v + \varepsilon)^{n-1} dF(v) + (1 - F(v^* - \varepsilon)) G(\mu_F(v^* - \varepsilon) + \varepsilon)^{n-1} \right) p_i.$$

Arranging the terms similarly to the above, we arrive at the same equation as equation (3).

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