# Optimal funding with assets: Collateralizing or selling assets

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June 6, 2018

**Abstract**

We develop a two-periods and two-agents model to study the effects of the costly information acquisition and fraudulent practice on the optimal way of funding with assets. When an asset holder faces the liquidity needs but cannot issue unsecured debt, the agent can use the asset as a medium of exchange either by making secured loan contracts or selling the asset. In the model, the future value of an asset is uncertain, but a lender (or a buyer of the asset) can acquire private information about the future value of the asset at a cost. The asset holder (a borrower), however, has an incentive to use a fraudulent asset as a medium of exchange at a cost. The model, then, is used to study the conditions under which collateralized debt contracts and asset sales are inequivalent, so one or the other emerges as an optimal contract for funding with the asset. Collateralized debt contracts can be optimal for two reasons. When the asset

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holder has a high incentive to use the fraudulent asset as a medium of exchanges, the asset cannot be sold directly but can be used as collateral because over-collateralization reduces the fraud incentive making collateralized debt contracts optimal. Furthermore, collateralized debt contracts can be also optimal because it reduces the lender's incentive to acquire costly information. However, under collateralized debt contracts, the borrower may default opportunistically. Thus, if both fraud incentive and information acquisition incentive are not severe, an asset sale can be optimal.

J.E.L. Classification: D8, D53, E0, E44, G12

Keywords: Asymmetric information, costly information acquisition, fraud, collateralized loan contract

# 1 Introduction

When an asset holder faces liquidiy needs but cannot issue unsecured debts, he or she can raise funds with the asset in two different ways: the agent can sell the asset directly or can use the asset as collateral to raise fund for liquidity. Many studies on collateralized debt contracts relies on asymmetric valuation on collateral assets. More precisely, a borrower values the asset more than a lender (See Antinolfi et. al 2014, Lacker 2001, Monnet and Narajabad 2012, Zhang 2014, Williamson forthcoming). This argument justifies the practice of collateralized debt contracts where collateral assets are real assets like houses, for example, because homeowners enjoy housing services by living there. However, this story cannot justify the practice of using financial assets as collateral well, because the intrinsic value of financial assets does not necessarily depend on the identity of asset holders. Furthermore, agents can simply repurchase or resell financial assets in a spot market if they want. However, trillions of dollars of financial assets, such as government bonds and asset backed securities (ABS), are traded daily in a repo and security lending markets as collateral, and at the same time, financial assets are also traded immediately on the spot without any repurchase agreements (see Gorton and Metrick 2012). Consequently, there is a tension between theory and practices in the real world.

In this paper, we attempt to make a progress in filling a gap between theory and practices in the real world. In particular, we answer to the following questions. Under which conditions economic agents choose to make secured loan contracts or to sell assets to raise funds when intrinsic value of assets are same across all agents. When are secured loan contracts over-collateralized and why? What determines the interest rate on secured loan contracts?

For this purpose, we construct a two-periods and two-agents model. In the model, the borrower makes an offer to get consumption goods from the lender in the first period. Because of limited commitment problem, unsecured loans are not possible. Instead, the borrower has Lucas trees that give dividend at the end of the second period, and he can use Lucas trees as a medium of exchanges. Dividend of Lucas trees follows a stochastic process. In a good state, Lucas trees yields dividend, and in a bad state, trees yield nothing. Dividend state is realized at the end of the second period. In principle, the borrower can raise funds for consumption with Lucas trees in two ways. First, he can sell trees to the lender directly (asset sale). Second, he can collateralize trees to borrow consumption goods from the lender in the first period. In general, these two types of trading would generate the same result in a standard exchange model (See Lagos 2010, and Rocheteau 2011).

One of key assumption in the model, however, is that the lender can acquire a private information about the dividend state of trees at a cost in the first period before trading with the borrower. The borrower does not have the technology to obtain information about the dividend state. However, the borrower receives the information about the dividend state at the begining of the second period with some probability. Thus, if the borrower made a secured loan contract in the first period and he learns that trees yield nothing, the borrower would default on the loan opportunistically. The other important assumption is that the borrower has an incentive to misrepresent the tree quality. More precisely, the borrower can produce fraudulent trees at some cost and use fraudulent trees as a medium of exchanges, which generates the fraud incentive constraint in the model.

We first focus on the terms of trade of secured loan contract that gives the highest payoff to the borrower among secured loan contracts. The type of loan contracts depend on whether the lender acquires the costly information about the dividend state or not and whether the fraud incentive constraint of the borrower binds or not. Depending on the information acquisition decision by the lender, the loan contract can be divided into two groups. First, information insensitive contract (IIS) does not induce the lender to acquire the costly information about the dividend state. Second, information sensitive contract (IS), on the other hand, triggers the information acquisition by the lender. Interest rates of IIS loan contract is a compensation for the opportunistic default by the borrower and may include the informational rent to deter the lender from the information acquisition. Under IS loan contracts, the lender accepts the borrower's offer only if the dividend state is good, so there is no opportunistic default by the borrower. Instead, the borrower must compensate the lender for information acquisition cost in the form of a positive interest rate. Obviously, it is more likely that IIS loan contracts dominate IS loan contracts as the information acquisition cost rises. In addition to the information acquisition incentive, the fraud incentive also matter for the type of secured loan contracts. In particular, if the fraud incentive constraint binds, the borrower over-collateralizes loans, so the value of collateral asset is higher than the size of repayment. This is because overcollateralization in the model lowers the payoff on the fraud, so it reduces the fraud incentive.

In the model, secured loan contracts can be optimal for two reasons. First, when the incentive problem of misrepresenting the quality of trees is severe, the borrower cannot sell trees directly to the lender. However, by over-collateralizing a secured loan contract, the borrower can mitigate the fraud incentive problem, so trees can be traded as a medium of exchanges. Thus, a secured loan contract is optimal. Second, even though the fraud incentive problem does not matter, a secured loan contract can be still optimal because it reduces the lender's incentive to acquire the information about the dividend state. Under a secured loan contract, the borrower takes collateral trees whenever he makes repayment on the loan. Thus, the lender has less incentive to get costly information about the dividend state compared to the direct sales of trees. However, when the lender does not have any incentive to acquire the information because of the high acquisition cost, a secured loan contract only allows the borrower to default in a profitable way whenever it is possible. Thus, the lender faces a risk of strategic default by the borrower. Because the borrower must compensate the lender for taking such risk to make him accept the offer, a secured loan contract can be suboptimal, and a direct sale of trees emerges as the optimal contract.

Related work on the optimal type of trading mechanism using asset includes Monnet and Narajabad (2012), Gottardi, Maurin, and Monnet (2015), Tomura(2015), Madison (2017), and Parlatore (2017). Most of these studies focus on the optimality of secured loan contract, but in our paper we also study the economic environment in which asset sale is optimal. Most relatedly, Dang, Gorton, and Holmstrom (2012) used costly information acquisition to study optimal contract for asset trading. However, the authors still focused on the optimality of repo contract and not the other way. Furthermore, we relax the model to allow a positive repo rate, and show that the lender can obtain positive profit even though the borrower has the whole bargaining power. We also show that costly information technology cannot explain over-collateralization practice in a secured loan contract different from Dang, Gorton, and Holmstrom (2012). Instead, we show that overcollateralization occurs to circumvent fraud incentive problem making secured loan contracts optimal. Our paper also related to the literature about information acquisition in an asset exchange models such as Andolfatto and Martin (2013), Andolfatto, et. al (2014), Gorton and Ordonez (2014), and the literature about fraudulent practices such as Li, Rocheteau, and Weill (2012), Kang (2017), and Williamson (forthcoming). However, in these literature, collateralized loan contracts are equivalent to asset sales, and they focused on the effects of information acquisition or fraudulent practices in asset markets on assets' liquidity.

The rest of the paper is organized as follows. Section 2 presents the environment of the model and section 3 solves bargaining problem to find the optimal contract. Section 4 is the conclusion.

## 2 The Model Economy

We consider an exchange economy that consists of two agents-a borrower ($b$) and a lender ($l$)- and two periods, $t \in \{0, 1\}$. The utility of each agent is

$$U^b = mc_{b0} - l_{b0} + c_{b1}$$

$$U^l = c_{l0} - l_0 + c_{l1},$$

where $m$ is the marginal value of consumption of the borrower at $t = 0$, $c_{it}$ and $l_{it}$ are utility from consumption of goods and disutility from labor of agent $i \in \{b, l\}$ in period $t$. We assume that $m > 1$, which can be interpreted as the borrower has liquidity needs at period $t = 0$. There is a single non-durable consumption good in each period, and the endowments of agents are as follows. The lender is endowed with a large amoung of consumption good $e$ at period $t = 0$ and receives nothing at period $t = 1$. On the other hand, the borrower does not receive any consumption goods at period $t = 0$ and is endowed with $e$ units of consumption goods in period $t = 1$ with probability $1 - \alpha \in [0, 1]$. In addition to consumptions goods, the borrower is also endowed with $a$ units of the divisible Lucas tree that can be interpreted as equity, bonds, asset-backed securities, or land at $t = 0$. One unit of tree yields $y$ units of consumption goods at the end of period $t = 1$ with probability of $\sigma \in [\frac{m-1}{m}, 1]$ (good state) while it yields nothing with complement probability (bad state). We assume that $e > ya$ so that there are enough consumption goods that can be traded with trees. The dividend state is realized at the end of period $t = 1$.

Given the utility functions and endowment process, there are gains from trade at period $t = 0$. However, because of a lack of commitment, the unsecured credit is not feasible since the borrower would always default on his obligation. Therefore, trees are necessary as a medium of exchanges for a trade to occur in period $t = 0$, and the borrower can finance his liquidity needs at period $t = 0$ using trees through one of two ways. On the one hand, the borrower can sell $a'$ units of trees in exchange of $q$ units of consumption goods to the lender (an asset sale). On the other hand, the borrower can borrow consumption goods from the lender by pledging trees as collateral (collateralized debt contract). A secured loan consists of three terms, $(q, p, a')$: At period $t = 0$, the borrower receives $q$ units of consumption goods from the lender, and promises to repay $p$ units of consumption goods in period $t = 1$. Thus, the interest rate on the secured loan is $r = \frac{p-q}{q}$. At the same time, the borrower post $a'$ units of trees as collateral at period $t = 0$. Thus, if the borrower fails to make repayments, then the lender seizes collateral trees. This transaction is akin to a repo contract where the borrower sell $a'$ units of trees with a repurchase agreement that the borrower can repurchase the collateral trees with $p$ units of consumption goods at period $t = 1$. In a bargaining at period $t = 0$, we assume that the borrower makes a take-it-or-leave-it offer to the lender.

In this environment, if there are no other frictions, then the borrower could purchase $\sigma ya$ units of goods from the lender in exchange of $a$ units of trees in period $t = 0$, given risk neutral preferences of each agent. At the same time, the borrower can borrow $\sigma ya$ units of goods at period $t = 0$ by pledging $a$ units of trees as collateral, and promises to repay $\sigma ya$ units of goods at period $t = 1$ if he receives the endowment. Therefore, the borrower is indifferent between a direct sale of trees and collateralized debt for financing choice.

**Costly Information Acquisition** In reality, an economic agent may want to acquire more information about future value of an asset than its expected value before purchasing the asset. For example, an agent may obtain analytic reports about the financial statements of the company and its future prospects of business that would give more precise information about the value of equity share even though there is a common perception about the expected value of company's equity share. Then, he makes investment decision based on that information. Similarly, when an agent considers purchasing houses, he may gather more detailed information about living environment and government policy direction that could affect the housing value in the neighborhood. Certainly, these types of information acquisition are costly. One may have to purchase research reports from analysts or must input his own efforts and time to obtain detailed information. In order to capture this practice, we assume that the lender, who is a buyer of trees under direct sale of trees or may seize collateral trees under collateralized debts, can acquire a private information about the dividend state of trees before trading with the borrower at period $t = 0$. To obtain this information, the borrower must incur a fixed cost of $\gamma > 0$ in terms of labor in period $t = 0$.

**Defaults on a secured loan** Under collateralized debt contracts, a borrower may default on the loan for two reasons. First, the borrower is not able to repay on the loan because he is not able to do. This is captured by parameter $\alpha$ in the model. With probability of $\alpha$, the borrower does not receive any consumption goods in period $t = 1$, so he cannot make repayment on the loan. Second, the borrower may default on the loan even though he has sufficient resources because it is profitable to him. Specifically, the borrower will compare the value of the collateral to the value of the avoided repayment, and will default optimally when the later is higher than the first. We introduce this type of default into the model in a very simple way. As Plantin (2009) argued, the owner of an asset may obtain private information about the future cash flow of the asset by holding the asset, which was dubbed as "learning by holding". Thus, we assume that the borrower receives a private information about the dividend state of trees at the end of period $t = 0$ with probability of $\eta \in [0, 1]$, after all transactions have take place. Therefore, if the borrower learns that trees yield nothing, then he will default on the loan at period $t = 1$. Note that the borrower receives the information about dividend state after trading with the lender, not before making an offer to the lender. Thus, the borrower does not have private information when he makes an offer to the lender at period $t = 0$.[1] We make this timing assumption to avoid signaling problem to make the

[1] Different from our study, Hopenhayn and Werner (1996), Velde, Weber, and Wright (1999), and Rocheteau (2011) study how informational asymmetries regarding the future value of assets affect assets' role in transaction and their liquidity when the owner of an asset has private information about the future cash

analysis as simple as possible but without compromizing the economic intuition.

**Fraud in financial affairs** Misrepresenting the quality of financial assets has been prevalent throughout history. Counterfeiting of money has a long history going back to the clipping of coins in ancient Rome, and the economic effects of counterfeiting of money has been one of the important topics in monetary economics (Williamson 2002, Nosal and Wallace 2007, Li and Rocheteau 2011, Kang 2017). However, money is not the only asset that has been a victim of frauds. For example, the complicated securitization process of asset backed securities (ABS) has made it difficult to peirce the veil of ABS and caused the lack of recognizability problem in ABS markets making ABS the target of fraudster. Anecdotes about fraudulent asset appraisals of asset backed securities (ABS) with rating deficiencies and false documentation concerning the underlying assets were common before the financial crisis and fraudulent activities in financial market were criticized as one of key factors in the financial crisis of 2008 (see Barnett 2012, Gourinchas and Jeanne 2012, and the Financial Crisis Inquiry Report 2011).[2] Furthermore, fraudulent practices are not restricted on financial assets. Mortgage markets are also susceptible to mortgage frauds such as misrepresenting the quality of collateral houses. One example is property flipping that involves the purchase and subsequent resale of property at artificially inflated price that enables the purchaser to obtain a greater loan and then default. Although no central repositor collects data of all mortgage frauds, Suspicious Activity Reports from financial institutions indicates the size of mortgage fraud is not negligible.

We introduce the incentive problem of misrepresenting the asset quality in the following way. The borrower can produce fraudulent trees that give no dividend at a proportional cost of $k$ units of labor, and exchange fake trees with the lender's goods at period $t = 0$, similar

flow of the asset.

[2] Robert Lucas, in his interview with the Wall Street Journal (Sep. 24, 2011) also emphasized this fraudulent practice in the financial market as the key factor of the financial crisis arguing that "Instead, the shock came because complex mortgage-related securities minted by Wall Street and "certified as safe" by rating agencies had become part of the effective liquidity supply of the system. All of a sudden, a whole bunch of this stuff turns out to be crap. It is the financial aspect that was instrumental in the meltdown of '08."

to the faking technology of Williamson (forthcoming). We assume that the lender can get the information about the dividend state of trees at a cost, but that information does not reveal the authenticity of trees. Obviously, checking the authenticity of assets would require different information and technology in reality.

To simplify the signaling problem, we assume that the borrower decides whether to produce fake trees or not after making an offer to the lender, but before the lender decides whether to accept the offer or not. Therefore, the borrower makes fraud decision given the terms of trade, which diciplines the lender's belief. However, the analysis and results do not hinge on this timing assumption. Even though the borrower makes faking decision first before making an offer, we get the same results as long as we use the reordering invariance equilibrium concept proposed by In and Wright (2017) to refine equilibria as demonstrated by Li, Rocheteau, and Weill (2012) and Kang (2017). In equilibrium, the borrower will not produce fake trees and the fraud possibility generates the incentive constraint (e.g. see Li, Rocheteau, and Weill, 2012 and Williamson, forthcoming). Figure 1 summarizes the sequence of actions in the economy.

# 3 Funding liquidity needs with trees in period $t = 0$

In this section, we study the terms of trade that the borrower offers to the lender in period $t = 0$. As discussed in the previous section, the borrower can obtain consumption goods from the lender with trees either by selling trees directly (asset sale) or collateralizing trees instead (secured loan contract). However, in the model, we can interpret the tree sale as a special case of a secured loan when $\alpha = 1$, because, in that case, the borrower cannot make repayment on the loan and the lender will seize collateral trees for sure. Therefore, we focus on a secured loan contract between the borrower and the lender in period $t = 0$ from now on, and compare secured loans and a tree sale later.

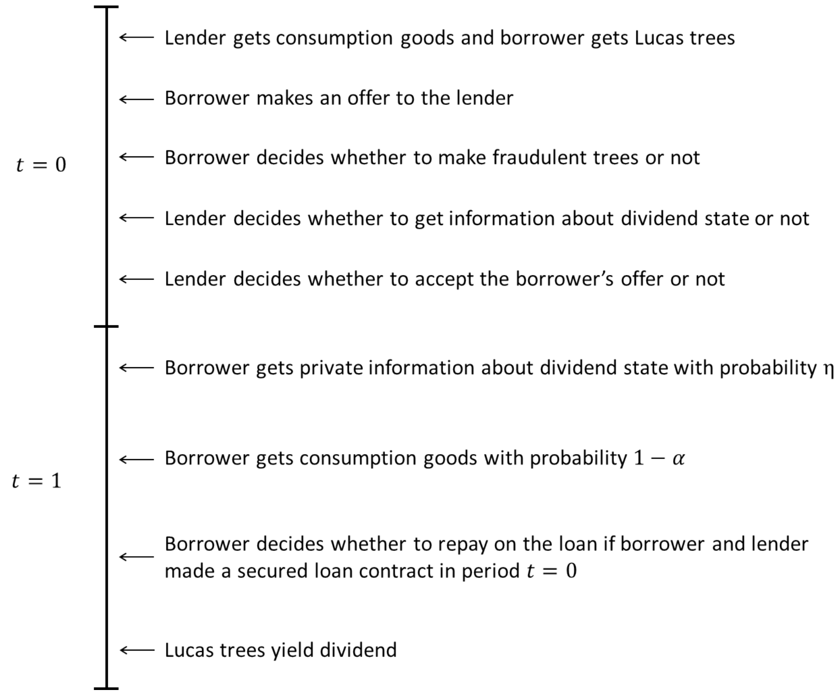


Figure 1: Timeline of borrower's and lender's decision

## 3.1 Secured loans

Under a secured loan, when the borrower defaults on the loan, the lender seizes the collateral trees. In that circumstance, the lender's final payoff from the secured loan contract depends on the dividend state of trees. Thus, the lender has an incentive to acquire information about the dividend state. However, if the lender believes that default would not occur or occurs with very low probability, he would not acquire the costly information about the dividend state of trees. On the other hand, if the lender has a belief that default would occur with a high probability, he may want to check the dividend state of trees before trading with the borrower and accepts the offer only if the dividend state is good. Thus, the lender's payoff from the loan contract $(q, p, a')$ depends on whether he acquires the costly information on the dividend state or not.

First, when the lender does not acquire the information, his expected payoff from a secured loan contract $(q, p, a')$ is

$$\pi_{IIS} = -q + (1-\alpha)[1-(1-\sigma)\eta]p + \alpha\sigma y a'. \tag{1}$$

Here, once the lender accepts the borrowers offer, the lender transfers $q$ units of consumption goods to the borrower in period 0. Then, he receives either $p$ units of consumption goods from the borrower or seizes $a'$ units of collateral trees in period 1. With probability of $\alpha$, the borrower does not receive any endowments in period 1, so he cannot make repayment of $p$. As a consequence, the lender seizes collateral trees that give $\sigma y a'$ units of dividend in expectation. On the other hand, with probability $1-\alpha$, the borrower recieves consumption goods in period $t = 1$, so he is able to make repayment $p$ on the loan and maintain the ownership on the collateral trees as long as it is profitable to him. However, if the borrower learns that trees give no dividend, then he will default on the loan even though he has sufficient consumption goods to make repayment on debts. The term $(1-\alpha)(1-\sigma)\eta p$ in equation (1) represents the lender's loss from this opportunistic default because the lender does not receive any dividend from collateral trees in that case.

Second, when the lender acquires information about the dividend state of trees, he will accept the borrower's offer only if the dividend state is good. Therefore, the lender's expected payoff is

$$\pi_{IS} = \sigma\left[-q + (1-\alpha)p + \alpha y a'\right] - \gamma. \tag{2}$$

Note that the borrower cannot default opportunistically in this case, because the lender only accept the borrower's offer when the dividend state is good. However, the lender must incur $\gamma$ units of labor cost to acquire private information.

Given terms of trade $(q, p, a')$, if $\pi_{IIS} \geq \pi_{IS}$ and $\pi_{IIS} \geq 0$, then the lender will accept the borrower's offer without information acquisition. On the other hand, if $\pi_{IIS} < \pi_{IS}$, then the lender has an incentive to acquire the costly information about the dividend state first and make acceptance decision based on the information about the dividend state as long as $\pi_{IS} \geq 0$. Following the language of Dang, Gorton, and Holstrom (2012), we say that a secured loan contract is information insensitive (IIS) if it does not trigger information acquisition by the lender and the contract is information sensitive (IS) otherwise.

Similarly, the borrower's surplus from trade depends on the lender's information acquisition decision. Specifically, an information insensitive contract ensures a trade with certainty as long as the lender's participation constraint is satisfied, so the borrower's surplus is but the borrower can achieve a trade only if the dividend state is good under the information sensitive contract. Thus, our strategy of finding a secured loan contract that the borrower would offer to the lender is as follows. First, we solve the borrower's problem under each type of loan contract - IIS and IS. Then, we compare the borrower's payoff under each type, and choose a secured loan contract that gives the highest surplus to the borrower.

### 3.1.1 Information Insensitive (IIS) loan contracts

We first start with an IIS secured loan contract under which the lender has no incentive to acquire information about the dividend state. Under a secured loan contract $(q, p, a')$, the borrower must transfer $p$ units of consumption goods to the lender in period $t = 1$ or has to cede $a'$ units of collateral trees to the lender. Note that information about the dividend state is not produced under IIS loan contracts, so the expected value of collateral trees at the beginning of date $t = 1$ is $\sigma y a'$. Therefore, a haircut $\theta$, defined as the difference between the collateral value and the size of granted loan, is $\theta = \frac{\sigma y a' - q}{\sigma y a'}$.

Next, as we explained above, the borrower could make fraudulent trees at the proportional cost of $k$. If the borrower pledges fraudulent trees as collateral, then he will default in the next period for sure without losing genuine trees. Thus, he can save $(1-\alpha)[1-\eta(1-\sigma)]p + \alpha\sigma y a'$ units of consumption goods from fraud, but he has to pay the $ka'$ units of labor to produce $a'$ units of fraudulent trees. Given the terms of contract $(q, p, a')$, the payoff from fraud should not be higher than the fraud cost. Otherwise, the lender would not accept the borrower's offer. This generates the fraud incentive constraint in the borrower's problem below. Then, the borrower's maximized value under IIS repos, $V_{IIS}$, is given by

$$V_{IIS} = \underset{q,p,a'}{Max} \{mq - (1-\alpha)[1-(1-\sigma)\eta]p - \alpha\sigma a' + \sigma y a\} \tag{3}$$

subject to

$$-q + (1-\alpha)[1-(1-\sigma)\eta]p + \alpha\sigma y a' \geq 0 \tag{4}$$

$$-(1-\sigma)q + (1-\eta)(1-\alpha)(1-\sigma)p + \gamma \geq 0 \tag{5}$$

$$ka' - (1-\alpha)[1-(1-\sigma)\eta]p - \alpha\sigma y a' \geq 0 \tag{6}$$

$$\sigma y a' - p \geq 0 \tag{7}$$

$$a - a' \geq 0 \tag{8}$$

$$q, p, a' \geq 0 \tag{9}$$

The objective function (3) consists the borrower's suplus from trade, $mq-(1-\alpha)[1-(1-\sigma)\eta]p-\alpha\sigma a'$, and the expected value of trees. The inequality (4) is the lender's participation constraint without information acquisition. (5) is the no information acquisition incentive constraint that deters the lender from information acquisition about the dividend state, which means that the lender's payoff with information acquisition (2) should not be higher than the payoff without information acquisition (1). Next, (6) is the fraud incentive constraint that prevents the borrower from producing fraudulent trees. The inequality (7) implies that the value of the avoided repayment is not higher than the expected value of collateral trees, so the borrower has an incentive to make repayment unless he receives a private information that tells the dividend state is bad. Here, if $p < \sigma y a'$, then we call the secured loan is over-collateralized and define $\frac{\sigma y a' - p}{\sigma y a'}$ as the over-collateralization rate. Note that if $p = \sigma y a'$, then the positive interest rate on the secured loan $r = \frac{p-q}{q}$ menifests itself as the positive haircut $\theta = \frac{\sigma y a' - q}{\sigma y a'}$ although the denominator is different. Thus, we focus on the analysis of the interest rate when $p = \sigma y a'$, and analyze the interest rate and haircut separately only if the secured loan is over-collateralized, i.e., $p < \sigma y a'$. Finally, (8) and (9) are the feasibility constraints. The following lemma describes the solution to the above maximization problem describing the terms of information insensitive (IIS) secured loan contracts.

**Lemma 1** *Define the level of information acquisition cost* $\gamma^*_{IIS} \equiv [\alpha + (1-\alpha)\eta\sigma](1-\sigma)\sigma y a$ *and* $\gamma^{**}_{IIS} \equiv \frac{[\eta k+(1-\eta)\alpha y](1-\sigma)\sigma a}{1-\eta(1-\sigma)}$, *and the probability that the borrower does not receive endowments* $\alpha^* \equiv \frac{(1-\eta)(m-1)-\eta\sigma}{1+(1-\eta)(m-1)-\eta\sigma}$ *and* $\alpha^{**} \equiv \frac{k[(m-1)(1-\eta)-\eta\sigma]}{m(1-\eta)\sigma y}$. *Then, terms of IIS secured loan constracts are as follows:*

1. *Suppose* $[1-\eta(1-\alpha)(1-\sigma)]\sigma y \leq k$.

   1-a) *[IIS-1] If* $\gamma^*_{IIS} \leq \gamma$, *then* $q = [1-(1-\alpha)(1-\sigma)\eta]\,\sigma y a$, $p = \sigma y a$, $a' = a$, *and* $V_{IIS} = [m-(m-1)(1-\alpha)(1-\sigma)\eta]\,\sigma y a$.

   1-b) *[IIS-2] If* $\gamma < \gamma^*_{IIS}$ *and* $\alpha \leq \alpha^*$, *then* $q = (1-\alpha)(1-\eta)\sigma y a + \frac{\gamma}{1-\sigma}$, $p = \sigma y a$, $a' = a$, *and* $V_{IIS} = m\left[(1-\eta)(1-\alpha)\sigma y a + \frac{\gamma}{1-\sigma}\right] + (1-\alpha)(1-\sigma)\eta\sigma y a$.

   1-c) *[IIS-3] If* $\gamma < \gamma^*_{IIS}$ *and* $\alpha > \alpha^*$, *then* $q = \frac{[1-(1-\alpha)(1-\sigma)\eta]\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)}$, $p = \frac{\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)}$ *where* $a' = \frac{\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)\sigma y}$, *and* $V_{IIS} = \frac{(m-1)[1-(1-\alpha)(1-\sigma)\eta]\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)} + \sigma y a$.

2. *Suppose* $\alpha\sigma y \leq k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$.

   2-a) *[IIS-4] If* $\gamma^{**} \leq \gamma$, *then* $q = ka$, $p = \frac{(k-\alpha\sigma y)a}{(1-\alpha)[1-\eta(1-\sigma)]}$, $a' = a$, *and* $V_{IIS} = (m-1)ka + \sigma y a$

   2-b) *[IIS-5] If* $\gamma < \gamma^{**}_{IIS}$ *and* $\alpha \leq \alpha^{**}$, *then* $q = \frac{(1-\eta)(k-\alpha\sigma y)a}{1-\eta(1-\sigma)} + \frac{\gamma}{1-\sigma}$, $p = \frac{(k-\alpha\sigma y)a}{(1-\alpha)[1-\eta(1-\sigma)]}$, $a' = a$, *and* $V_{IIS} = m\left\{\frac{(1-\eta)(k-\alpha\sigma y)a}{1-\eta(1-\alpha)} + \frac{\gamma}{1-\sigma}\right\} - ka + \sigma y a$

   2-c) *[IIS-6] If* $\gamma < \gamma^{**}_{IIS}$ *and* $\alpha > \alpha^{**}$, *then* $q = \frac{[1-\eta(1-\sigma)]\gamma k}{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma}$, $p = \frac{(k-\alpha\sigma y)\gamma}{[\eta k+\alpha(1-\eta)y](1-\alpha)(1-\sigma)\sigma}$, $a' = \frac{[1-\eta(1-\sigma)]\gamma}{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma}$, *and* $V_{IIS} = \frac{(m-1)[1-\eta(1-\sigma)]\gamma k}{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma} + \sigma y a$

3. *Suppose* $k < \alpha\sigma y$. *Then, an IIS secured loan contract is not feasible*

**Proof.** See Appendix ■

As one can see from lemma 1, the solution to the problem (3) depends on the fraud cost $k$, the information acquisition cost $\gamma$, and the probability that the borrower does not receive endowments $\alpha$ because of the effects of those variables on the no information acquisition incentive constraint (5) and the fraud incentive constraint (6), and there are 7 cases. In the following, we analyze each type of IIS secured loan contracts and provide economic intuitions for the results.

When $[1-\eta(1-\alpha)(1-\sigma)]\sigma y \leq k$, the fraud cost is sufficiently high, so the fraud incentive constraint (6) does not bind. In this case, as one can see from lemma 1, $p = \sigma y a'$. The type and terms of secured loan contract depend on parameters that affect the lender's incentive to acquire information about the dividend state: the information acquisition cost $\gamma$ and the probability $\alpha$ that the borrower does not receive endowments in period $t = 1$. Here, the effect of $\gamma$ on the information acquisition incentive is straightforward. As $\gamma$ increases, the lender has less incentive to acquire the information. Next, $\alpha$ is the probability that the borrower is not able to make repayment on the loan in the period $t = 1$. Thus, as $\alpha$ increases, it is more likely that the lender ends up holding the ownership of the collateral trees, so he has higher incentive to get the information.

If $\gamma \geq \gamma^*_{IIS}$, then the information acquisition cost is so high that the lender has no incentive to acquire information, so the no information acquisition incentive constraint (5) does not bind. Under a secured loan contract, the borrower can default on the loan in a profitable way when he receives the private information about the dividend state at the beginning of period $t = 1$. Because the lender knows the possibility of opportunistic default, the borrower has to compensate the lender for taking such risk. This is given by the positive interest rate on the loan $r = \frac{(1-\alpha)(1-\sigma)\eta}{1-(1-\alpha)(1-\sigma)\eta}$. Note that if $\eta = 0$ so the borrower cannot default in a profitable way, then the interest rate is zero. As the probability that the borrower does not receive endowment $\alpha$ or the probability of good dividend state $\sigma$ increases, there is less chance for the borrower to default opportunistically. Thus, the interest rate and haircut fall as $\alpha$ and $\sigma$ rise.

However, when the information acquisition cost $\gamma$ is low such that $\gamma < \gamma^*_{IIS}$, the incentive constraint (5) starts to bind. Specific type of IIS loan contracts depends on the probability $\alpha$ that the borrower is not endowed in period $t = 1$ because of its effects on the information acquisition incentive of the lender. More precisely, as $\alpha$ decreases, the borrower will cede collateral trees to the lender with higher probability. Thus, the lender's incentive to acquire information rises.

If $\alpha \leq \alpha^*$ (IIS-2 type), then the borrower will be able to repay on the loan with relatively high probability, so the information acquisition incentive is relatively low. In this case, the borrower posts all trees as collateral, i.e., $a' = a$, and reduces the lender's information acquisition by lowering the loan size $q$ in (5). Note, that a decrease of $q$ relaxes the lender's participation constraint (4) more than no information incentive constraint (5). Therefore, in order to discourage the lender from the information acquisition, the borrower must provide a positive surplus, which is an informational rent, to the lender under IIS-2 loan contract, and this rent rises as the information acquisition incentive increases. Hence, the interest rate, $r = \frac{[\eta(1-\alpha)+\alpha]a - \frac{\gamma}{1-\sigma}}{(1-\alpha)(1-\eta)a + \frac{\gamma}{1-\sigma}}$, contains the compensation for the risk of opportunistic default and informational rent. Note that even if $\eta = 0$, the interest rate of IIS-2 loan contract is still positive. The effects of $\eta$ on the interest rate is same with IIS-1 case above: higher $\eta$ means higher probability of opportunistic default, so the borrower must provide a higher interest rate similar to IIS-1 loan contract. However, because of the binding no-information incentive constraint (5), the probability of exogenous default $\alpha$ has two opposing effects on the interest rate. First, as explained above, the borrower has less chance to default on the loan in an opportunistic way as $\alpha$ increases, which pushes down the interest rate. Second, as $\alpha$ increases, the probability that the lender ends up holding the ownership of the collateral trees increases. Therefore, the lender has more incentive to acquire the costly information about the future value of trees, which pushes up the interest rate. In the IIS-2 case, the second effect dominates the first effect, so the interest rate increases with $\alpha$. By the same reasoning, as the quantity of collateral trees $a$ increases or the information acquisition cost $\gamma$ decreases, the lender's information acquisition incentive rises, so the interest rate increases. Finally, because the lender concerns the bad dividend state of trees when he owns the collateral trees, the information acquisition incentive decreases with $\sigma$. Thus, the interest rate on the loan falls as $\sigma$ rises. In the extreme case, if trees yeild dividend always, i.e., $\sigma = 1$, then the information has no bites, and (5) does not bind always.

On the other hand, when $\gamma < \gamma^*_{IIS}$ and $\alpha > \alpha^*$, the borrower defaults on the loan with the relatively high probability. Thus, the lender has a high incentive to acquire the information about the dividend state. In this circumstance, it is too costly to discourage the lender from information acquisition by rendering the informational rent to the lender. Instead, the borrower posts only a fraction of trees as collateral, i.e., $a' < a$, and reduce $q$ and $p$, to reduce the lender's information acquisition incentive. Given all other things equal, as the quantity of collateral trees falls, the benefit of the information about the future value of trees decreases. Thus, the lender has less incentive to acquire the information given the fixed cost of information acquisition. The quantity of collateral trees $a' = \frac{\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)}$, here, increases with $\gamma$ and decreases with $\eta$ and $\alpha$ because of their effects on the lender's information acquisition incentive. Because the borrower does not give the informational rent to the lender in this case, the interest rate on the collateral loan, $r = \frac{(1-\alpha)(1-\sigma)\eta}{1-(1-\alpha)(1-\sigma)\eta}$ is same with the IIS-1 case.

Now consider the case where $k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$, so the borrower has an incentive to use fraudulent trees as a medium of exchangs, and hence the lender may be reluctant to trade with the borrower because of threat of fraud. In this environment, the borrower can mitigate the fraud incentive problem in the following way. As explained above, the borrower can save $(1-\alpha)[1-\eta(1-\sigma)]p + \alpha\sigma y a'$ units of consumption goods in expectation by giving fraudulent trees to the lender. Thus, given the quantity of collateral trees, $a'$, the benefit from fraud decreases as $p$ falls, while the cost of producing fraudulent mortgages does not change. Therefore, the borrower can give a signal about the quality of his collateral trees to the lender by over-collateralizing the loan, i.e., $p < \sigma y a'$. However, when $k < \alpha\sigma y$, the fraud incentive problem is so severe that the borrower cannot circumvent the fraud incentive problem and cannot issue IIS secured debts. On the other hand, if $\alpha\sigma y \leq k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$, the fraud incentive is not too high, and, hence, the IIS secured loan contracts are feasible with the binding fraud incentive constraint (6) and over-collateralization. The ratio of over-collateralization, $\frac{\sigma y a' - p}{\sigma y a'}$, is same for all types (IIS-4 to IIS-6) as $\frac{[1-\eta(1-\alpha)(1-\sigma)]\sigma y - k}{(1-\alpha)[1-\eta(1-\sigma)]\sigma y}$, and it increases (decreases) with $\alpha$ and $\sigma$ ($k$ and $\eta$), because of the effects of each parameter on the fraud incentive: the payoff from fraud, $(1-\alpha)[1-\eta(1-\sigma)]p + \alpha\sigma y a'$, increases with $\alpha$ and $\sigma$ (decreases with $\eta$), and the borrower has less incentive to commit fraud as the fraud cost $k$ increases.

Although the binding fraud incentive constraint (6) generates the over-collateralization, the effects of $\gamma$, $\alpha$, and $a'$ on the lender's information acquisition incentive and the structure of IIS repo contract, as one can see from lemma 1, are similar to the case with the non-binding fraud incentive constraint (6). For example, as the lender's incentive to acquire information about the dividend state increases, the borrower attempts to deter information acquisition by providing informational rent first (IIS-5) and then cuts the quantity of collateral trees instead of resorting on informational rent (IIS-6). In the following, we focus on the interest rate and haircuts that might provide new insights.

As explained above, IIS-4 and IIS-6 loan contracts do not contain informational rent. Thus, the interest rate is the same as $\frac{[\alpha+\eta(1-\alpha)(1-\sigma)]k - \alpha\sigma y}{(1-\alpha)[1-\eta(1-\sigma)]k}$ for both cases, and it increases with $\eta$ and decreases with $\alpha$ and $\sigma$ because of their effects on the opportunistic default of the borrower similar with IIS-1 and IIS-3 contracts. However, in these cases, fraud cost $k$ affects the interest rate owing to the binding fraud incentive constraint (6). More precisely, an increase of the fraud cost $k$ raises the repayment $p$ through the binding fraud incentive constraint (6). This, in turn, increases the loan size $q$ by relaxing the lender's participation constraint (4) or the no information acquisition incentive constraint (5). Then, as one can see from (4) and (5), an increase of $k$ raises $p$ more than $q$, so the interest rate of IIS-4 and IIS-6 increases with $k$. One interesting result is that if Lucas trees are a safe asset, i.e., $\sigma = 1$, the interest rate is $\frac{\alpha(k-y)}{(1-\alpha)k} < 0$ given $k < y$. Then, by continuity, the interest rate of IIS-4 or IIS-6 debt contracts collateralized by an asset with $\sigma \approx 1$ might feature the negative interest rate. This result provides an explanation for some indicative evidence on the negative interest rate in repo markets.

The interest rate of IIS-5 is a bit more complicated as $\frac{[\alpha+\eta(1-\alpha)](k-\alpha\sigma y)(1-\sigma)a-(1-\alpha)[1-\eta(1-\sigma)]\gamma}{(1-\alpha)\{(1-\eta)(k-\alpha\sigma y)(1-\sigma)a+[1-\eta(1-\sigma)]\gamma\}}$ because of the informational rent. The effects of parameters such as $\eta$, $a$, $\gamma$, and $\sigma$ on the interest rate are same with those of IIS-2 by the similar reasoning. The interest rate also increases with the fraud cost $k$ because of its effect on the value of repayment $p$ and loan size $q$. However, the effects of $\alpha$ on the interest rate is unclear in contrast to the IIS-2 type. As we explained above, when the incentive constraint of no information acquisition binds, an increase of $\alpha$ has two opposing effects on the interest rate. First, it lowers an opportunistic default possibility which pushes down the interest rate. Second, it tightenes the incentive constraint of no information acquisition by the lender, which pushes up the interest rate. In IIS-2 case, the second effect dominates the first one. In the IIS-5 type, there is a third effect: An increase of $\alpha$ lowers the repurchase price, $p$, through the binding fraud incentive constraint, which works as a force of lowering the interest rate. Combined all together, the effects of changing $\alpha$ on the interest rate is ambiguous.

Next, when the fraud cost $k$ is high enough that the fraud incentive constraint (6) does not bind, $p = \sigma y a'$. Thus, the interest rate on the loan $r = \frac{p-q}{q}$ and haircut $\theta = \frac{\sigma y a' - q}{\sigma y a'}$ show a positive correlation in responding to various source of frictions. However, when the fraud incentive constraint (6) binds, the secured loan is over-collateralized, i.e., $p < \sigma y a'$, and hence the interest rate on the loan $r = \frac{p-q}{q}$ does not represent the haircut $\theta = \frac{\sigma y a' - q}{\sigma y a'}$ precisely. Therefore, it seems worthwhile to spend a little time on the haircut. From the part 2 of lemma 1, we have $\theta = \frac{\sigma y - k}{\sigma y}$ for IIS-4 and IIS-6 types, and $\theta = 1 - \frac{(1-\eta)(k-\alpha\sigma y)(1-\sigma)a+[1-\eta(1-\sigma)]\gamma}{(1-\sigma)[1-\eta(1-\sigma)]\sigma y a}$ for IIS-5 type. Thus, the haircut and the interest rate on secured loan contracts can move in a different way in response to the change of parameters that represent economic environment and asset's properties. For example, as $\alpha$ or $\sigma$ rises, the interest rate on IIS-4 loan contract falls, but the haircut does not change. In particular, as the fraud cost $k$ increases, the interest rates rise but haircuts fall for all types (IIS-4 to IIS-6).

### 3.1.2 Information sensitive loan contracts (IS)

If the information acquisition cost $\gamma$ is too low, then the information insensitive (IIS) secured loan contracts may not be attractive to the borrower. Instead, an offer that triggers the information acquisition by the lender can be a better option. Notice that the borrower can always decide not to trade with the lender, which gives the payoff of $V = \sigma y a$ in expectation. Thus, if the information acquisition cost, $\gamma$, is not low enough, then the information sensitive (IS) contract is not profitable or, even it is not feasible. Here, we focus on IS loan contracts that give higher payoff to the borrower than no trade option, and impose the necessary conditions for those type of contracts to exist.

Under IS loan contracts, the lender trades with the borrower only if the dividend state is good, so the borrower cannot default opportunistically. Thus, the borrower's expected payment to the lender is $\sigma\left[(1-\alpha)p + \alpha y a'\right]$. Given our timing assumption that the borrower makes faking decision after making an offer to the lender but before the lender's acceptance decision, this expected payment should not be higher than the fraud cost $ka'$ to prevent the borrower from fraud, which generates the fraud incentive constraint for IS loan contract. Then, the borrower's maximized value under IS loan contracts, $V_{IS}$, is obtained by solving the following maximization problem:

$$V_{IS} = \underset{q,p,a'}{Max} \left\{\sigma[mq - (1-\alpha)p - \alpha y a'] + \sigma y a\right\} \tag{10}$$

subject to

$$(11) \qquad -\sigma q + (1-\alpha)\sigma p + \alpha\sigma y a' - \gamma \geq 0$$

$$(12) \qquad (1-\sigma)q - (1-\eta)(1-\alpha)(1-\sigma)p - \gamma \geq 0$$

$$(13) \qquad ka' - (1-\alpha)\sigma p - \alpha\sigma y a' \geq 0$$

$$(14) \qquad ya' - p \geq 0$$

$$(15) \qquad a - a' \geq 0$$

$$(16) \qquad q, p, a' \geq 0.$$

The objective function (10) is sum of borrower's expected suplus from trade and the expected value of trees. The inequality (11) is the lender's non-negative profit constraint with the information acquisition. (12) is the information acquisition incentive constraint that induces the lender to acquire the information about the dividend state of trees, and (13) is the fraud incentive constraint which states that the benefit from fraud with the information acquisition by the lender must be less than the cost of producing fraudulent trees. The inequality (14) is the incentive constraint for the borrower to make repayments on the loan, and (15) and (16) are the feasibility constraints. Notice, from (14), that the borrower has an incentive to repay on the loan instead of abandoning the collateral trees as long as $p \leq ya'$ because the lender trades with the borrower only if the dividend state of trees is good.

As explained above, when the lender acquires the information about the tree's dividend state, the borrower must compensate the lender for the information acquisition cost to make the lender accept the offer. Therefore, for IS loan contracts to exist, the information acquisition cost, $\gamma$, must be sufficiently low. In other words, there is no $\{q, p, a'\}$ that satisfies (11) - (16) if $\gamma$ is sufficiently high. For example, if $\gamma > [\alpha + (1-\alpha)\eta](1-\sigma)\sigma ya$, then (11) and (12) cannot be satisfied at the same time. Moreover, $V_{IS}$ must be higher than $\sigma ya$ because no trade is always a feasible option. Given these arguments, the next lemma describes the information sensitive (IS) secured loan contracts.

**Lemma 2** *Define the cutoff level of $\gamma$ as $\gamma_{IS}^{**} \equiv [\eta k + \alpha(1-\eta)\sigma y](1-\sigma)a$. Then, an IS secured loan contract has one of the following forms:*

1) *(IS-1) If $\sigma y \leq k$ and $\gamma \leq Min\left\{[\alpha + (1-\alpha)\eta](1-\sigma)\sigma ya, \frac{(m-1)\sigma ya}{m}\right\}$, then $q = ya - \frac{\gamma}{\sigma}$, $p = ya$, $a' = a$, and $V_{IS} = m(\sigma ya - \gamma)$*

2) *(IS-2) If $\alpha\sigma y \leq k < \sigma y$ and $\gamma \leq Min\left\{\gamma_{IS}^{**}, \frac{(m-1)ka}{m}\right\}$ where $\gamma_{IS}^{**} \equiv [\eta k + \alpha(1-\eta)\sigma y](1-\sigma)a$, then $q = \frac{ka-\gamma}{\sigma}$, $p = \frac{(k-\alpha\sigma y)a}{(1-\alpha)\sigma}$, $a' = a$, and $V_{IS} = (m-1)ka - m\gamma + \sigma ya$*

3) *Otherwise, IS secured loan contracts are not feasible or worse than no trade, i.e., $a' = 0$ and $V_{IS} = \sigma ya$*

**Proof.** See Appendix ■

Similar to the IIS loan contracts, if the fraud cost $k$ is lower than $\alpha\sigma y$, IS loan contracts are not feasible, and when $k$ is in a moderate range as $\alpha\sigma y \leq k < \sigma y$ (IS-2 type), the IS loan contract is over-collateralized. The difference from IS loan contracts is that for IS loan contracts to be feasible or profitable, the information acquisition cost $\gamma$ should be sufficiently low as explained above. More precisely, if $\gamma > [\eta Min\{k, \sigma y\} + \alpha(1-\eta)\sigma y]\,(1-\sigma)a$, then IS loan contracts are not feasible because it cannot satisfy the lender's participation constraint and induce the lender to acquire information at the same time. On the other hand, if $\gamma > \frac{Min\{k,\sigma y\}(m-1)a}{m}$, then IS loan contracts are worse than no trade even though they may be feasible.

Given the information acquisition, trade occurs only if the dividend state is good, and hence, the lender's acceptance of the offer reveals this information. Thus, the expected value of the collateral is $ya'$, and the ratio of over-collateralization and haircut are defined as $\frac{ya'-p}{ya'}$ and $\frac{ya'-q}{ya'}$ respectively. First, the over-collateralization ratio, which is same for IS-1 and IS-2 as $\frac{\sigma y-k}{(1-\alpha)\sigma y}$, increases with $\alpha$ and $\sigma$, and is decreasing in $k$ similar to IIS loan contracts. Second, the haircut is $\frac{\gamma}{\sigma ya}$ for IS-1 and $\frac{\sigma ya-ka+\gamma}{\sigma ya}$ for IS-2. Thus, it increases with the information acquisition cost $\gamma$ for both cases but decreases with the fraud cost $k$ for the

IS-2 loan contract. As one can see, over-collateralization ratio and haircuts do not depend on $\eta$ because the opportunistic default is not possible.

Next, under IS loan contracts, there is no reason for the borrower to provide informational rent to deter information acquisition, and the expected surplus of the lender is zero. Interest rates on the loan, $\frac{\gamma}{\sigma y a-\gamma}$ for IS-1 and $\frac{(1-\alpha)\gamma-\alpha(\sigma y-k)a}{(1-\alpha)(ka-\gamma)}$ for IS-2, respectively, are a compensation for the cost of information acquisition. Thus, the interest rate increases with the information acquisition cost, $\gamma$, and decreases with the quantity of trees, $a$, given the fixed cost of the information acquisition, in contract to IIS loan contracts. Note that the interest rate on IS-2 loan contract that is over-collateralized can be negative if $\gamma \approx 0$. Because trade occurs only if the dividend state is good under IS loan contracts, Lucas trees can be interpreted as safe asset if $\gamma \approx 0$. Thus, the negative interest rate when the fraud incentive constraint (13) binds is consistent with IIS loan contract cases. Further, similar to IIS loan contracts, a correlation between the interest rate and haircuts on IS-1 loan contract is positive, but the interest rate and haircut change in an opposite direction for IS-2 loan contract when $k$ changes.

Finally, the borrower's maximized value $V_{IS}$ does not depend on the counter-party risk $\alpha$ and $\eta$ under IS loan contracts. The intuition is as follows. First, because the borrower cannot default opportunistically under IS loan contracts, the probability $\eta$ that the borrower receives the information about the dividend state at the end of $t = 0$ does not matter for terms of contracts and maximized value $V_{IS}$. Second, under IS-1 contract, the probability $\alpha$ that the borrower does not receive endowments does not affect terms of contract because trade occurs when the dividend state is good and $p = ya'$. However, terms of IS-2 contract when the fraud incentive constraint (13) binds depends on $\alpha$ because of its effects on the fraud incentive. More precisely, whenever the loan is over-collateralized, i.e., $p < ya'$, it is more costly for the borrower to cede collateral trees $a'$ than to make repayment $p$. Therefore, as $\alpha$ increases, the borrower's expected payment to the lender, $\sigma\left[(1-\alpha)p + \alpha y a'\right]$, increases raising the borrower's incentive to produce fraudulent trees. In order to satisfy the binding fraud incentive constraint (13), the repayment $p$ falls. Therefore, an increase of $\alpha$ has two opposing effects on the borrower's expected payment: It increases the probability of losing the ownership of collateral trees but it lowers the size of repayment. These two effects are exactly cancelled out, so the initial loan size $q$ and borrower's maximized value $V_{IS}$ do not change.

#### 3.1.3 Induce information acquisition or not?

In the model, the borrower has a whole bargaining power and makes an offer to the lender at $t = 0$. Therefore, the borrower will optimally chooses collateralized debt contracts between IIS contracts and IS contracts by comparing the maximized value $V_{IIS}$ and $V_{IS}$ given in equations (3) and (10), respectively. Thus, the borrower's maximized value with collateralized debts is given as $V = Max\{V_{IIS}, V_{IS}\}$. If $V_{IIS} \geq V_{IS}$, then an IIS loan contract dominates an IS loan contract, and vice versa. What type of loan contract that the borrower would choose depends on the information acquisition cost $\gamma$, fraud cost $k$, and exogenous default probability $\alpha$, which is described in the following proposition.

**Proposition 1** *The type of secured loan contract that gives the highest payoff to the borrower among secured loan contracts is as follows:*

*1) Suppose* $\sigma y \leq k$*.*

*1-a) If* $\alpha \leq \alpha^*$*, there is* $\widehat{\gamma}_1 < \gamma^*_{IIS}$ *such that the secured loan type is i) IS-1 for all* $\gamma \in (0, \widehat{\gamma}_1)$*, ii) IIS-2 for all* $\gamma \in [\widehat{\gamma}_1, \gamma^*_{IIS})$*, and iii) IIS-1 for all* $\gamma \geq \gamma^*_{IIS}$*.*

*1-b) If* $\alpha > \alpha^*$*, there is* $\widetilde{\gamma}_1 < \gamma^*_{IIS}$ *such that the secured loan type is i) IS-1 for all* $\gamma \in (0, \widetilde{\gamma}_1)$*, ii) IIS-3 for all* $\gamma \in [\widetilde{\gamma}_1, \gamma^*_{IIS})$*, and iii) IIS-1 for all* $\gamma \geq \gamma^*_{IIS}$*.*

*2) Suppose* $[1 - \eta(1 - \alpha)(1 - \sigma)]\sigma y \leq k < \sigma y$*.*

*2-a) If* $\alpha \leq \alpha^*$*, there is* $\widehat{\gamma}_2 < \gamma^*_{IIS}$ *such that the secured loan type is i) IS-2 for all* $\gamma \in (0, Min\{\widehat{\gamma}_2, \gamma^{**}_{IS}\})$*, ii) IIS-2 for all* $\gamma \in [Min\{\widehat{\gamma}_2, \gamma^{**}_{IS}\}, \gamma^*_{IIS})$*, and iii) IIS-1 for all* $\gamma \geq \gamma^*_{IIS}$*.*

*2-b) If $\alpha > \alpha^*$, then there is $\widetilde{\gamma}_2 < \gamma^*_{IIS}$ such that the secured loan type is i) IS-2 for all $\gamma \in (0, Min\{\widetilde{\gamma}_2, \gamma^{**}_{IS}\})$, ii) IIS-3 for all $\gamma \in [Min\{\widetilde{\gamma}_2, \gamma^{**}_{IS}\}, \gamma^*_{IIS})$, and iii) IIS-1 for all $\gamma \geq \gamma^*_{IIS}$.*

*3) Suppose $\alpha\sigma y \leq k < [1 - \eta(1-\alpha)(1-\sigma)]\sigma y$.*

*3-a) If $\alpha \leq \alpha^{**}$, there is $\widehat{\gamma}_3 < \gamma^{**}_{IIS}$ such that the secured loan type is i) IS-2 for all $\gamma \in (0, \widehat{\gamma}_3)$, ii) IIS-5 for all $\gamma \in [\widehat{\gamma}_3, \gamma^{**}_{IIS})$, and iii) IIS-4 for all $\gamma \geq \gamma^{**}_{IIS}$.*

*3-b) If $\alpha > \alpha^{**}$, there is $\widetilde{\gamma}_3 < \gamma^{**}_{IIS}$ such that the secured loan type is i) IS-2 for all $\gamma \in (0, \widetilde{\gamma}_3)$, ii) IIS-2 for all $\gamma \in [\widetilde{\gamma}_3, \gamma^{**}_{IIS})$, and iii) IIS-4 for all $\gamma \geq \gamma^{**}_{IIS}$*

*4) If $k < \alpha\sigma y$, then a secured loan contract is not feasible.*

**Proof.** See Appendix ■

Although the specific type of the secured loan contract depends on the fraud cost $k$ in the proposition 1, it is more likely that IIS loan contracts give a higher payoff to the borrower than IS loan contracts as $\gamma$ increases. The intuition is is in line with our earlier observations. As one can see from lemmas 1 and 2, $V_{IIS}$ weakly increases with the information acquisition cost, $\gamma$, while $V_{IS}$ weakly decreases with $\gamma$. This is because an increase of $\gamma$ relaxes the no information acquisition incentive constraint (5) for IIS loan contracts while it means a higher information acquisition cost for IS loan contracts. Thus, the borrower makes an offer that induces the lender to acquire the information only if the information acquisition cost $\gamma$ is sufficiently low. This is illustrated in Figure 2 that describes $V_{IIS}$ and $V_{IS}$ with respect to $\gamma$ when $\sigma y \leq k$, and the maximized value of the borrower under secured loan contracts, $V$, is given by the upper line of both graphs. Because we analyzed terms of each contract such as an interest rate, haircuts, and over-collateralization in the previous subsection, we focus on other issues.

First, as $k$ decreases, the fraud incentive starts to matter for IS loan contracts first as one can see from proposition 1. This is because under IIS loan contracts, the borrower can default opportunistically, which lowers the expected repayment on the loan. This, in turn, implies

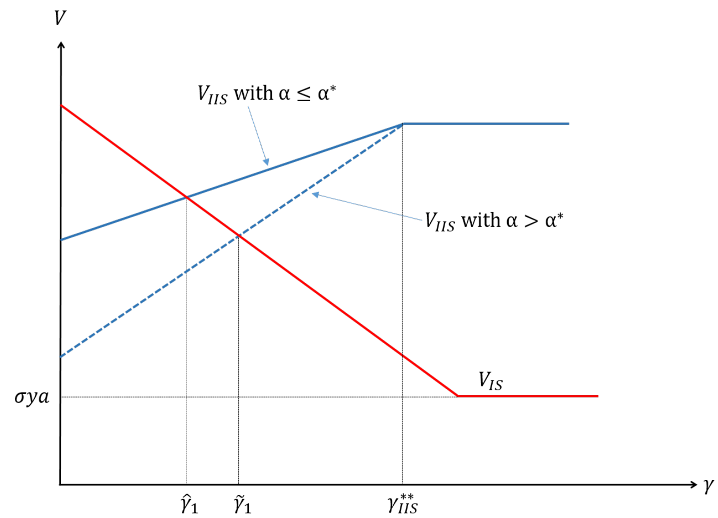


Figure 2: $V_{IIS}$ and $V_{IS}$ with respect to $\gamma$ when $\sigma y \leq k$

less benefit from producing fraudulent trees. Note that if $\eta = 0$ so the borrower cannot default opportunistically, then the second case in proposition 1, where $[1-\eta(1-\alpha)(1-\sigma)]\sigma y \leq k < \sigma y$, disappears, and the fraud incentive constraint binds for both IIS loan contracts and IS loan contracts when $k < \sigma y$.

Second, consider two limiting cases: 1) $\gamma \to \infty$ and 2) $\gamma \to 0$. When $[1-\eta(1-\alpha)(1-\sigma)]\sigma y \leq k$, the fraud incentive constraint of IIS loan contracts does not bind. In that case, as $\gamma \to \infty$, IIS-1 loan type is optimal among secured loan contracts and $V = [m - (m-1)(1-\alpha)(1-\sigma)\eta]\,\sigma ya$. On the other hand, $\lim_{\gamma \to 0} V = m\sigma ya$ under IS-1 loan contract if $\sigma y \leq k$ or $\lim_{\gamma \to 0} V = (m-1)ka + \sigma ya$ under IS-2 loan contract if $k < \sigma y$. In either cases, the borrower obtains higher surplus when $\gamma \to 0$ than when $\gamma \to \infty$. The reason is as follows. Under IIS-1 loan contract, the borrower can default opportunistically if he receives private signal about the dividend state. Thus, the borrower has to compensate the lender for taking such risk. When the lender acquires the information, there is no asymmetric information. Thus, the borrower cannot default in an opportunistic way, and the borrower does not need to pay for opportunistic default risk. Note that if $\eta = 0$ so the borrower does not receive private signal, then $\lim_{\gamma \to \infty} V = m\sigma ya$, which equals $\lim_{\gamma \to 0} V$ if $\sigma y \leq k$ and is strictly higher than $\lim_{\gamma \to 0} V$ if $k < \sigma y$ because of the binding fraud incentive constraint under IS-2 loan contract.

On the other hand, when $\alpha\sigma y \leq k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$, the fraud incentive constraint of IIS-4 loan contract binds as $ka = (1-\alpha)[1-\eta(1-\sigma)]p+\alpha\sigma ya$. Thus, the value of repayment, $p$, is adjusted as $\eta$ changes, and the borrower's expected payment is fixed as $ka$, which equals the loan size $q$ given the binding lender's participation constraint (4). Therefore, $V_{IIS}$ under IIS-4 loan contract does not depend on $\eta$, and $\lim_{\gamma\to\infty} V = \lim_{\gamma\to 0} V = (m-1)ka + \sigma ya$.

Third, if the fraud cost $k$ is sufficiently high, for instance, higher than $\sigma y$ as in the first case in proposition1, the fraud incentive constraints - equation (6) for IIS loan contracts and (13) for IS loan contracts - do not bind. In that case, over-collateralization does not exist, as one can see from lemmas 1 and 2. Therefore, default risk and lender's information acquisition incentive cannot explain the existence of over-collateralization practice in secured loan contracts in the model. Instead, interest rates reflect default risk and the information acquisition incentive.

On a related point, it seems worthwhile to discuss recent work on repo contracts in the context of information acquisition presented by Dang, Gorton, and Holmstrom (2012) who derived over-collateralization, which equals haircuts in their model, in a similar economic environment. However, they assumed that the interest rate should be zero, and this assumption drives a positive haircut in secured loan contracts. Given that the repo rate is zero, haircuts are same with the over-collateralization ratio. More precisely, a lender, in Dang, Gorton, and Holmstrom (2012), may have to sell collateral assets at a discounted price to the third party because the third party can learn the exact value of collateral assets at some cost similar to the information acquisition technology of our model. Obviously, when a borrower and a lender make a secured loan contract, the borrower must compensate, in any shape or form, the lender for the possibility that the lender has to sell collateral asset at a discounted price. However, Dang, Gorton, and Holmstrom (2012) did not allow a positive interest rate in their model, so the compensation was embodied in a secured loan contract in the form of over-collateralization, which is the haircut in their model. On the other hand, we extend the model to allow the interest rate can be positive, and show that over-collateralization occur not because of information acquisition incentive but because of the threat of fraudulent practice in financial markets.

Finally, if $k < \alpha\sigma y$, secured loan contracts are not feasible because of the threat of fraud. Furthermore, trees cannot be traded under the direct asset sale because an asset sale is a special case of a secured loan contract where $\alpha = 1$. Thus, trees are illiquid when the fraud incentive problem is severe similar to Li, Rocheteau, and Weill (2012) although a secured loan contract and an asset sale were treated equivalently in their model. The difference from Li, Rocheteau and Weill (2012) is that when the fraud incentive problem is severe, the entire trees are not traded in our model, while only a fraction of an illiquid asset is not traded in Li, Rocheteau and Weill (2012).

## 3.2 Collateralized debts vs. Asset sale

So far, we have focused on secured loan contracts. However, as argued before, the borrower can potentially sell trees to the lender on the spot in period 0. This asset sale can be interpreted as a special case of a secured loan contract with $\alpha = 1$ because, in that case, the borrower can never be able to repay on the loan so the lender always seizes the collateral trees. Thus, the borrower's maximized value with an asset sale, which is denoted as $V_{\alpha=1}$, can be obtained by plugging $\alpha = 1$ into proposition 1. Because trees are illiquid when $k < \alpha\sigma y$, we focus on the cases where $k \geq \alpha\sigma y$.

As one can see from lemma 1, necessary conditions to have the binding fraud incentive constraints are $\alpha\sigma y \leq k < [1 - \eta(1-\alpha)(1-\sigma)]\sigma y$ for IIS-type loan contracts and $\alpha\sigma y \leq k < \sigma y$ for IS-type loan contracts. If $\alpha = 1$, either of these two conditions cannot be satisfied. Therefore, if the optimal secured loan contract among secured loan contracts is over-collateralized, then an asset sale is not feasible option for trading trees, so the secured loan is the optimal contract. More precisely, by over-collateralizing secured loan contracts, which is not possible under a direct sale of trees, the borrower can give the lender signal about the authenticity of collateral trees he offers, circumventing the fraud incentive problem.

From a technical point of view, this result implies that it sufficies to look at the first case of proposition 1, where $\sigma y \leq k$, for comparing secured loan contracts and tree sales because, in other cases, tree sales are not a feasible option. The next lemma shows a property of IS-1 loan contract, which is a general property of information sensitive (IS) loan contracts, providing a useful intermediate step.

**Lemma 3** *Suppose $k \geq \sigma y$ and $\gamma < Min\{\widehat{\gamma}_1, \widetilde{\gamma}_1\}$ given $\alpha = \alpha_0$. Then, IS-1 is the best among collateralized debts for all $\alpha \geq \alpha_0$, and the borrower is indifferent between secured loan contracts and a direct sale of trees.*

**Proof.** See Appendix ■

In the proof of Lemma 3, we show that $\widehat{\gamma}_1 \leq \widetilde{\gamma}_1$ if and only if $\alpha \leq \alpha^*$. Therefore, when $k \geq \sigma y$ and $\alpha = \alpha_0$, IS-1 loan contract is optimal among secured loan contracts for all $\gamma < Min\{\widehat{\gamma}_1, \widetilde{\gamma}_1\}$. Then, Lemma 3 states that IS-1 loan contract is also optimal for all $\alpha \geq \alpha_0$, which implies that a direct sale of trees is also information sensitive because tree sale equals secured loan contract with $\alpha = 1$. Furthermore, under IS-1 loan contract, there is no asymmetric information problem, and the borrower's maximized value does not depend on $\alpha$ (see lemma 2). Therefore, whenever IS-1 loan contract is optimal among secured loan contracts, the borrower is indifferent between a secured loan contract and a tree sale. In the following, we focus on the case in which the borrower strictly prefers one type to the other.

First, suppose that $\eta = 0$, so the borrower cannot default on the loan in an opportunistic way in period 1. Then, under IIS-1 loan contract, when the no information acquisition incentive constraint does not bind, the borrower's maximized value, $V_{IIS}$, does not depend on $\alpha$ similar to the case with the information sensitive (IS) loan contracts. On the other hand, $V_{IIS}$ strictly decreases with $\alpha$ under the IIS loan contracts with the binding no information acquisition incentive constraint. That means $V = Max\{V_{IIS}, V_{IS}\}$ is weakly decreasing in $\alpha$, so secured loan contracts are either better than or equivalent to tree sales. Thus, secured loan contracts are always optimal contract similar with previous studies like Gottardi, Maurin, and Monnet (2015), Tomura(2015), Parlatore (2017). The intuitive explanation for this finding is as follows. Under a secured loan contract, the borrower takes the collateral trees when he receives endowment and makes repayment on the loan at $t = 1$. Hence, the lender has less incentive to acquire the costly information about the dividend state because the dividend state matters to the lender only if he becomes the owner of collateral trees. Therefore, the borrower can relax the no information acquisition incentive constraint by offering a secured loan contract rather than a direct sale of trees.

On the other hand, if the borrower receives a private information about the dividend state with a positive probability $\eta > 0$, the borrower can default on loans in a profitable way at period 1 even though he receives the endowments. The borrower must compenstate the lender for this opportunistic default on a loan contract. Because the borrower can default opportunistically only if he receives his endowment $e$, an increase of the probability $\alpha$ that the borrower does not receive the endowments lowers the possibility that the borrower takes an advantage of private information under IIS loan contracts. However, when the no information acquisition incentive constraint binds, an increase of $\alpha$ tightens the binding incentive constraint because as $\alpha$ increases, it is more likely that the lender ends up having collateral trees. This effect of an increase of $\alpha$ dominates the first effect on opportunistic default possibility as explained in the previous section. Thus, an increase of $\alpha$ reduces $V$ under IIS loan contracts with the binding no information acquisition incentive constraint.

However, when the information acquisition cost $\gamma$ is sufficiently high such that the no information acquisition incentive constraint does not bind as the IIS-1 loan contract, an increase of $\alpha$ only reduces the opportunistic default possibility. Hence, the borrower's maximized value, $V$, under the IIS-1 loan contract strictly increases with $\alpha$. Furthermore, note, from the first case with $\sigma y \leq k$ in proposition 1, that if $\gamma \geq \gamma^*_{IIS}$ at $\alpha = \alpha_0$, then the IIS-1 loan contract is optimal among secured loan contracts for all $\alpha \leq \alpha_0$ because $\gamma^*_{IIS} \equiv [\alpha + (1 - \alpha)\eta\sigma]\,(1 - \sigma)\sigma y a$, defined in lemma 1, increases with $\alpha$. Now suppose that there exists $\alpha' \in (0, 1)$ such that when $\alpha = \alpha'$, the IIS-1 contract is optimal among

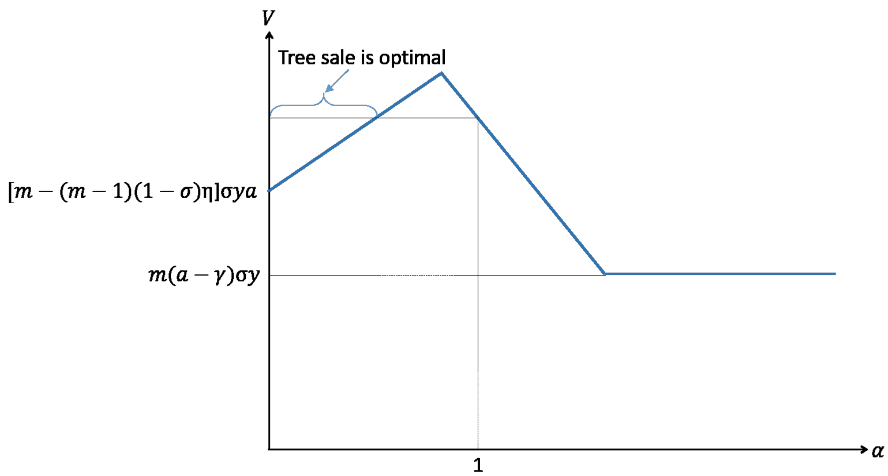


Figure 3: $V$ with respect to $\alpha$ when $\sigma y \leq k$

secured loan contracts and the borrower's maximized value $V$ equals the maximized value with tree sales $V_{\alpha=1}$. Then, a direct sale of trees is optimal for all $\alpha < \alpha_0$ because $V_{IIS}$ under IIS-1 contract increases with $\alpha$, as illustrated in Figure 3. Otherwise, secured loan contracts are optimal or the borrower is indifferent between both types of trading arrangement if the secured loan contract is information sensitive. The above analysis leads to the next proposition.

**Proposition 2** *1. Suppose $\alpha\sigma y \leq k < \sigma y$. Then an asset sale is not feasible, and, hence, secured loan contracts are optimal.*

*2. Suppose $\sigma y \leq k$. 2-a) If $\gamma < Min\{\widehat{\gamma}_1, \widetilde{\gamma}_1\}$, then secured loan contracts and asset sales are information sensitive and the borrower is indifferent between them. 2-b) If $Max\left\{\frac{(m-1)(1-\sigma)\sigma ya}{m-1+m(1-\sigma)}, [1-\eta(1-\sigma)](1-\sigma)\sigma ya\right\} \leq \gamma$, then for all $\alpha \in \left[0, \frac{\gamma-[1-\eta(1-\sigma)](1-\sigma)\sigma ya}{\eta(1-\sigma)^2\sigma ya}\right)$, tree sales are optimal. 2-c) Otherwise, the borrower prefers secured loan contracts to tree sales because secured loan contracts reduces the lender's information acquisition incentive.*

**Proof.** See Appendix ■

The above proposition show that secured loan contracts can be the optimal contract for two reasons. First, when the fraud incentive problem of misrepresenting the quality of trees is severe, a secured loan contract is optimal because over-collateralization in a secured loan contract mitigates the fraud incentive problem, allowing trees tradeable as a medium of exchanges. Thus, whenever a secured loan contract is over-collateralized, it must be the optimal contract. Second, even though the fraud incentive problem does not matter, a secured loan contract can be still optimal because it reduces the lender's incentive to acquire the costly information about the dividend state.

However, when the lender does not have any incentive to acquire the information because of the high acquisition cost, a secured loan contract only allows the borrower to default in a profitable way whenever it is possible. Thus, the lender faces a risk of opportunistic default by the borrower. Because the borrower must compensate the lender for taking such risk to make him accept the offer, a secured loan contract can be suboptimal, and a direct sale of trees emerges as the optimal contract.

# 4 Conclusion

In this paper, we construct a simple model to study the effects of the costly information acquisition and fraudulent practice on an optimal type of funding with an asset. In the model, the dividend of an asset follows a stochastic process, but a lender can acquire private information about the future dividend state at a cost. A borrower who owns an asset at the first period has an incentive to fake the quality of asset at a cost. The borrower can use the asset as a medium of exchange either by making secured loan contracts or selling the asset. The model is used to study the conditions under which secured loan contracts and asset sales are inequivalent, so one or the other emerges as an optimal trading arrangement for funding with the asset. Secured loan contracts can be optimal for two reasons. When the fraud incentive problem is severe, over-collateralization reduces the incentive to misrepresent the asset quality, and, hence, a secured loan contract is optimal. A secured loan contract can be also optimal because it reduces the lender's incentive to acquire costly information. However, under a secured loan contract, the borrower may default opportunistically. Thus, if both faking incentive and information acquisition incentive do not matter, an asset sale can be optimal.

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# Appendix: Omitted proofs

**Proof of Lemma 1.** We define the Lagrangian function for the optimal information insensitive repo contract problem (3) as

$$\begin{aligned}
L &= mq - (1-\alpha)[1-(1-\sigma)\eta]p - \alpha\sigma a' + \sigma ya + \lambda_1\left[-q + (1-\alpha)[1-(1-\sigma)\eta]p + \alpha\sigma ya'\right] \\
&\quad + \lambda_2\left[-(1-\sigma)q + (1-\eta)(1-\alpha)(1-\sigma)p + \gamma\right] + \lambda_3\left[ka' - (1-\alpha)[1-(1-\sigma)\eta]p - \alpha\sigma ya'\right] \\
&\quad + \lambda_4\left[\sigma ya' - p\right] + \lambda_5\left[a - a'\right] + \lambda_6 q + \lambda_7 p + \lambda_8 a'
\end{aligned}$$

where $\lambda_i$ for $i \in \{1, \dots 8\}$ are the Lagrange multipliers. The first order conditions are

$$\{q\} : m + \lambda_6 = \lambda_1 + \lambda_2(1-\sigma) \tag{17}$$

$$\{p\} : \lambda_4 - \lambda_7 = (1-\alpha)\left\{(\lambda_1 - \lambda_3 - 1)[1-\eta(1-\sigma)] + \lambda_2(1-\eta)(1-\sigma)\right\} \tag{18}$$

$$\{a'\} : \lambda_5 - \lambda_8 = (\lambda_1 - \lambda_3 - 1)\alpha\sigma y + \lambda_3 k + \lambda_4 \sigma y. \tag{19}$$

**Case 1 (IIS-1).** $\lambda_2 = \lambda_3 = 0$

From (17) - (19), we obtain $\lambda_1 > 0$, $\lambda_4 - \lambda_7 > 0$, and $\lambda_5 - \lambda_8 > 0$. Thus, $a' = a$, $p = \sigma ya$, $q = [1-\eta(1-\alpha)(1-\sigma)]\sigma ya$, and $V_{IIS} = (m-1)[1-\eta(1-\alpha)(1-\sigma)]\sigma ya + \sigma ya$. To have $\lambda_2 = \lambda_3 = 0$, it must be $\gamma \geq [\alpha + (1-\alpha)\eta\sigma](1-\sigma)\sigma ya$ and $k \geq [1-\eta(1-\alpha)(1-\sigma)]\sigma y$. ■

**Case 2 (IIS-2).** $\lambda_2 > 0$, $\lambda_1 = \lambda_3 = 0$

Because (5) binds in this case, $q > 0$ and $\lambda_6 = 0$. Then from (17) and (18), $\lambda_4 - \lambda_7 = (1-\alpha)[m-1-\eta(m-1+\sigma)]$. Suppose $\eta > \frac{m-1}{m-1+\sigma}$. Then, $\lambda_4 = 0$, $\lambda_7 > 0$. Then, $q = \frac{\gamma}{1-\sigma}$ by (5), and it must be $a' > 0$ to satisfy (4). However, from (19), we obtain $\lambda_5 - \lambda_8 < 0$, so $a' = 0$, a contradiction. Thus, we assume that $\eta \leq \frac{m-1}{m-1+\sigma}$ from now on. Then, $\lambda_4 - \lambda_7 \geq 0$. Thus, $p = \sigma ya'$. Suppose $a' = 0$. Then, $q = 0$ by (4) and it must be $\lambda_2 = 0$, which is a contradiction. Thus, $a' > 0$ must hold which implies $\lambda_7 = \lambda_8 = 0$. . From (19), we obtain

$$\lambda_5 = \sigma y \left\{(m-1)(1-\eta) - \eta\sigma - \alpha\left[(m-1)(1-\eta) - \eta\sigma + 1\right]\right\}. \tag{20}$$

Thus, $\alpha \leq \frac{(m-1)(1-\eta)-\eta\sigma}{(m-1)(1-\eta)-\eta\sigma+1}$ must hold and, hence, $a' = a$ and $p = \sigma ya$. Then, from the binding (5), $q = (1-\eta)(1-\alpha)\sigma ya + \frac{\gamma}{1-\sigma}$, and the maximized value is given as $V_{IIS} = m\left[(1-\eta)(1-\alpha)\sigma ya + \frac{\gamma}{1-\sigma}\right] + (1-\alpha)(1-\sigma)\eta\sigma ya$. Finally, to have $\lambda_1 = \lambda_3 = 0$, it must be $\gamma \leq [\alpha + (1-\alpha)\eta\sigma](1-\sigma)\sigma ya$ and $k \geq [1-\eta(1-\alpha)(1-\sigma)]\sigma y$. ■

**Case 3 (IIS-3).** $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 = 0$

Because $\lambda_2 > 0$, it must be $q > 0$ by (5) so $\lambda_6 = 0$. Further, if $a' = 0$ so $p = 0$, then (4) cannot satisifed. Thus, $a' > 0$ and $\lambda_8 = 0$ must hold. Next, from (17) - (19), we obtain

(21)
$$\lambda_4 - \lambda_7 = (1-\alpha)\left[(m-1)(1-\eta) + \eta\sigma(\lambda_1 - 1)\right]$$

(22)
$$\lambda_5 - \lambda_8 = \sigma y \left\{(m-1)(1-\eta) - \eta\sigma - \alpha[1 + (m-1)(1-\eta) - \eta\sigma] + \lambda_1[\alpha + (1-\alpha)\eta\sigma]\right\}.$$

Suppose $\lambda_7 \geq 0$, so $p = 0$ and $\lambda_4 = 0$. Then, it must be $\lambda_1 \leq 1 - \frac{(m-1)(1-\eta)}{\eta\sigma}$ to satisfy (21). Then, $\lambda_5 - \lambda_8 < 0$ so $a' = 0$, which contradicts with the binding (18). Thus, it must be $\lambda_4 \geq 0$, and $p = \sigma y a'$. Then, from the binding (4) and (5), we obtain $q = \frac{[1-\eta(1-\alpha)(1-\sigma)]\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)}$ and $a' = \frac{\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)\sigma y}$. Thus, $p = \frac{\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)}$ and $V_{IIS} = \frac{(m-1)[1-(1-\alpha)(1-\sigma)\eta]\gamma}{[\alpha+(1-\alpha)\eta\sigma](1-\sigma)} + \sigma y a$. Thus, if $\alpha \leq \frac{(m-1)(1-\eta)-\eta\sigma}{(m-1)(1-\eta)-\eta\sigma+1}$, then $a' = a$, it is the knife edge case of case 2. Thus, focus on the case where $a' < a$ which requires $\alpha > \frac{(m-1)(1-\eta)-\eta\sigma}{(m-1)(1-\eta)-\eta\sigma+1}$. Given this, one can always find $\lambda_1 \in (0, m)$ that makes $\lambda_4 \geq 0$ and $\lambda_5 = 0$ in (22). Finally, to have $\lambda_3 = 0$, $k \geq [1-\eta(1-\alpha)(1-\sigma)]\sigma y$ must hold. ■

**Case 4 (IIS-4).** $\lambda_2 = 0$, $\lambda_3 > 0$, $p > 0$

Given $\lambda_2 = 0$, $\lambda_1 = m + \lambda_6 > 0$, and we obtain, from (17 - (19),

(23)
$$\lambda_4 - \lambda_7 = (m + \lambda_6 - \lambda_3 - 1)(1-\alpha)[1 - \eta(1-\sigma]$$

(24)
$$\lambda_5 - \lambda_8 = \sigma y(m + \lambda_6 - \lambda_3 - 1)[1 - \eta(1-\alpha)(1-\sigma)] + \lambda_3 k.$$

Suppose $\lambda_4 > 0$, so $p = \sigma y a'$ and $\lambda_7 = 0$. Because $m + \lambda_6 - \lambda_3 - 1 > 0$ to have $\lambda_4 > 0$, $\lambda_5 - \lambda_8 > 0$. Thus, $a' = a$ and $q = [1-\eta(1-\alpha)(1-\sigma)]\sigma y a$ by the binding (4). To have $\lambda_3 > 0$, it must be $k = [1-\eta(1-\alpha)(1-\sigma)]\sigma y$. This is exactly the knife edge case of case 1. Thus, from now on, we assume that $\lambda_4 = 0$, which implies $p < \sigma y a'$. From (23) and (24), we get $\lambda_5 - \lambda_8 = \lambda_3 k > 0$. Thus, $a' = a$ and $\lambda_8 = 0$. Next, from the binding constraint (4), we obtain

$$q = (1-\alpha)[1-\eta(1-\sigma)]p + \alpha\sigma y a > 0, \tag{25}$$

so $\lambda_6 = 0$. Given $\lambda_3 > 0$, the constraint (6) must bind, which gives

$$p = \frac{(k-\alpha\sigma y)a}{(1-\alpha)[1-\eta(1-\sigma)]}. \tag{26}$$

Thus, it must be $\alpha\sigma y < k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$. Substituting (26) into (25), we obtain $q = ka$, and $V_{IIS} = (m-1)ka + \sigma y a$. Finally, to have $\lambda_2 = 0$, it must be $\frac{[\eta k + (1-\eta)\alpha y](1-\sigma)\sigma a}{1-\eta(1-\sigma)} \leq \gamma$. ■

**Case 5 (IIS-5).** $\lambda_1 = 0$, $\lambda_2 > 0$, $\lambda_3 > 0$

From the binding (5), we obtain $q = (1-\eta)(1-\alpha)p + \frac{\gamma}{1-\sigma} > 0$, so $\lambda_6 = 0$ and $\lambda_2 = \frac{m}{1-\sigma}$ by (17). Further, to satisfy the constraint (4), $a' > 0$ must hold given $q > 0$, so $\lambda_8 = 0$. Now suppose $\lambda_4 > 0$ so $p = \sigma y a'$ and $\lambda_7 = 0$. Next, from the binding constraint (6), we obtain

$$k = [1-\eta(1-\alpha)(1-\sigma)]\sigma y. \tag{27}$$

Then, from (18), (19), and (27), we get equation (20). Thus, $\alpha \leq \frac{(m-1)(1-\eta)-\eta\sigma}{(m-1)(1-\eta)-\eta\sigma+1}$ must hold and, hence, $a' = a$ and $p = \sigma y a$. This is exactly same with case 2 above.

Thus, focus on the case where $\lambda_4 = 0$ and, hence, $p < \sigma y a'$. If $p = 0$, then $k = \alpha\sigma y$ by (6). Then, from (19), $\lambda_5 = -\alpha\sigma y$, which is a contradiction. Therefore, it must be $p > 0$ and $\lambda_7 = 0$. From (18) and (19), we obtain $\lambda_5 \approx k - \frac{m(1-\eta)\alpha\sigma y}{(m-1)(1-\eta)-\eta\sigma}$. Therefore, $\alpha \leq \frac{[(m-1)(1-\eta)-\eta\sigma]k}{m(1-\eta)\sigma y}$ must hold, and, then, $a' = a$. Next, from the binding (5) and (6), we get $q = \frac{(1-\eta)(k-\alpha\sigma y)a}{1-\eta(1-\sigma)} + \frac{\gamma}{1-\sigma}$ and $p = \frac{(k-\alpha\sigma y)a}{(1-\alpha)[1-\eta(1-\sigma)]}$, and the maximized value of the objective is given as $V_{IIS} = m\left\{\frac{(1-\eta)(k-\alpha\sigma y)a}{1-\eta(1-\alpha)} + \frac{\gamma}{1-\sigma}\right\} - ka + \sigma y a$. Because $p \in (0, \sigma y a)$, $\alpha\sigma y < k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$ must hold. Finally, to satisfy the constraint (4) with $\lambda_1 = 0$, it must be $\gamma \leq \frac{[\eta k + \alpha(1-\eta)y](1-\sigma)\sigma a}{1-\eta(1-\sigma)}$. ■

**Case 6 (IIS-6).** $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$

Since $\lambda_2 > 0$ and the constraint (5) binds $q > 0$ and $\lambda_6 = 0$ must hold. Therefore, it must be $a' > 0$ and $\lambda_8 = 0$ to satisfy (4) and (7). Next, the binding constraint (6) gives $ka' = (1-\alpha)[1-\eta(1-\sigma)]p + \alpha\sigma y a'$. Now suppose $\lambda_4 > 0$ so $p = \sigma y a'$. If $a' = a$, then the result is same with case 1. On the other hand, if $a' < a$, then the result is same with case 3. Therefore, we focus on the case where $\lambda_4 = 0$ and $p < \sigma y a'$.

From (4) - (6), we obtain $q = \frac{[1-\eta(1-\sigma)]\gamma k}{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma}$, $p = \frac{(k-\alpha\sigma y)\gamma}{[\eta k+\alpha(1-\eta)y](1-\alpha)(1-\sigma)\sigma}$, and $a' = \frac{[1-\eta(1-\sigma)]\gamma}{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma}$. Next, because $p \in [0, \sigma y a')$, it must be $\alpha\sigma y \leq k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$, and, hence, $\lambda_7 = 0$. Then, from (17) - (19), we get

$$\lambda_1 = \frac{\lambda_3[1-\eta(1-\sigma)] - (m-1)(1-\eta) + \eta\sigma}{\eta\sigma} \tag{28}$$

$$\lambda_5 = \frac{\lambda_3[(1-\eta)\alpha y + \eta k] - (m-1)(1-\eta)\alpha y}{\eta}. \tag{29}$$

Because $\lambda_1 > 0$, then $\lambda_3 > \frac{(m-1)(1-\eta)-\eta\sigma}{1-\eta(1-\sigma)}$ must hold by (28). Then, substituting $\lambda_3 = \frac{(m-1)(1-\eta)-\eta\sigma}{1-\eta(1-\sigma)}$ into (29), we obtain

$$\lambda_5 > \frac{[(m-1)(1-\eta)-\eta\sigma]k - \alpha m(1-\eta)\sigma y}{1-\eta(1-\sigma)}.$$

Therefore, if $\alpha \leq \frac{[(m-1)(1-\eta)-\eta\sigma]k}{m(1-\eta)\sigma y}$, then $a' = a$, and the result is same with case 6. Thus, focus on the case where $\alpha > \frac{[(m-1)(1-\eta)-\eta\sigma]k}{m(1-\eta)\sigma y}$ and $a' < a$, which requires $\gamma < \frac{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma a}{1-\eta(1-\sigma)}$ by definition of $a'$ in this case. Then, the value of objective is given as $V_{IIS} = \frac{(m-1)[1-\eta(1-\sigma)]\gamma k}{[\eta k+\alpha(1-\eta)y](1-\sigma)\sigma} + \sigma y a$ ■

**Case 7 (No trade).** $\lambda_2 = 0$, $\lambda_3 > 0$, $p = 0$

In this case, we get, from (6), $ka' = \alpha\sigma y a'$. Thus, if $a' > 0$, it must be $k = \alpha\sigma y$ and $q = ka$ given $\lambda_1 = m + \lambda_6 > 0$, which is the knife edge case of case 4 above. Assume $a' = 0$, and hence, $q = 0$, which means that trees are not traded in period $t = 0$. Then, (24) becomes $-\lambda_8 = (m + \lambda_6 - 1)\alpha\sigma y - \lambda_3(\alpha\sigma y - k)$. Therefore, the necessary condition for this case to exist is $k < \alpha\sigma y$. ■

By defining $\gamma^*_{IIS} \equiv [\alpha + (1-\alpha)\eta\sigma]\,(1-\sigma)\sigma ya$, $\gamma^{**}_{IIS} \equiv \frac{[\eta k+(1-\eta)\alpha y](1-\sigma)\sigma a}{1-\eta(1-\sigma)}$, $\alpha^* \equiv \frac{(1-\eta)(m-1)-\eta\sigma}{1+(1-\eta)(m-1)-\eta\sigma}$, and $\alpha^{**} \equiv \frac{k[(m-1)(1-\eta)-\eta\sigma]}{m(1-\eta)\sigma y}$ and reorganizing the cases 1-7 above, we obtain lemma 1. ■

**Proof of Lemma 2.** If $a' = 0$, then $p = 0$ by (14). Then, (11) cannot be satisfied. Further, $q > 0$ must hold by the constraint (12). Therefore, it must be $q > 0$ and $a' > 0$. Suppose $p = 0$. Then, $k = \alpha\sigma y$ by (13). We will show that $p = 0$ is the limiting case where $p < ya'$ when $k$ goes to $\alpha\sigma y$. Thus, we focus on the case where $p > 0$. Then, the first-order conditions are

$$\lambda_1 = m + \frac{(1-\sigma)\lambda_2}{\sigma} \tag{30}$$

$$\lambda_4 = (1-\alpha)\left[(m-1)\sigma + \eta(1-\sigma)\lambda_2 - \sigma\lambda_3\right] \tag{31}$$

$$\lambda_5 = (\lambda_1 - \lambda_3 - 1)\alpha\sigma y + k\lambda_3 + y\lambda_4, \tag{32}$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, and $\lambda_5$ are the Lagrange multipliers for (11) - (15), respectively. Note, from (13), that $k \geq \frac{(1-\alpha)\sigma p}{a'} + \alpha\sigma y$. Thus, $\lambda_5 > 0$, and, hence, $a' = a$. Next, because $\lambda_1 > 0$ by (30). Thus, the constraint (11) must bind, so

$$q = (1-\alpha)p + \alpha ya - \frac{\gamma}{\sigma}. \tag{33}$$

**Case 1 (IS-1).** $\lambda_3 = 0$

In this case, $\lambda_4 > 0$ by (31), so $p = ya$, $q = ya - \frac{\gamma}{\sigma}$, and $V_{IS} = m(\sigma ya - \gamma)$. Then, constraints (12) and (13) requires $\gamma \leq [\alpha + (1-\alpha)\eta](1-\sigma)\sigma ya$ and $k \geq \sigma y$, respectively. Finally, the borrower can always choose no trade, which gives the borrower the payoff $\sigma ya$. Thus, it must be $V_{IS} = m(\sigma ya - \gamma) \geq \sigma ya$, which requires $\gamma \leq \frac{(m-1)\sigma ya}{m}$. Combined together, it must be $\gamma \leq Min\left\{[\alpha + (1-\alpha)\eta](1-\sigma)\sigma ya, \frac{(m-1)\sigma ya}{m}\right\}$ for this case to be the optimal IS repo contract. ■

**Case 2 (IS-2).** $\lambda_3 > 0$

Suppose $\lambda_4 > 0$. Then $p = ya$ and $k = \sigma y$, which is same with the previous case. Thus, focus on the case where $\lambda_4 = 0$ and $p < ya$. From the binding (13), $p = \frac{(k-\alpha\sigma y)a}{(1-\alpha)\sigma}$. Thus, $\alpha\sigma y \leq k < \sigma y$ mus hold. From (33), we get $q = \frac{ka-\gamma}{\sigma}$. Then, $V_{IS} = (m-1)ka - m\gamma + \sigma ya$. Finally, to satisfy the constraint (12) and $V_{IS} \geq \sigma ya$, it must be $\gamma \leq Min\left\{[\eta k + \alpha(1-\eta)\sigma y](1-\sigma)a, \frac{(m-1)ka}{m}\right\}$ ■

Note that we impose the condition to have IS repo contract gives a higher payoff than no trade, i.e., $V_{IS} \geq \sigma ya$. Thus, expect the above two cases, IS repo contract is either infeasible or worse than no trade, so the borrower would not make IS repo contract with the lender. By defining the cutoff level for $\gamma$ as $\gamma^{**}_{IS} \equiv [\eta k + \alpha(1-\eta)\sigma y](1-\sigma)a$ and summarizing the cases 1 and 2 above, we obtain lemma 2. ■

**Proof of Proposition 1.** To save space, let $V_{IIS-i}$, where $i \in \{1, \dots 6\}$, denote the maximized value of the borrower under the IIS-$i$ type repo contracts. For example, $V_{IIS-1} = [m - (m-1)(1-\alpha)(1-\sigma)\eta]\,\sigma ya$. Similarly, let $V_{IS-1}$ and $V_{IS-2}$ denote the maximized value of the borrower under the IS-1 and IS-2 type repo contract, respectively. The optimal repo contract can be obtained by comparing $V_{IIS}$ and $V_{IS}$. The key is that $V_{IIS}$ is weakly increasing in $\gamma$ while $V_{IS}$ is decreasing in $\gamma$, which implies the following remark.

If $V_{IIS} = V_{IS}$ at $\gamma = \gamma_0$, then $V_{IIS} \gtreqless V_{IS}$ for all $\gamma \gtreqless \gamma_0$

From lemmas 1 and 2, we can divide analysis into three groups: 1) $\sigma y \leq k$, 2) $[1-\eta(1-\alpha)(1-\sigma)]\sigma y \leq k < \sigma y$, and 3) $\alpha\sigma y \leq k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$.

**Case 1.** $\sigma y \leq k$.

In this case, IIS-1, IIS-2, IIS-3, and IS-1 are candidates for the optimal repo contracts. First, note that for all $\gamma \geq \gamma^{*}_{IIS}$, $V_{IIS-1} \geq V_{IS-1}$ given $\sigma \geq \frac{m-1}{m}$. Note, from lemmas 1 and

2, that

(34)

$$V_{IIS-2} \geq V_{IS-1} \text{ iff } \gamma \geq \frac{\{m[\alpha + (1-\alpha)\eta] - (1-\alpha)(1-\sigma)\eta\}(1-\sigma)\sigma ya}{(2-\sigma)m} \equiv \widehat{\gamma}_1$$

(35)

$$V_{IIS-3} \geq V_{IS-1} \text{ iff } \gamma \geq \frac{(m-1)[\alpha + (1-\alpha)\eta\sigma](1-\sigma)\sigma ya}{(m-1)[1-(1-\alpha)(1-\sigma)\eta] + m(1-\sigma)[\alpha + (1-\alpha)\eta\sigma]} \equiv \widetilde{\gamma}_1.$$

A simple algebra shows that $\frac{(m-1)\sigma ya}{m} > \widetilde{\gamma}_1$. Next, it can be verified that if $\alpha \leq \alpha^*$, $\frac{(m-1)\sigma ya}{m} > \widehat{\gamma}_1$. Finally, note that $\gamma^*_{IIS} < [\alpha + (1-\alpha)\eta](1-\sigma)\sigma ya$ by definition of $\gamma^*_{IIS}$, and, hence, $\widehat{\gamma}_1$ and $\widetilde{\gamma}_1$ are lower than the value, $[\alpha + (1-\alpha)\eta](1-\sigma)\sigma ya$. From the above analysis, we can obtain Figure 2, which proves the part 1 of the proposition 1. ■

**Case 2.** $[1-\eta(1-\alpha)(1-\sigma)]\sigma y \leq k < \sigma y$

In this case, IIS-1, IIS-2, IIS-3, and IS-2 are candidates for the optimal repo contracts. Similarly, $V_{IIS-1} > V_{IS-2}$ when $\gamma = \gamma^*_{IIS}$, and from lemmas 1 and 2,

$$V_{IIS-2} \geq V_{IS-2} \text{ iff } \gamma \geq \frac{\{\{1-(1-\alpha)[m(1-\eta) + (1-\sigma)\eta]\}\sigma ya + (m-1)ka\}(1-\sigma)}{(2-\sigma)m} \equiv \widehat{\gamma}_2$$

$$V_{IIS-3} \geq V_{IS-2} \text{ iff } \gamma \geq \frac{(m-1)[\alpha + (1-\alpha)\eta\sigma](1-\sigma)ka}{(m-1)[1-(1-\alpha)(1-\sigma)\eta] + m(1-\sigma)[\alpha + (1-\alpha)\eta\sigma]} \equiv \widetilde{\gamma}_2.$$

In this case, we need to compare $\frac{(m-1)ka}{m}$, $\widehat{\gamma}_2$, and $\widetilde{\gamma}_2$. First, a simple algebra shows $\frac{(m-1)ka}{m} > \widetilde{\gamma}_2$. Next, using the condition that $\alpha \leq \alpha^*$, one can show that $\frac{(m-1)ka}{m} > \widehat{\gamma}_2$. However, in this case, it is possible that $\gamma^{**}_{IS}$, defined in lemma 2, is smaller than $\widehat{\gamma}_2$ and $\widetilde{\gamma}_2$. If $\gamma^{**}_{IS} < \widehat{\gamma}_2$, for example, then for all $\gamma \in [\gamma^{**}_{IS}, \gamma^*_{IIS})$, IIS-2 is optimal when $\alpha \leq \alpha^*$. On the other hand, if $\gamma^{**}_{IS} \geq \widehat{\gamma}_2$, then for all $\gamma \in [\widehat{\gamma}_2, \gamma^*_{IIS})$, IIS-2 is optimal when $\alpha \leq \alpha^*$. The similar argument applies when $\alpha > \alpha^*$. Then, using the remark 1, we obtain the part 2 of the proposition 1.

■

**Case 3.** $\alpha\sigma y \leq k < [1-\eta(1-\alpha)(1-\sigma)]\sigma y$

In this case, we have to compare IIS-4, IIS-5, and IIS-6 with IS-2. Note that $V_{IIS-4} >$

$V_{IS-2}$ for any $\gamma > 0$. By definition of $V_{IIS-5}$, $V_{IIS-6}$, and $V_{IS-2}$, we obtain

$$V_{IIS-5} \geq V_{IS-2} \text{ iff } \gamma \geq \frac{[k\eta + \alpha(1-\eta)y](1-\sigma)\sigma a}{(2-\sigma)[1-(1-\sigma)\eta]} \equiv \widehat{\gamma}_3$$
$$V_{IIS-6} \geq V_{IS-2} \text{ iff } \gamma \geq \frac{(m-1)[k\eta + \alpha(1-\eta)y]k(1-\sigma)\sigma a}{(m-1)[1-\eta(1-\sigma)]k + m[k\eta + \alpha(1-\eta)y](1-\sigma)\sigma} \equiv \widetilde{\gamma}_3.$$

Observe that $\frac{(m-1)ka}{m} > \widetilde{\gamma}_3$. Similar to the above cases, it can be verified that $\frac{(m-1)ka}{m} > \widehat{\gamma}_3$ if $\alpha \leq \alpha^{**}$, where $\alpha^{**}$ is defined in lemma 1. Finally, a simple algebra shows $\gamma_{IS}^{**} > \widehat{\gamma}_3$ and $\gamma_{IS}^{**} > \widetilde{\gamma}_3$ given $k \geq \alpha\sigma y$. Thus, if $\alpha \leq \alpha^{**}$, for example, IS repo contract - either IIS-4 or IIS-5, is optimal for all $\gamma \geq \widehat{\gamma}_3$. Using the remark 1 and the above results, we can prove the part 2 of the proposition 1.Then, using the remark 1, we obtain the part 3 of the proposition 1. ■

**Case 4.** $k < \alpha\sigma y$

From lemmas 1 and 2, one can see that repo contracts - either IIS or IS- are not feasible if $k < \alpha\sigma y$. ■

Finally, by reorganizing the analysis of the above 4 cases, we obtain the proposition 1. ■

**Proof of Lemma 3.** From (34) and (35), it can be verified that $\widehat{\gamma}_1$ and $\widetilde{\gamma}_1$ increases with $\alpha$. Thus, if $\gamma \leq Min\{\widehat{\gamma}_1, \widetilde{\gamma}_1\}$ when $\alpha = \alpha_0$, then $\gamma < Min\{\widehat{\gamma}_1, \widetilde{\gamma}_1\}$ for all $\alpha \geq \alpha_0$. Next, from (34) and (35), we obtain $\widehat{\gamma}_1 \geq \widetilde{\gamma}_1$ if and only if

$$(1-\sigma)\left\{\begin{array}{c} \alpha\left[1+(1-\eta)(m-1)-\eta\sigma\right] \\ -(1-\eta)(m-1)+\eta\sigma \end{array}\right\}\{m\alpha + \eta(1-\alpha)(m\sigma - m + 1)\} \geq 0.$$

Then, because $\sigma \geq \frac{m-1}{m}$, $\widehat{\gamma}_1 \geq \widetilde{\gamma}_1$ if and only if $\alpha \geq \alpha^* \equiv \frac{(m-1)(1-\eta)-\eta\sigma}{(m-1)(1-\eta)-\eta\sigma+1}$. Thus, if $\gamma < Min\{\widehat{\gamma}_1, \widetilde{\gamma}_1\}$, then IS-1 type is the best among collateralized debt contracts by the first part of proposition 1 when $k \geq \alpha\sigma y$. Combined together, IS-1 is the best among collateralized debts for all $\alpha \geq \alpha_0$. Furthermore, the borrower's maximized value does not depend on $\alpha$ under IS-1 loan contract (see lemma 2). Therefore, whenever IS-1 loan contract is optimal among secured loan contracts, the borrower is indifferent between a secured loan contract and a tree sale. ■

**Proof of Proposition 2.** We already proved the first part in the main body and lemma 3 proves 2-a) part. Furthermore, once we prove 2-b) part of proposition 2, then 2-c) part is straightforward. Thus, we focus on the proof of 2-b) part of proposition 2 here. As one can see from proposition 1, the type of loan contract with $\alpha = 1$ can be IIS-1, IIS-3, or IS-1 when $\sigma y \leq k$. For tree sales to be optimal, IIS-1 loan contract should be best among secured loan contracts because if the other type of secured loan contract is the best among secured loan contracts, then loan contract always gives higher payoff than tree sales (or at least the same payoff with tree saes). Thus, we compare the borrower's maximized value $V$ with IIS-1 loan contract and the maximized value with tree sales, $V_{\alpha=1}$. For the IIS-1 to be the best loan contract among secured loan contracts, it must be $\gamma \geq \gamma^*_{IIS} \equiv [\alpha + (1-\alpha)\eta\sigma](1-\sigma)\sigma y a$, which requires $\alpha \leq \frac{\gamma - \eta(1-\sigma)\sigma^2 y a}{(1-\eta\sigma)(1-\sigma)\sigma y a}$. Then, it must be

$$\gamma > \eta(1-\sigma)\sigma^2 y a, \tag{36}$$

because $\alpha \geq 0$. Now, we consider three possiblce cases for the type of loan contract with $\alpha = 1$ which is same with direct sales of trees.

First, suppose that the type of loan contact with $\alpha = 1$ is IIS-1. Then, tree sales are optimal because $V$ is increasing in $\alpha$. When $\alpha = 1$, $\gamma^*_{IIS} = (1-\sigma)\sigma y a$. Thus, the necessary condition for this case is

$$\gamma \geq (1-\sigma)\sigma y a. \tag{37}$$

Second, suppose that the type of loan contract with $\alpha = 1$ is IIS-3, which requires $\gamma \in [\widetilde{\gamma}_1, \gamma^*_{IIS})$ with $\alpha = 1$, and the borrower's maximized value is given by $V_{\alpha=1} = \frac{(m-1)\gamma}{1-\sigma} + \sigma y a$. By substituting $\alpha = 1$ into the definitions of $\widetilde{\gamma}_1$ and $\gamma^*_{IIS}$, which are given in the proof of case

1 of proposition 1 and lemma 1, respectively, we obtain the necessary condition for IIS-3 to be the best with $\alpha = 1$ as

$$\frac{(m-1)(1-\sigma)\sigma ya}{m-1+m(1-\sigma)} \leq \gamma < (1-\sigma)\sigma ya. \tag{38}$$

Note, from lemma 1 and lemma 2, that the borrower's maximized value, $V$, under IIS-1 loan contract is $(m-1)[1-\eta(1-\alpha)(1-\sigma)]\sigma ya + \sigma ya$, and tree sales delievers the payoff $V_{\alpha=1} = \frac{(m-1)\gamma}{1-\sigma} + \sigma ya$ to the borrower. Thus, tree sales are better than secured loan contract so are optimal only if $\alpha \leq \frac{\gamma - [1-\eta(1-\sigma)](1-\sigma)\sigma ya}{\eta(1-\sigma)^2\sigma ya}$. Since $\alpha \geq 0$, the necessary condition is

$$[1-\eta(1-\sigma)](1-\sigma)\sigma ya \leq \gamma. \tag{39}$$

Third, suppose that the type of loan contract with $\alpha = 1$ is IS-1, and the maximized value is $V_{\alpha=1} = (m-1)\sigma ya - m\gamma + \sigma ya$. Then, the the borrower's maximized value $V$ under IIS-1 contract is

$$\begin{aligned} V &= [m-(m-1)(1-\alpha)(1-\sigma)\eta]\,\sigma ya \\ &> [m-(m-1)(1-\sigma)\eta]\,\sigma ya \\ &> (m-1)\sigma ya - m\gamma + \sigma ya, \end{aligned}$$

where the second inequality comes from the condition of (36). Thus, if the type of loan contract with $\alpha = 1$ is IS-1, then the IIS-1 type loan contract dominates tree sales.

Combining (36) - (39), we obtain the necessary condition for tree sales to be optimal as $Max\left\{\frac{(m-1)(1-\sigma)\sigma ya}{m-1+m(1-\sigma)}, [1-\eta(1-\sigma)](1-\sigma)\sigma ya\right\} \leq \gamma$. Given this condition, tree sales are optimal for all $\alpha \leq \frac{\gamma-[1-\eta(1-\sigma)](1-\sigma)\sigma ya}{\eta(1-\sigma)^2\sigma ya}$, which finishes the the proof of 2-b). ■