Population Sustainability of Social and Economic Networks

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Abstract

In the traditional framework of the network formation model, the set of agents is assumed to be fixed. In this paper, we investigate how the network changes in the variation of the set of agents. More precisely, for a given set of agents, suppose that a pairwise stable network is initially obtained. If a new agent enters the society, then the initial network may not be pairwise stable anymore. Starting from the initial network, a new network will be constructed by an improving path. Eventually, for the new set of agents, a pairwise stable network or networks in a closed cycle will be obtained. We define four different notions of population sustainabilities which relates between the initial and the new networks: *link sustainability, distance sustainability, connection sustainability,* and *graph sustainability.* First, we show that pairwise stability is not compatible with *link sustainability* under mild assumptions on allocation rules. However, if we consider specific models such as the connections and the coauthor models, the complete network is the only network which is *link sustainable* all the time.

JEL classification: C70, D70, D85.

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1. Introduction

An analysis on the social and economic network is concerned with the following situation: given a set of agents, two agents can add a link if they agree on, and any agent can sever a link if he wants to; the link is beneficial, but costly. In most studies (Jackson and Wolinsky [1996], Watts [2001], Jackson and Watts [2002], and others), agents are assumed to behave myopically: agents do not consider how their behavior might affect others' decision.[1] An improving path is a sequence of adjacent networks that can be obtained when agents add and sever links based on the myopic expectation. A pairwise stable network is the one in which agents do not have an incentive to add or sever the links.[2] Jackson and Wolinsky [1996] introduce and analyze the *pairwise stable* network in a static setting, and show that it may not be compatible with efficiency. Watts [2001] studies the dynamic process of network formation in a specific model, the connections model. Jackson and Watts [2002] extend the dynamic network formation model to a general setting in which agents could form or sever links by mistake and investigate the implications of stochastic stability.

In this paper, we ask how a pairwise stable network changes in the variation of the set of agents assuming the agents behave myopically. More precisely, for a given set of agents, suppose that a pairwise stable network is initially obtained. If a new agent enters the society, then the initial network may not be pairwise stable anymore. Consequently, starting from the initial network, a new network for the new set of agents will be constructed by an improving path. This dynamic process of network formation will lead to a pairwise stable network or a closed cycle. For a given network, we define its limiting networks as the pairwise stable network or the networks in a closed cycle obtained through the dynamic process. We investigate whether the relations between the original agents in the initial network remain un-

[1]Exceptions are Dutta, Ghosal, and Ray [2005], Watts [2002], Deroian [2001], and others.

[2]For other definitions of stability, see for example Dutta and Mutuswami [1997], Jackson and van den Nouweland [2005], and Slikker and van den Nouweland [2001].

changed in its limiting networks. Our analysis can be applied to analyze how a pairwise stable network changes upon an arrival of a new neighbor in a town.

We introduce four different notions of population sustainabilities to make a comparison between two networks; *link sustainability, distance sustainability, connection sustainability,* and *graph sustainability. Link sustainability* requires that upon the arrival of a new agent, the direct connection between the original agents would not be affected. *Distance sustainability* requires that the distance between any two original agents be unaffected. *Connection sustainability* requires that the connectedness between any two original agents would not be affected. *Graph sustainability* requires that the arrival of a new agent would not affect the graph at all: no severance of the existing links and no addition of new links.

First, we show that for a certain class of allocation rules, pairwise stability is not compatible with *link sustainability.*

Second, we analyze their implications in the contexts of the connections and the coauthor models (Jackson and Wolinsky [1996]). In those models, a complete pairwise stable network is always *link sustainable,* but any other pairwise stable network may not be *link sustainable.* Moreover, in the symmetric connections model, any pairwise stable network is *connection sustainable.* Also, depending on the values of the parameters in the model, we can identify their ranges when a pairwise stable network is either *graph sustainable* or *distance sustainable.*

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 introduces four notions of population sustainability and establishes the logical relations between them. Section 4 presents impossibility results on a general domain. Section 5 investigates their implications on specific models such as the connections and the coauthor models. Concluding remarks follow in Section 6. All the proofs are in the Appendix.

2. A Model of Networks

We follow the terminology of Jackson and Wolinsky [1996] and Jackson and Watts [2002], but modify the notation to allow variations in the set of agents.

Agents

Let $\mathbb{N} \equiv \{1, 2, \ldots\}$ be a (finite or infinite) universe of "potential" agents. Let $\mathcal{N}$ be the family of nonempty finite subsets of $\mathbb{N}$, with elements denoted by N and N'.

Graphs

The relation between agents in a network is represented by a graph in which a node represents an agent and a link captures a pairwise relation. For all $N \in \mathcal{N}$, let L^N be the set of all subsets of N with size 2. A *network*, or a *graph*, is $g \equiv (N, L)$, where $N \in \mathcal{N}$ and $L \subseteq L^N$. For $N \in \mathcal{N}$, let $\mathcal{G}^N$ be the set of all graphs for N, and $\mathcal{G} \equiv \{\mathcal{G}^N | N \in \mathcal{N}\}$. For all i, $j \in N$, the (undirected) *link* between i and j is denoted ij. For all $g = (N, L) \in \mathcal{G}$, if $ij \in L$, then nodes i and j are directly connected under g, and if $ij \notin L$, then nodes i and j are not directly connected under g. For all $g \in \mathcal{G}$, let $N(g)$ be the set of nodes under g with at least one link, and $L(g)$ be the set of links under g. For all $N \in \mathcal{N}$, let $e^N \equiv (N, \emptyset)$ be the *empty network for* N, and $g^N \equiv (N, L^N)$ be the *complete network for* N. A network $g = (N, L)$ is a *singleton network* if $|N| = 1$.

For all $N \in \mathcal{N}$, let $N^c \equiv \mathbb{N} \setminus N$ be the set of agents not in N. For all $g = (N, L) \in \mathcal{G}$ and all $k \in N^c$, let $g \oplus k \in \mathcal{G}^{N \cup \{k\}}$ be the graph obtained by adding a new agent k to N without affecting the set of links, that is, $g \oplus k \equiv (N \cup \{k\}, L)$. For all $g = (N, L) \in \mathcal{G}$ and all i, $j \in N$ such that $ij \notin L$, let $g + ij = (N, L \cup \{ij\})$ be the graph obtained by adding the link ij to g. And for all $g = (N, L) \in \mathcal{G}$ and all i, $j \in N$ such that $ij \in L$, let $g - ij = (N, L \backslash \{ij\})$ be the graph obtained by deleting the link ij from g. If $g' = g + ij$ or $g' = g - ij$, then g and g' are *adjacent*.

A *chain* in $g = (N, L) \in \mathcal{G}$ connecting i_1 and i_n is a set of distinct nodes $\{i_1, i_2, \ldots, i_n\} \subseteq N$ such that $\{i_1 i_2, i_2 i_3, \ldots, i_{n-1} i_n\} \subseteq L$. If such a chain exists, then nodes i_1 and i_n are *connected*. A graph $g = (N, L) \in \mathcal{G}$ is *connected* if for any two distinct nodes $i, j \in N$ there is a chain in g connecting i and j. A graph $g = (N, L) \in \mathcal{G}$ is a *subgraph* of $g' = (N', L') \in \mathcal{G}$ if $N \subseteq N'$ and $L \subseteq L'$, written as $g \subseteq g'$. A connected subgraph g' of $g \in \mathcal{G}$ is a *component of* g if for any $g'' \in \mathcal{G}$ with $g' \subset g'' \subseteq g$ and $g'' \neq g'$, g'' is not connected. In other words, a component of g is a maximal connected subgraph of g. We note that an isolated node is included as a component in our definition. Let $C(g)$ be the set of all components of g.

For $g = (N, L) \in \mathcal{G}$ and $i, j \in N$, the (geodesic) *distance* between i and j under g, $d(i, j; g)$, is the smallest number of links connecting i and j. If there is no chain connecting i and j in g, we set $d(i, j; g) = \infty$, and if $i = j$, then $d(i, j; g) = 0$.

Value Function and Allocation Rule

A *value function* is a function $v : \mathcal{G} \to \mathbb{R}$, which associates with any $g = (N, L) \in \mathcal{G}$ a value in $\mathbb{R}$. We normalize v by setting the value of singleton components equal to zero. For $N \in \mathcal{N}$, let $\mathcal{V}^N$ be the set of all value functions for N, and $\mathcal{V} \equiv \{\mathcal{V}^N | N \in \mathcal{N}\}$.

An allocation rule, or a *rule,* is a function $Y : \mathcal{G} \times \mathcal{V} \to \cup_{N \in \mathcal{N}} \mathbb{R}^N$, which associates to any $((N, L), v) \in \mathcal{G} \times \mathcal{V}$ a vector in $\mathbb{R}^N$. It allocates the value of a network to agents.

A rule Y is *budget balanced* if, for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $g = (N, L) \in \mathcal{G}^N$, $\sum_{i \in N} Y_i(g, v) = v(g)$. A value function v is *component additive* if for all $N \in \mathcal{N}$ and all $g \in \mathcal{G}^N$, $v(g) = \sum_{h \in C(g)} v(h)$. A rule Y is *component balanced* if for all $N \in \mathcal{N}$, all component additive $v \in \mathcal{V}^N$, all $g \in \mathcal{G}^N$, and all $h \in C(g)$, $v(h) = \sum_{i \in N(h)} Y_i(g, v)$.

For all $N \in \mathcal{N}$, let Π^N be the class of permutations from N to N. For all $g = (N, L) \in \mathcal{G}$ and all $\pi \in \Pi^N$, let $g^\pi \equiv \{ij \mid i = \pi(k), j = \pi(l),\ \text{for } kl \in L\}$ be the graph obtained by permutating nodes in g by π. Let $v^\pi : \mathcal{G} \to \mathbb{R}$ be defined by $v^\pi(g^\pi) \equiv v(g)$. A value function v is *anonymous* if for all $g = (N, L) \in \mathcal{G}$ and all $\pi \in \Pi^N$, $v(g^\pi) = v(g)$. A rule Y is *anonymous* if for all $g = (N, L) \in \mathcal{G}$, all $\pi \in \Pi^N$, and all $i \in N$, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.

Pairwise Stability

As in Jackson and Wolinsky [1996], we assume that the formation of a link requires the consent of both parties, but the severance can be done unilaterally. Moreover, agents are assumed to behave myopically, that is, they do not forecast how adding and severing a link might affect the future formation of a network. A network is *pairwise stable* if no player would benefit by severing an existing link and no two players would benefit by forming a new link. Formally, a network $g = (N, L) \in \mathcal{G}^N$ *defeats* an adjacent network $g' = (N, L') \in \mathcal{G}^N$ if either (i) for some $ij \in L'$, $g = g' - ij$ and $Y_i(g, v) > Y_i(g', v)$, or (ii) for some $ij \in L$, $g = g' + ij$, $Y_i(g, v) > Y_i(g', v)$ and $Y_j(g, v) \geq Y_j(g', v)$. A network is *pairwise stable* if it is not defeated by any adjacent network.

Improving Path and Cycle

An *improving path* (Jackson and Watts [2002]) from $g \in \mathcal{G}^N$ to $g' \in \mathcal{G}^N$ is a finite sequence of adjacent networks $\{g_1, \ldots, g_l\}$ with $g_1 = g$ and $g_l = g'$ such that, for any $t = 1, \ldots, l-1$, g_t is defeated by g_{t+1}. It captures the sequence of improvement in networks when agents form and sever links based on the myopic expectation.

For all $N \in \mathcal{N}$, a set of networks $C \subseteq \mathcal{G}^N$ is a *cycle* if for any g, $g' \in C$ there exists an improving path from g to g'. A cycle C is a *closed cycle* if no network in C lies on an improving path leading to a network that is not in C.

Limiting Network

For all $N \in \mathcal{N}$, $g' \in \mathcal{G}^N$ is a *limiting network* of $g \in \mathcal{G}^N$ if g' is a pairwise stable network or a network in a closed cycle that can be obtained by an improving path from g. If g' is pairwise stable, then g can improve to g'. When g' is obtained, it will be sustained. If g' is in a closed cycle, then g can improve to g', but g' will be repeatedly obtained. If a network is pairwise stable, then its limiting network is itself. Using the limiting networks, we can discuss a dynamic process of network formation through improving paths.

From the proof of Lemma 1 in Jackson and Watts [2002], for any v and Y, each network that is not pairwise stable lies on an improving path to a pairwise stable network or a network in a closed cycle. This establishes the existence of the limiting network. For all $g \in \mathcal{G}$, let $\mathcal{L}(g)$ be the set of all limiting networks of g.

3. Population Sustainability

A pairwise stable network $g = (N, L) \in \mathcal{G}$ is expected to remain unchanged when the set of agents is fixed. However, if a new agent $k \in N^c$ enters the problem, then the network may not be pairwise stable. Therefore, we may have an improving path from $g \oplus k \in \mathcal{G}^{N \cup \{k\}}$ to another network $g' \in \mathcal{G}^{N \cup \{k\}}$. If g' is pairwise stable or in a closed cycle for the new society, then we can expect that g' is sustained or repeatedly obtained. We investigate how the relations between the original agents will be affected when the pairwise stable network changes due to the arrival of a new agent. We propose four different population sustainabilities analyzing the relations between original agents.

3.1. Link Sustainability

Link sustainability requires that upon the arrival of a new agent, the links between all the original agents would not be affected. In other words, it requires that the direct relation between the original agents remains to be unchanged even though the set of agents is changed. All the initial links should not be severed and an additional link, if any, should connect between the new agent and an original agent.

Link Sustainability: For all $N \in \mathcal{N}$, all $k \in N^c$, all pairwise stable network $g = (N, L) \in \mathcal{G}^N$, and all $g' \in \mathcal{L}(g \oplus k)$, $L(g) = L(g') \cap L^N$.

Now we present an example of a *link sustainable* network by introducing the connections model in Jackson and Wolinsky [1996].

Example 1: *Connections model.* For all $g = (N, L) \in \mathcal{G}$ and all $i,\ j \in N$, let $\bar{u}_i$ be his reservation utility he can obtain when he does not form any link, $w_{ij} \in [0, \infty)$ be the *intrinsic value* of j to i, and $c_{ij} \in [0, \infty)$ be the *cost* to i of maintaining the link ij. If i and j make a link, then i's utility gain depends on the intrinsic value and the distance between them. More specifically, for some $\delta \in]0, 1[$, the utility gain is equal to w_{ij} discounted by $\delta^{d(i,j;g)}$. Then, i's utility under g is obtained by adding up his reservation utility, and the sum of utilities that he obtains by making links with other agents minus the link cost.

$$u_i(g) = \bar{u}_i + \sum_{j \neq i,\ j \in N} \delta^{d(i,j;g)} w_{ij} - \sum_{j : ij \in L} c_{ij}. \tag{1}$$

And the value of a network g is $v(g) \equiv \sum_{i \in N} u_i(g)$ and the allocation rule is $Y_i(g, v) = u_i(g)$. For each $N \in \mathcal{N}$, let $\mathcal{O}^N$ be the set of all connections model for N, and $\mathcal{O} \equiv \cup \mathcal{O}^N$.

The *symmetric connections* model is of particular interest. In this model, all reservation utilities are equal to zero, all intrinsic values are equal to one, and all agents have the same cost of making a link. Therefore, i's utility is given as follows:

$$u_i(g) = \sum_{j \neq i,\ j \in N} \delta^{d(i,j;g)} - \sum_{j : ij \in L} c. \tag{2}$$

Now, let $N \equiv \{1, 2, 3, 4\}$, $\delta = .9$, and $c = .2$. Let $g^* = (N, \{12, 23, 34\})$. Note that g^* is pairwise stable. Let $k \in N^c$ be a new agent, and $g_0 \equiv g^* \oplus k$. We can show that improving paths from g_0 to a pairwise stable network are $\{g_0, g_0 + 2k\}$, $\{g_0, g_0 + 3k\}$, $\{g_0, g_0 + 1k, g_0 + 1k + 4k\}$, and $\{g_0, g_0 + 4k, g_0 + 1k + 4k\}$ as shown in Figure 1. Since $L(g^*) = L(g_0 + 2k) \cap L^N = L(g_0 + 3k) \cap L^N = L(g_0 + 1k + 4k) \cap L^N = \{12, 23, 34\}$, g^* is *link sustainable.* $\square$

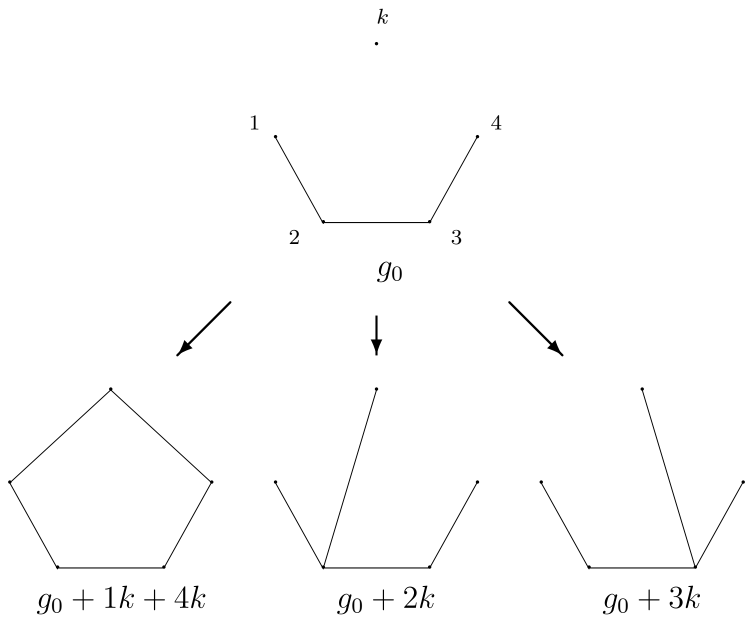

Figure 1 Link sustainability

3.2. Distance Sustainability

Link sustainability is concerned with the *direct* relation between the initial agents. However, when a new agent enters the problem, the *indirect* relation between the initial agents may have been changed. In Example 1, for any $k \in N^c$, $d(1,4;g^*) = 3 \neq 2 = d(1,4;g_0 + 1k + 4k)$, so that the distance between 1 and 4 has been decreased from 3 to 2. Even though the direct relation between two agents are not changed, the indirect relation has been affected.

To analyze such a situation, we propose a stronger notion of sustainability, which requires that the distance between any two original agents be unaffected upon the arrival of a new agent.

Distance Sustainability: For all $N \in \mathcal{N}$, all i, $j \in N$, all $k \in N^c$, all pairwise stable network $g \in \mathcal{G}^N$, and all $g' \in \mathcal{L}(g \oplus k)$, $d(i,j;g) = d(i,j;g')$.

Next, we present an example of a *distance sustainable* network.

Example 2: A *distance sustainable* network in the symmetric connections model. Let $N \equiv \{1,2,3,4\}$, $\delta = .9$ and $c = .5$. Once again, g^* is pairwise stable. Let $g_0 \equiv g^* \oplus k$. For any $i \in N$, the path $\{g_0, g_0 + ik\}$ is an improving path and $g_0 + ik$ is pairwise stable (Figure 2). Moreover, there does not exist any other improving path. Since for all $i \in N$ and all $h, j \in N$, $d(h,j;g^*) = d(h,j;g_0 + ik)$, g^* is *distance sustainable.* $\square$

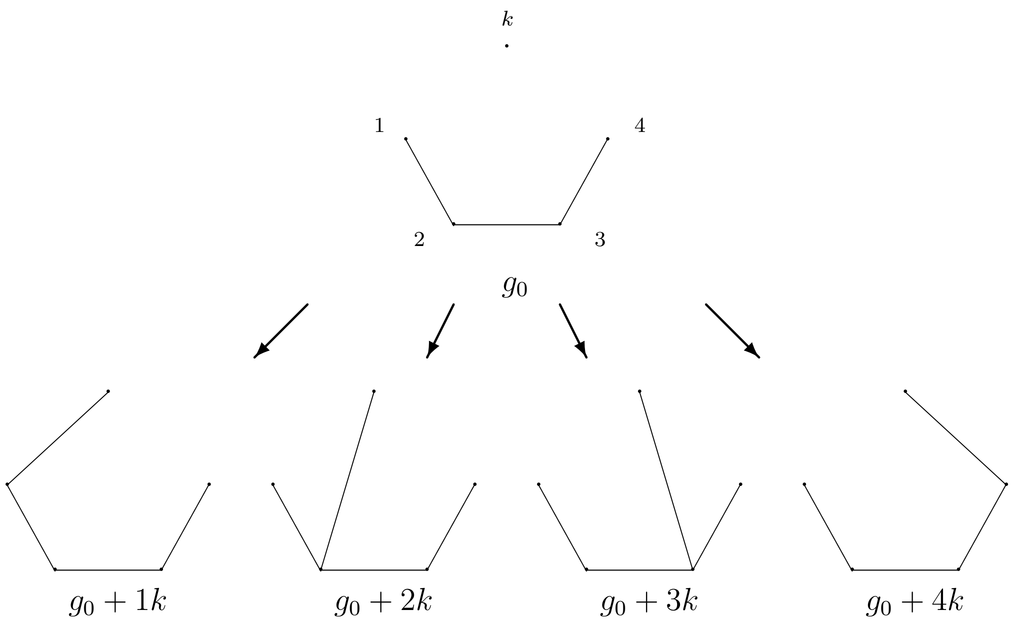

Figure 2 Distance sustainability

3.3. Connection Sustainability

Our third sustainability is concerned with the connectedness between two agents. *Connection sustainability* requires that the connectedness between any two agents would not be affected upon the arrival of new agent. In fact, we can easily check that g^* in Examples 1 and 2 are *connection sustainable.* For all $g = (N, L) \in \mathcal{G}$, let $N_i(g) \equiv \{j \in N | d(i,j;g) < \infty\}$.

Connection Sustainability: For all $N \in \mathcal{N}$, all $k \in N^c$, all pairwise stable network $g \in \mathcal{G}^N$, all $g' \in \mathcal{L}(g \oplus k)$, and all $i \in N$, $N_i(g) = N_i(g') \setminus \{k\}$.

3.4. Graph Sustainability

Our last notion, *graph sustainability,* requires that an arrival of a new agent would not affect the graph at all: the new agent does not make a link with the original agent and the existing links between original agents are not affected. It can easily be shown that *graph sustainability* implies that other three sustainabilities.

Graph Sustainability: For all $N \in \mathcal{N}$, all $k \in N^c$, and all pairwise stable network $g \in \mathcal{G}^N$, $g \oplus k$ is pairwise stable in $\mathcal{G}^{N \cup \{k\}}$.

Example 3: A *graph sustainable* network in the symmetric connections model. Let $N \equiv \{1, \ldots, 16\}$, $c = 1$ and $\delta = .9$. In this model, Jackson and Wolinsky [1996] show that the tetrahedron in Figure 3 is pairwise stable. Let g be the tetrahedron. Let $k \in N^c$ be a new agent. Since $c > \delta$, any $i \in N$ becomes worse-off if he adds ik to $g \oplus k$. Thus, k is unconnected. Since g is pairwise stable in $\mathcal{G}^N$, any $i \in N$ does not have an incentive to sever or form a link with any other agent in N. Therefore, g is *graph sustainable.* $\square$

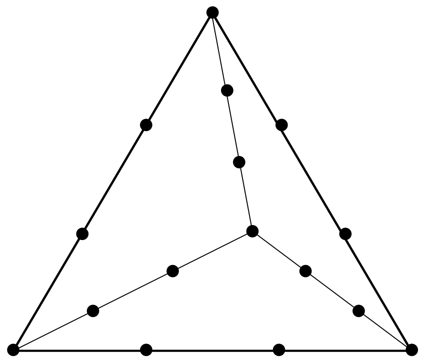

Figure 3 Graph sustainability

Remark 1: As noted earlier, *graph sustainability* implies *distance sustainability,* which in turn implies *link sustainability* and *connection sustainability.* For a graph with one component, *link sustainability* implies *connection sustainability.* Otherwise, there is no direct logical relation between them.

4. Impossibility Results

Now we investigate whether a pairwise stable network can be *link sustainable* for a general allocation rule. Our main result here is negative: we show that a pairwise stable network can not be *link sustainable* if an allocation rule satisfies some properties. For this result, we introduce two more axioms.

Our first axiom, *weak link symmetry,* requires that if a new link is profitable to one of the two agents forming the link, then it must be profitable to the other agent. It was introduced by Dutta, van den Nouweland, and Tijs [1998] in the context of communication games, where the value of a network primarily depends only on the connectivity of a graph. It is much weaker than *fairness* (Myerson [1977]), which requires that a new link should affect the two agents forming the link by the same amount.

Weak Link Symmetry: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $g = (N, L) \in \mathcal{G}^N$, and all $i, j \in N$ such that $ij \notin L$, $Y_i(g+ij, v) > Y_i(g, v)$ implies $Y_j(g+ij, v) > Y_j(g, v)$.

Improvement, also introduced by Dutta, van den Nouweland, and Tijs [1998] in the context of communication games, requires that the formation of a new link cannot benefit a player who is not involved in the link without benefitting at least one of the two players involved in the link. Our second axiom, *weak improvement,* requires that *improvement* should hold only when the formation of a new link increases the value of the network.

Weak Improvement: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $g = (N, L) \in \mathcal{G}^N$, and all $i, j \in N$ such that $ij \notin L$, if $v(g + ij) > v(g)$, and for some $k \in N \setminus \{i, j\}$, $Y_k(g+ij, v) > Y_k(g, v)$, then $Y_i(g+ij, v) > Y_i(g, v)$ or $Y_j(g+ij, v) > Y_j(g, v)$.

We are now ready to present our main negative result. A pairwise stable network with more than two nodes is not *link-sustainable* provided that an allocation rule satisfies certain axioms.

Theorem 1: *There is no allocation rule which satisfies component balance, anonymity, weak link symmetry, and weak improvement, and that for all*

$N \in \mathcal{N}$ and all component additive $v \in \mathcal{V}^N$, at least one pairwise stable network g with more than two nodes is link sustainable.

5. Applications: Possibility Results

Despite the negative result, if we restrict our attention to specific models, we can have positive results. As in Jackson and Wolinsky [1996], we will consider the connections and the coauthor models.

5.1. The general connections model

To figure out whether a pairwise stable network in a model is *link-sustainable,* we should know all the parametric values of the model. However, in some models, we can answer the question only looking at the network. We will show that in the general connections model, the complete network is *link sustainable* all the time. Moreover, if a network is not complete, then it may not be *link sustainable.* For all $g = (N, L) \in \mathcal{G}$, let $\mathcal{O}(g)$ be the class of all connections models in which g is pairwise stable.

Theorem 2: *A pairwise stable network $g = (N, L) \in \mathcal{G}$ is link sustainable for all $O \in \mathcal{O}(g)$ if and only if g is the complete network.*

As discussed in Remark 1, *distance sustainablity* implies *link sustainability.* Moreover, for the complete network, its converse is true. Altogether, we have the following corollary.

Corollary: *A pairwise stable network $g = (N, L) \in \mathcal{G}$ is distance sustainable for all $O \in \mathcal{O}(g)$ if and only if g is the complete network.*

5.2. The symmetric connections model

In the symmetric connections model, we can establish additional positive results. All the pairwise stable networks are *connection sustainable.* Moreover, depending on the values of parameters, a pairwise stable network can be *graph sustainable* or *distance sustainable.*

Theorem 3: *In the symmetric connections model,*

(i) *Every pairwise stable network is connection sustainable.*

(ii) *If $c > \delta$, every pairwise stable network is graph sustainable.*

(iii) *If $c < \delta - \delta^2$, every pairwise stable network is distance sustainable.*

5.3. The coauthor model

Now we consider the coauthor model in Jackson and Wolinsky [1996]. For all $g = (N, L) \in \mathcal{G}$ and all $i \in N$, let $n_i \geq 0$ be the number of projects that i is involved in. If $n_i = 0$, $u_i(g) \equiv 0$. Otherwise, i's utility is defined to be:

$$u_i(g) \equiv \sum_{j \in N : ij \in L} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right].$$

Furthermore, $v(g) = \sum_{i \in N} u_i(g)$ and $Y_i(g, v) = u_i(g)$.

As in the connections model, the complete network is *link sustainable,* but any incomplete pairwise stable network is not *link sustainable.*

Theorem 4: *In the coauthor model, a pairwise stable network $g = (N, L) \in \mathcal{G}$ is link sustainable if and only if g is the complete network.*

Once again, as in Theorem 2, *link sustainablity* can be replaced by *distance sustainability.*

6. Conclusion

In this paper, we consider the situation when only one new agent enters into a pairwise stable network. In other words, we consider two pairwise stable networks merging into one when one of them is singleton. It would be interesting if we can develop a general theory describing what happens if two pairwise stable networks merge.

Another interesting question is to ask what happens to a pairwise stable network if an agent leaves. The answer depends on the role of the leaving agent in the network. For example, suppose that a star network is pairwise stable in a symmetric connections model. Since all agents have the same intrinsic value and cost, it does not matter who enters the network as a new agent. However, the network obtained after deleting the center node is completely different from the network obtained by deleting one of any other nodes.

We hope to address these issues in our future research.

Appendix

Now we present the proofs for all theorems.

Proof of Theorem 1: Let Y be a rule satisfying *component balance, anonymity, weak link symmetry,* and *weak improvement.* It suffices to show that, for some component additive value function, any pairwise stable network with more than two nodes is not *link sustainable.* Let $N \in \mathcal{N}$ be such that $|N| > 2$, and $g \in \mathcal{G}^N$, and $\alpha > 0$. For $h \in C(g)$, let $\bar{v}^\alpha$ be defined by:

$$\bar{v}^\alpha(h) = \begin{cases} 0, & \text{if } h = g^{N(h)} \text{ and } |N(h)| > 2, \\ \alpha \cdot |L(h)|, & \text{otherwise.} \end{cases}$$

Now we define the value function v^α as follows:

$$v^\alpha(g) = \sum_{h \in C(g)} \bar{v}^\alpha(h).$$

Note that v^α satisfies *component additivity* and *anonymity.*

Step 1: *Let i, $j \in N$ be such that $ij \notin L(g)$. If $v^\alpha(g + ij) > v^\alpha(g)$, then $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g + ij, v^\alpha) > Y_j(g, v^\alpha)$.*

Proof. Let i, $j \in N$ be such that $ij \notin L(g)$ and $v^\alpha(g + ij) > v^\alpha(g)$. First, suppose that for some $k \in N \setminus \{i, j\}$, $Y_k(g + ij, v^\alpha) > Y_k(g, v^\alpha)$. By *weak improvement,* $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ or $Y_j(g + ij, v^\alpha) > Y_j(g, v^\alpha)$. By *weak link symmetry,* we have both $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g + ij, v^\alpha) >$

$Y_j(g, v^\alpha)$. Next, suppose that for all $k \in N \setminus \{i, j\}$, $Y_k(g+ij, v^\alpha) \leq Y_k(g, v^\alpha)$. Since for a component additive value function, *component balance* implies *budget balance,* $\sum_{i\in N} Y_i(g + ij, v^\alpha) = v^\alpha(g + ij) > v^\alpha(g) = \sum_{i\in N} Y_i(g, v^\alpha)$. Therefore, we must have either $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ or $Y_j(g + ij, v^\alpha) > Y_j(g, v^\alpha)$. By *weak link symmetry,* we have both $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g + ij, v^\alpha) > Y_j(g, v^\alpha)$.

Step 2: *If g is a graph obtained by deleting more than one link from the complete network g^N with $|C(g)| = 1$, then for all i, $j \in N$ such that $ij \notin L(g)$, g is defeated by $g + ij$.*

Proof. Let g be a graph obtained by deleting more than one link from the complete network g^N with $|C(g)| = 1$. By definition of v^α, it is obvious that for all i, $j \in N$ such that $ij \notin L(g)$, $v^\alpha(g + ij) > v^\alpha(g)$. By Step 1, $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g + ij, v^\alpha) > Y_j(g, v^\alpha)$. Therefore, g is defeated by $g + ij$.

Step 3: *Let g be such that $|C(g)| > 1$ and i, $j \in N$ belong to two different components of g. Then, g is defeated by $g + ij$.*

Proof. Let g be such that $|C(g)| > 1$ and h_i, $h_j \in C(g)$ be the components of g that contains i and j, respectively. Let $h_i \equiv (N_i, L_i)$ and $h_j \equiv (N_j, L_j)$. Let $h \equiv (N_i \cup N_j, L_i \cup L_j \cup ij)$. Since all the components other than h, h_i, and h_j are not changed, we can compare the values of g and $g + ij$ by focusing only on the values of h, h_i and h_j. First, suppose that both h_i and h_j are singleton, which implies that $\bar{v}^\alpha(h_i) = \bar{v}^\alpha(h_j) = 0$. Since h is complete with only two nodes, $\bar{v}^\alpha(h) = \alpha$, so that $\bar{v}^\alpha(h) > \bar{v}^\alpha(h_i) + \bar{v}^\alpha(h_j)$. Next, suppose that at least one of h_i and h_j has more than one node. Since h is obtained by adding only one link between h_i and h_j, h is not complete. Therefore, $\bar{v}^\alpha(h) > \bar{v}^\alpha(h_i) + \bar{v}^\alpha(h_j)$. In any case, $\bar{v}^\alpha(h) > \bar{v}^\alpha(h_i) + \bar{v}^\alpha(h_j)$, which implies that $v^\alpha(g + ij) > v^\alpha(g)$. By Step 1, $Y_i(g + ij, v^\alpha) > Y_i(g, v^\alpha)$ and $Y_j(g + ij, v^\alpha) > Y_j(g, v^\alpha)$. Therefore, g is defeated by $g + ij$.

Step 4: *If $g = g^N$, then for all $ij \in L(g)$, g is defeated by $g - ij$.*

Proof. Let $g = g^N$. Since $|N| > 2$, g has more than two nodes. Therefore, $v^\alpha(g) = 0$. By *component balance* and *anonymity,* for all $i \in N$, $Y_i(g, v^\alpha) = 0$. For some $i \in N$, let $g_0 = (N, \{lm \in g | l \neq i \text{ and } m \neq i\})$ be the network obtained by deleting all of i's links from g. By *component balance,* $Y_i(g_0, v^\alpha) = 0$. Let $N \setminus \{i, j\} = \{l_1, \ldots, l_{|N|-2}\}$ and let $g_1 = g_0 + il_1$, $g_2 = g_1 + il_2$, $\ldots$, $g_{|N|-2} = g_{|N|-3} + il_{|N|-2}$. Note that $g_{|N|-2} = g - ij$. Since $v^\alpha(g_0) < v^\alpha(g_1) < \cdots < v^\alpha(g_{|N(g)|-2})$, by Step 1, $Y_i(g_0, v^\alpha) < \cdots < Y_i(g_{|N(g)|-2}, v^\alpha) = Y_i(g - ij, v^\alpha)$. Since $Y_i(g, v^\alpha) = 0 = Y_i(g_0, v^\alpha)$, $Y_i(g, v^\alpha) < Y_i(g - ij, v^\alpha)$. Therefore, g is defeated by $g - ij$.

Step 5: *A network g is pairwise stable if and only if for some i, $j \in N$, $g = g^N - ij$.*

Proof. We prove the "only if" part. Suppose by way of contradiction that there does not exist i, $j \in N$ such that $g = g^N - ij$. Then g is either g^N or a network obtained by deleting more than one link from g^N. If $g = g^N$, by Step 4, for each $ij \in L(g)$, g is defeated by $g - ij$. Otherwise, by Steps 2 and 3, for some $ij \notin L(g)$, g is defeated by $g + ij$, a contradiction.

We now prove the "if" part. For some i, $j \in N$, let $g = g^N - ij$. Let $kl \in L(g)$. If $|N| > 3$, then $g - kl$ has one component. By Step 2, $g - kl$ is defeated by g. If $|N| = 3$, then $g - kl$ has two components and k and l belong to two different components. By Step 3, $g - kl$ is defeated by g. In either case, g is not defeated by $g - kl$. Since $g + ij = g^N$, by Step 4, g is not defeated by $g + ij$. Altogether, we conclude that g is pairwise stable.

Step 6: Now we will prove the theorem. Let $g \in \mathcal{G}^N$ be a pairwise stable network. By Step 5, for some $i, j \in N$, $g = g^N - ij$. Let $k \in N^c$ be a new agent and $g_0 \equiv g \oplus k$. Since i and k belong to different components in g_0, by Step 3, g_0 is defeated by $g_1 \equiv g_0 + ik$. Since g_1 is obtained by deleting more than one link from $g^{N \cup \{k\}}$ and has one component, by Step 2, g_1 is defeated by $g_2 \equiv g_1 + ij$. Applying Step 2 repeatedly, we generate an improving path $\{g_2, \ldots, g_r\}$, where for each $t = 2, \ldots, r-1$, g_{t+1} is obtained by adding one link to g_t, and for some $l \in N$, $g_r = g^{N \cup \{k\}} - lk$. Then $\{g_0, g_1, g_2 \ldots, g_r\}$ is an improving path. By Step 5, g_r is pairwise stable and also, a limiting network of g_0. Since $ij \in L(g_2)$ and g_r is obtained by adding links to g_2, $ij \in L(g_r)$. Therefore, $L(g) \neq L(g_r) \cap L^N$, which implies that g is not *link sustainable.* $\square$

Proof of Theorem 2: For all $g = (N, L) \in \mathcal{G}$ and all i, $j \in N$, let $I_g(ij) = 1$ if $ij \in L$ and $I_g(ij) = 0$ if $ij \notin L$. Then the equation (1) can be rewritten as follows:

$$u_i(g) = \bar{u}_i + \sum_{j \neq i,\, j \in N} \left[\delta^{d(i,j;g)} w_{ij} - I_g(ij) c_{ij}\right]. \tag{3}$$

(i) We prove the "if" part. Since the proof is obvious when $|N| < 3$, we assume that $|N| \geq 3$. Let $g = g^N$ be pairwise stable. Let $k \in N^c$ be a new agent and $g_0 = g \oplus k$. For all $N' \subseteq N$, let $L(k, N') \equiv \{ik | i \in N'\}$ be the set of links connecting k and each agent in N', and $g(k, N') \equiv (N \cup \{k\}, L^N \cup L(k, N'))$. Since for all i, $j \in N$, $d(i, j; g) = d(i, j; g(k, N'))$ and $d(i, j; g - ij) = d(i, j; g(k, N') - ij)$, $u_i(g(k, N') - ij) - u_i(g(k, N')) = u_i(g - ij) - u_i(g) + [\delta^{d(i,k;g(k,N')-ij)} - \delta^{d(i,k;g(k,N'))}]w_{ik}$. Since $d(i, k; g(k, N') - ij) \geq d(i, k; g(k, N'))$, $\delta^{d(i,k;g(k,N')-ij)} - \delta^{d(i,k;g(k,N'))} \leq 0$, and therefore, $u_i(g(k, N') - ij) - u_i(g(k, N')) \leq u_i(g - ij) - u_i(g)$. From the pairwise stability of g, $u_i(g - ij) - u_i(g) \leq 0$, which implies that $u_i(g(k, N') - ij) - u_i(g(k, N')) \leq 0$. Similarly, we can show that $u_j(g(k, N') - ij) - u_j(g(k, N')) \leq 0$. Since the choice of N', i, and j is arbitrary, for all $N' \subseteq N$ and i, $j \in N$, $g(k, N') - ij$ does not defeat $g(k, N')$. Therefore, if $\{g_0, g_1, \ldots, g_l\}$ is an improving path, then for all $s \leq l - 1$, $g_{s+1} = g_s + ik$ or $g_{s+1} = g_s - ik$ for some i. Since for all $s = 0, \ldots, l$, $L(g) = L(g_s) \cap L^N = L^N$, we have for all $g^* \in \mathcal{L}(g_0)$, $L(g) = L(g^*) \cap L^N$, so that g is *link sustainable.*

(ii) We now prove the "only if" part. For all $N \in \mathcal{N}$, all $O \in \mathcal{O}^N$, all $g' \in \mathcal{G}^N$, and $i \in N$, let $u_i(g'; O)$ be i's utility from g' in O. Let $g \in \mathcal{G}^N$ be *link sustainable* for all $O' \in \mathcal{O}(g)$. Suppose, by way of contradiction, that g is not complete. Then, there exist i, $j \in N$ such that $ij \notin L(g)$. Let $O \in \mathcal{O}(g)$. Since g is pairwise stable in O, we have either (1) $u_i(g + ij; O) - u_i(g; O) = 0$

and $u_j(g+ij;O) - u_j(g;O) = 0$ or (2) $u_i(g+ij;O) - u_i(g;O) < 0$ or $u_j(g+ij;O) - u_j(g;O) < 0$.

Case 1: $u_i(g+ij;O) - u_i(g;O) = 0$ and $u_j(g+ij;O) - u_j(g;O) = 0$. Let $k \in N^c$ be a new agent and $g_0 = g \oplus k$. For sufficiently large $p > 0$ and sufficiently small $q > 0$, let $O' \in \mathcal{O}^{N \cup \{k\}}$ be obtained from O by setting new parameters as follows: $w_{ik} = w_{jk} = w_{kj} = p$, for all $h \notin \{j,k\}$ $w_{kh} = 0$, $c_{jk} = q$, and for all $h \neq k$, $c_{kh} = q$. For all $i_1, i_2 \in N$, $w_{i_1 i_2}$, $\delta_{i_1 i_2}$, $c_{i_1 i_2}$, and $\bar{u}_{i_1}$ in O' are equal to those of O. Therefore, for all $h \in N$ and all $g' \in \mathcal{G}^N$, $u_h(g';O') = u_h(g';O)$.

Since w_{jk} and w_{kj} are sufficiently large and c_{jk} and c_{kj} are sufficiently small in O', j and k have incentives to add jk to g_0. Therefore, $\{g_0, g_0 + jk\}$ is an improving path in O'. Let $g_1 \equiv g_0 + jk$.

Now suppose that the limiting networks from g_1 form a closed cycle in O'. We show that there is at least one $\bar{g}$ in the cycle such that $L(\bar{g}) \cap L^N \neq L(g)$. Suppose by way of contradiction that for all $\bar{g}$ in the cycle, $L(\bar{g}) \cap L^N = L(g)$. Let the cycle be $\{\bar{g}_1, \ldots, \bar{g}_l, \bar{g}_{l+1}\}$, where $\bar{g}_1 = \bar{g}_{l+1}$. Since $L(\bar{g}_1) \cap L^N = \cdots = L(\bar{g}_l) \cap L^N = L(g)$, for all $s = 2, \ldots, l+1$, g_s is obtained by adding fk to g_{s-1} or deleting fk from g_{s-1} for some $f \neq k$. Moreover, from $\bar{g}_1 = \bar{g}_{l+1}$, there exist $t \in \{2, \ldots, l+1\}$ and $h \neq k$ such that $\bar{g}_t = \bar{g}_{t-1} + hk$. Since w_{jk} and w_{kj} are sufficiently large, and c_{jk} and c_{kj} are sufficiently small in O', j and k do not have an incentive to sever jk in O' even though other links are changed. Therefore, for all $s = 1, \ldots, l$, $jk \in L(\bar{g}_s)$, which implies that $h \neq j$. On the other hand, since for all $f \notin \{j,k\}$, $w_{kf} = 0$ and $c_{kh} = q > 0$, k becomes worse-off if he adds hk to $\bar{g}_{t-1}$, which is impossible. Therefore, there is at least one $\bar{g}$ in the cycle such that $L(\bar{g}) \cap L^N \neq L(g)$. Taken together, g is not *link-sustainable,* which is a contradiction.

Next, suppose that the limiting network g^* from g_1 is pairwise stable in O'. Let $\{g_1, \ldots, g_l\}$ be an improving path from g_1 in O', where $g_l = g^*$. Then, $g_l \in \mathcal{L}(g_0)$. For $g' \in \mathcal{G}^{N \cup \{k\}}$, let $L(k, g') \equiv \{ik \mid ik \in g'\}$ be the set of all links that k has in g'. Since w_{jk} and w_{kj} are sufficiently large, and c_{jk} and c_{kj} are sufficiently small in O', j and k do not have an incentive to sever jk regardless of changes in other links, so that for all $s = 2, \ldots, l$, $jk \in L(g_s)$. Since for all $h \notin \{j, k\}$, $w_{kh} = 0$ and $c_{kh} = q > 0$, for all $s = 1, \ldots, l-1$, k does not want to have a link with h in g_s. Therefore, $L(k, g_1) = L(k, g_2) = \cdots = L(k, g_l) = jk$. Now suppose by way of contradiction that g is *link-sustainable.* Then $L(g_l) \cap L^N = L(g) = L(g_1) \cap L^N$. Since $L(k, g_l) = L(k, g_1)$ and $L(g_l) \cap L^N = L(g_1) \cap L^N$, we have $L(g_l) = L(g_1)$. Altogether, $g^* = g_l = g_1$, which implies that g_1 is pairwise stable. On the other hand, for all $h \in N$, $d(i, h; g_1) = d(i, h; g)$ and $d(i, h; g_1 + ij) = d(i, h; g + ij)$, $u_i(g_1 + ij; O') - u_i(g_1; O') = u_i(g + ij; O') - u_i(g; O') + [\delta^{d(i,k;g_1+ij)} - \delta^{d(i,k;g_1)}]p$. Since $d(i, k; g_1 + ij) = 2$, $d(i, k; g_1) \geq 3$, and $u_i(g + ij; O') - u_i(g; O') = u_i(g + ij; O) - u_i(g; O) = 0$, $u_i(g_1 + ij; O') - u_i(g_1; O') \geq u_i(g + ij; O') - u_i(g; O') + (\delta_{ik}^2 - \delta_{ik}^3)p = (\delta_{ik}^2 - \delta_{ik}^3)p$. Since $p > 0$ and $0 < \delta_{ik} < 1$, $u_i(g_1 + ij; O') - u_i(g_1; O') > 0$. Since for all $h \in N$, $d(j, h; g_1 + ij) = d(j, h; g + ij)$, $d(j, h; g_1) = d(j, h; g)$, and $d(j, k; g_1 + ij) = d(j, k; g_1)$, $u_j(g_1 + ij; O') - u_j(g_1; O') = u_j(g + ij; O') - u_j(g; O') = u_j(g + ij; O) - u_j(g; O) = 0$. Therefore, $g_1 + ij$ defeats g_1, contradicting to the pairwise stability of g_1.

Case 2: $u_i(g + ij; O) - u_i(g; O) < 0$ (the proof is similar for $u_j(g + ij; O) - u_j(g; O) < 0$). First, suppose that $u_j(g + ij; O) - u_j(g; O) \leq 0$. Let $O(p) \in \mathcal{O}^N$ be obtained from O by replacing w_{ji} by p. If $p = w_{ji}$, then $u_j(g + ij; O(p)) - u_j(g; O(p)) \leq 0$, and if $p \to \infty$, then $u_j(g + ij; O(p)) - u_j(g; O(p)) \to \infty$. Since $u_j(g + ij; O(p)) - u_j(g; O(p))$ is continuous in p, there exists p^* such that $u_j(g + ij; O(p^*)) - u_j(g; O(p^*)) = 0$. Since $u_i(g + ij; O(p^*)) - u_i(g; O(p^*)) = u_i(g + ij; O) - u_i(g; O) < 0$, i has no incentive to add ij to g in $O(p^*)$. From the pairwise stability of g, for all k_1, $k_2 \in N$ such that k_1, $k_2 \notin \{i, j\}$, k_1 and k_2 have no incentive to add a new link to g or delete an existing link from g in $O(p^*)$. Therefore, g is also pairwise stable in $O(p^*)$ and thus, $O(p^*) \in \mathcal{O}(g)$. Now let $O(p^*, q)$ be obtained from $O(p^*)$ by replacing w_{ij} by q. Similarly, there is q^* such that $u_i(g + ij; O(p^*, q^*)) - u_i(g; O(p^*, q^*)) = 0$ and that $O(p^*, q^*) \in \mathcal{O}(g)$. Since $u_i(g + ij; O(p^*, q^*)) - u_i(g; O(p^*, q^*)) = 0$ and $u_j(g + ij; O(p^*, q^*)) - u_j(g; O(p^*, q^*)) = 0$, by applying Case 1, we have a contradiction.

Next, suppose that $u_j(g + ij; O) - u_j(g; O) > 0$. Let $O(r) \in \mathcal{O}^N$ be obtained from O by replacing c_{ji} by r. If $r = c_{ji}$, then $u_j(g + ij; O(r)) - u_j(g; O(r)) > 0$, and if $r \to \infty$, then $u_j(g + ij; O(r)) - u_j(g; O(r)) \to -\infty$. Since $u_j(g + ij; O(r)) - u_j(g; O(r))$ is continuous in r, there is r^* such that $u_j(g+ij; O(r^*)) - u_j(g; O(r^*)) = 0$. Since $u_i(g+ij; O(r^*)) - u_i(g; O(r^*)) < 0$ and $O(r^*) \in \mathcal{O}(g)$, as before, we have a contradiction. $\square$

Proof of Theorem 3: We prove (ii) first, and then (iii) and (i).

(ii) Let $g = (N, L) \in \mathcal{G}^N$ be a pairwise stable network, $k \in N^c$ be a new agent, and $g_0 = g \oplus k \in \mathcal{G}^{N \cup \{k\}}$. Since for all $i \in N$, $u_i(g_0) - u_i(g_0 + ik) = -\delta + c > 0$, g_0 is not defeated by $g_0 + ik$. For all i, $j \in N$ with $ij \notin L$, $u_i(g_0) - u_i(g_0 + ij) = u_i(g) - u_i(g + ij)$ and $u_j(g_0) - u_j(g_0 + ij) = u_j(g) - u_j(g + ij)$, which implies that g_0 is defeated by $g_0 + ij$ if and only if g is defeated by $g + ij$. Since g is pairwise stable, $u_i(g) - u_i(g + ij) \geq 0$ and $u_j(g) - u_j(g + ij) \geq 0$. Therefore, for all i, $j \in N$ with $ij \notin L$, g is not defeated by $g + ij$, or equivalently, g_0 is not defeated by $g_0 + ij$. Similarly, we can show that for all i, $j \in N$ with $ij \in L$, g_0 is not defeated by $g_0 - ij$. Since g_0 is not defeated by any adjacent network, it is pairwise stable, and therefore, g is *graph sustainable.*

(iii) Let $g = (N, L) \in \mathcal{G}^N$ be a pairwise stable network and $k \in N^c$ be a new agent. By Proposition 2 in Jackson and Wolinsky [1996], if $c < \delta - \delta^2$, then the only pairwise stable network is g^N. Therefore, it suffices to show that g^N is *distance sustainable.* Let $g' \in \mathcal{G}^{N \cup \{k\}}$ be such that $g' \neq g^{N \cup \{k\}}$. For all i, $j \in N \cup \{k\}$ such that $ij \notin L(g')$, $u_i(g' + ij) - u_i(g') \geq \delta - \delta^2 - c > 0$ and $u_j(g' + ij) - u_j(g') \geq \delta - \delta^2 - c > 0$, so that g' is defeated by $g' + ij$. Since this is true for all $g' \in \mathcal{G}^{N \cup \{k\}}$ such that $g' \neq g^{N \cup \{k\}}$, the only limiting network from $g^N \oplus k$ is $g^{N \cup \{k\}}$. Therefore, for all i, $j \in N$, $d(i, j; g^N) = d(i, j; g^{N \cup \{k\}}) = 1$, which implies that g^N is *distance sustainable.*

(i) The proof is divided into three cases. Let $g = (N, L) \in \mathcal{G}^N$ be a pairwise stable network and $k \in N^c$ be a new agent.

Case 1: $c < \delta$. First, suppose that for some $ij \notin L$, $|C(g + ij)| < |C(g)|$. Since for i, the cost of ij is c and the benefit of ij is more than or equal to

δ, $u_i(g+ij) - u_i(g) \geq \delta - c > 0$, which implies $u_i(g+ij) > u_i(g)$. Similarly, $u_j(g+ij) > u_j(g)$. Therefore, g is defeated by $g+ij$, which implies that a pairwise stable network has only one component, that is, $|C(g)| = 1$.

Let $g_0 \equiv g \oplus k$. Since $|C(g_0)| = 2$, g_1 defeats g_0 if and only if $g_1 = g_0 + hk$ for some $h \in N$. Let $\{g_0, g_1, g_2, \ldots, g_l\}$ be an improving path from g_0. Then, for $t = 1, \ldots, l-1$, $|C(g_t)| \geq |C(g_{t+1})|$. Since $|C(g_1)| = 1$, for all $t = 1, \ldots, l$, $|C(g_t)| = 1$. Since this is true for any improving path, g is *connection sustainable.*

Case 2: $c = \delta$.

Step 1: *If* $|C(g+ij)| < |C(g)|$, *then* g *does not defeat* $g+ij$.

Proof. Let h_i, $h_j \in C(g)$ be the components of g that contains i and j, respectively. Let $h_i \equiv (N_i, L_i)$ and $h_j \equiv (N_j, L_j)$. Let $h \equiv (N_i \cup N_j, L_i \cup L_j \cup ij)$. First, suppose that h_i and h_j are both singletons. Since for i, the cost of ij is c and the benefit of ij is δ, $u_i(g+ij) - u_i(g) = \delta - c = 0$. Similarly, $u_j(g+ij) - u_j(g) = 0$. Therefore, g does not defeat $g+ij$. Now suppose that at least one of h_i and h_j, say h_i, is a non-singleton. Since h_i is a non-singleton, $u_j(g+ij) - u_j(g) > \delta - c = 0$. And $u_i(g+ij) - u_i(g) \geq \delta - c = 0$. Altogether, $g+ij$ defeats g, which implies that g does not defeat $g+ij$.

Step 2: *If* $g = (N, L) \in \mathcal{G}^N$ *is pairwise stable, then* g *is the empty network or* $|C(g)| = 1$.

Proof. Since the proof for $|N| \leq 2$ is obvious, we assume that $|N| > 2$. Suppose by way of contradiction that there is a nonempty pairwise stable network g with $|C(g)| > 1$. Then there exist a non-singleton component $h_1 \in C(g)$ and another component $h_2 \in C(g)$. Let $i \in N(h_1)$ and $j \in N(h_2)$. By Step 1, $g+ij$ defeats g, which contradicts to the pairwise stability of g.

Step 3: *A pairwise stable network is connection sustainable.*

Proof. By Step 2, g is either the empty network or $|C(g)| = 1$. If g is the empty network, we can easily show that $g \oplus k$ is pairwise stable, which implies that g is *graph sustainable.* By Remark 1, g is *connection sustainable.*

Next, suppose that $|C(g)| = 1$. Since $|N| > 2$, the component is non-singleton. Let $g_0 \equiv g \oplus k$. Since $|C(g_0)| = 2$, by the pairwise stability of g and Step 1, g_1 defeats g_0 if and only if $g_1 = g_0 + hk$ for some $h \in N$. Let $\{g_0, g_1, g_2, \ldots, g_l\}$ be an improving path from g_0. By Step 1, for $t = 1, \ldots, l-1$, $|C(g_{t+1})| \leq |C(g_t)|$. Since $|C(g_1)| = 1$, for all $t = 1, \ldots, l$, $|C(g_t)| = 1$. Since this is true for any improving path, g is *connection sustainable.*

Case 3: $c > \delta$. By (ii), any pairwise stable network is *graph sustainable.* By Remark 1, it is *connection sustainable.* □

Proof of Theorem 4: From Proposition 4 in Jackson and Wolinsky [1996], (i) a pairwise stable network $g = (N, L) \in \mathcal{G}^N$ can be partitioned into fully intraconnected components, each of which has a different number of members, and that $m > n^2$, where m is the number of nodes of one component of g and n is the next largest in size. In the proof, it is also shown that (ii) if for some $i, j \in N$, $n_j \leq \max\{n_h \mid ih \in L\}$, then i strictly prefers to have a link with j. In addition, it can easily be shown that (iii) if for some $i \in N$, $n_i = 0$, then he wants to have a link with any other agents.

First, we prove the "if" part. Let $N \in \mathcal{N}$, g^N be the initial network, and $k \in N^c$ be a new agent. Note that the complete network is always pairwise stable. For $N' \subsetneq N$, let $L(k, N') \equiv \{ik | i \in N'\}$ be the set of all links connecting k and any other agents in N', and $g(k, N') \equiv (N \cup \{k\}, L^N \cup L(k, N'))$. By (ii) and (iii), for all $N' \subsetneq N$, g' defeats $g(k, N')$ if and only if $g' = g(k, N') + ij$ for some $i, j \in N \cup \{k\}$ with $ij \notin L(k, N')$. Therefore, an improving path from $g_0 \equiv g^N \oplus k$ is of the form $\{g_0, \ldots, g_l\}$, where for all $t = 0, \ldots, l-1$, $g_{t+1} = g_i + ik$ for some $i \in N$. Hence $\mathcal{L}(g_0) = \left\{g^{N \cup \{k\}}\right\}$, which implies that g is *link sustainable.*

We now prove the "only if" part. Let $g \in \mathcal{G}^N$ be a pairwise stable network and $k \in N^c$ be a new agent. By way of contradiction, suppose that g is *link sustainable,* but not complete. By (i) and (iii), this can happen only when $|N| > 4$, and moreover, there are at least two components of g. Let i be in the largest component of g and j be in the next largest component of g. Let $g_0 = g \oplus k$. By (ii) and (iii), k and i want to make a link, so that g_0 is defeated by $g_0 + ik$. In $g_0 + ik$, j and k also want to make a link. Therefore, $g_0 + ik + jk$ defeats $g_0 + ik$. By applying the argument repeatedly, we have an improving path $\{g_0, g_0 + ik, g_0 + ik + jk, \ldots, g^{N \cup \{k\}}\}$. Since $L(g) \neq L(g^{N \cup \{k\}}) \cap L(g^N)$, g is not *link sustainable,* which is a contradiction. $\square$

References

[1] F. Deroian, Farsighted strategies in the formation of a communication network, *Economics Letters* 80 (2001), 343-349.

[2] B. Dutta, A. van den Nouweland, and S. Tijs, Link formation in cooperative situations, *International Journal of Game Theory* 27 (1998), 245-256.

[3] B. Dutta, S. Ghosal, and D. Ray, Farsighted network formation, *Journal of Economic Theory* 122 (2005), 143-164.

[4] B. Dutta and S. Mutuswami, Stable networks, *Journal of Economic Theory* 76 (1997), 322-344.

[5] M. O. Jackson and A. van den Nouweland, Strongly stable networks, *Games and Economic Behavior* 51 (2005), 420-444.

[6] M. O. Jackson and A. Watts, The evolution of social and economic networks, *Journal of Economic Theory* 106 (2002), 265-295.

[7] M. O. Jackson and A. Wolinsky, A strategic model of social and economic networks, *Journal of Economic Theory* 71 (1996), 44-74.

[8] R. Myerson, Graphs and cooperation in games, *Mathemtics of Operations Research* 2 (1977), 225-229.

[9] M. Slikker and A. van den Nouweland, A one-stage model of link formation and payoff division *Games Economic Behavior*, 34 (2001), 153-175.

[10] A. Watts, A dynamic model of network formation, *Games and Economic Behavior* 34 (2001), 331-341.

[11] A. Watts, Non-myopic formation of circle networks, *Economics Letters* 74 (2002), 277-282.