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Two-Dimensional Self-Selection of Borrowers*

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The multi-dimensional heterogeneity of agents can provide interesting insights. To illustrate this point in the loan market context, we examine the borrowers' decision to switch from a variable rate loan to a fixed rate loan, using a model of two-dimensional borrower types (risk aversion and riskiness). Among high risk borrowers, more risk averse ones are selected out of the loan market and less risk averse ones are not tempted by the fixed rate loan. Switchers are more risk averse but have lower default risk. The Financial Services Commission's 2015 Mortgage Refinancing Program in Korea is discussed under our model's framework.

JEL Classification: D82, C46, G21
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I. Introduction

This paper aims to show that the multi-dimensional heterogeneity of agents is a useful element in economic models. To illustrate this in the loan market modeling context, we outline a tractable model of borrowers of two-dimensional heterogeneity and argue that the two-dimensional self-selection offers an interesting take on a recent policy episode in the Korean mortgage market. Specifically, we offer a theoretical analysis of the borrowers' decision to switch from a variable rate loan to a fixed rate loan.

We use the term "self-selection" (or simply "selection") to refer to an endogenous representation of private types in an economic model. Selection plays important roles in many areas of economics. Akerlof's market for "lemons" is a perennial

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example. Selection and the sample bias caused by it are often discussed in econometric and empirical research.[1]

Theoretical models involving the self-selection of agents often have a single parameter representing the private type, such as product quality, taste for quality, ability, risk status, etc. But in many situations, there may be a number of distinct type dimensions that are relevant. The models with multi-dimensional types have proven difficult to analyze, although some general results are available in the literature (Rochet and Stole, 2003; Armstrong and Rochet, 1999), wherein the optimum typically has a complex menu of contracts offered to agents.

Our aim in this paper is more modest. Rather than derive general results on optimal contracts, our model focuses on the self-selection of agents in response to the (exogenous) offer of contract forms. Even this simple framework can be a useful tool for discussing a case in the Korean mortgage market. The policy episode involves giving the current borrowers of a variable rate loan a chance to switch to a fixed rate loan. If we consider the degree of risk aversion of the borrowers as the only type dimension, we can expect that the more risk averse borrowers would select into the switch.

On the other hand, if we consider the degree of riskiness (default risk) as the only type dimension, the variable rate vs the fixed rate comparison does not suggest an obvious direction of selection. Hence we cannot say much about the riskiness of the switchers in a one-dimensional type model.

We will show that in our two-dimensional type model, the borrowers' self-selection occurs two dimensionally, so that the switchers from the variable rate happen to be more risk averse as well as less risky. The intuition for this prediction can grasped by observing that highly risk averse agents will be holding variable rate loans only if they happen to know themselves to be low risks.[2] We formalize this intuition in our model.

This prediction has policy implications and raises further questions. If a fixed rate loan selects good risks out of the borrower pool, then it results in "adverse selection" for the bank who sold the variable rate loan. Was the policy intervention beneficial to the mortgage market? If offering a fixed rate loan can "cream skim" the borrowers, why didn't banks offer such loan contracts more actively? This paper does not claim to give conclusive answers to all of these sweeping questions. Our hope is to show that our model provides an intuitive framework for addressing such questions in a meaningful way.

[1] *The New Palgrave of Dictionary of Economics* has articles on adverse selection, selection bias and self-selection, market competition and selection, and group selection. (https://link.springer.com/referencework/ 10.1057/978-1-349-95121-5).

[2] This observation is not entirely novel. The classic paper by Rothschild and Stiglitz (1976) noted a similar possibility in the insurance market setting, while Cutler *et al*. (2008) examined this issue empirically.

The rest of the paper is organized as follows. In Section 2, we provide background information, with a brief literature review. Section 3 gives an overview of the model. In Section 4, we analyze the demand for a variable rate loan, characterizing two-dimensional self-selection of borrowers. In Section 5, we introduce the chance to switch to a fixed rate loan and analyze the consequences. Section 6 indicates limitations of our analysis and discusses the possible extensions. Section 7 concludes the paper with a recapitulation and some policy discussion.

II. Background

As background information, this section explains the distinction between two forms of loan contracts and describes the Korean mortgage market. We also review the literature on mortgage choices.

2.1. The *Form* of Loan Contracts: Fixed Rate versus Variable Rate

A loan contract's interest rate may be *fixed or variable*. The contrast between the two kinds of loan contracts and the borrowers' responses to each are the focus of this paper. Hereafter, we refer to this distinction as the *form* of loan contracts.

For a fixed rate loan, the interest rate remains fixed over its entire term. For a variable rate loan, the interest rate is subject to periodic changes, based on some index (*e.g.*, a benchmark or a prime market rate). Alternative terms for variable rates include floating rates and adjustable rates. For mortgage loans (*i.e.*, long-term loans backed by real estate properties), the abbreviations ARMs (adjustable rate mortgages) and FRMs (fixed rate mortgages) are often used. Our model is not a complete model of mortgage loans, which should encompass the real estate market, financial market, and monetary policy, etc, but is inspired by a recent policy episode in the Korean mortgage market. So our discussion will be mainly in terms of mortgages.

The majority of the current loans in Korea has variable rates. In 2017, fixed rate loans accounted for less than 35% of all (corporate and household) outstanding loan balances in Korea (Table 1, the first two columns). If we restrict our attention to mortgage loans, which took up 55% of all household loans (excluding merchandise credit) in 2014, ARMs had been strikingly dominant in the past but the share of FRMs is steadily increasing over the last five years (Figure 1).

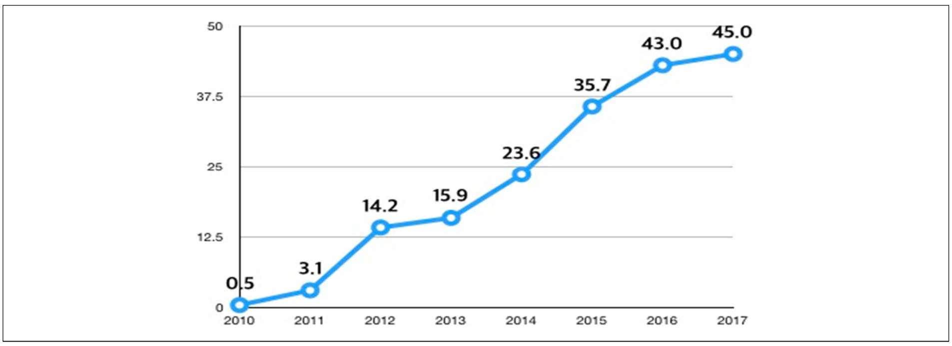
If we focus on the new loans issued to households (Table 1, the last column) instead of outstanding balances, recent popularity of fixed rate loans somewhat dwindled in 2017.

[Table 1] Proportion of fixed rate loans among all loans (%)

	Outstanding		Newly issued	
	Corporate	Household	Corporate	Household
2013	33.8	21.3	35.3	30.6
2014	32.7	28.4	33.2	39.8
2015	34.1	31.3	36.4	48.1
2016	35.3	34.5	34.7	49.3
2017	32.5	33.2	32.4	35.6

Source: Bank of Korea Economic Statistics System, https://ecos.bok.or.kr.

[Figure 1] Proportion of FRMs among mortgage loans (%)



Source: Financial Services Commission, http://www.fsc.go.kr.

2.2. The Mortgage Loans and the 2015 *Mortgage Refinancing Program* of Korea

In the US mortgage market, FRMs were originally the norm until the mid-1980s when regulatory changes led to introduction of ARMs. The new ARMs were immensely popular, with the share of ARMs in the number of closed loan deals reaching 70% during the late 1980s and the early 1990s (see Figure 1 in Mori et al., 2009).[3] Since then, the share of ARMs has decreased yet remains non-negligible and routinely reported and discussed by the trade press.[4] Moreover, Mori et al. (2009) note that the average ARM is 1.3 times larger in size than the average FRM, so these figures understate the actual weight of ARMs in the mortgage market.

On the other hand, Korean mortgage market is understood to have commenced

[3] Other notable periods of popularity of ARMs in the US occur around the years 1996 (60%) and 2006 (40%), both of which happen to be coincident with the onset of international financial crises.

[4] See "MBA Chart of the Week: Adjustable-Rate Mortgage Share," available at the Mortgage Bankers Association website, https://www.mba.org/publications/insights/archive. Also see "Adjustable-rate mortgages make a comeback as rate rises loom" published in March 2017 at http://www.marketwatch.com.

with the establishment of the Korea Housing-Finance Corporation (HF) in 2004. At the time, most household loans backed by real estate properties were short-term with variable rates and were not amortized in repayment schemes (Koh and Ju, 2011).

The major objective of establishing HF and the mortgage market in 2004 was to promote long-term, fixed rate, and amortized mortgage loans because financial authorities deemed variable rates and delayed repayment of principals as posing significant financial risks on a national scale (*e.g.*, variable rate loans are believed to have higher default rates; see Koh and Ju, 2011). But as noted in Figure 1 above, fixed rate mortgages did not gain much popularity until recently.

The *Mortgage Refinancing Program*[5] conducted by the Financial Services Commission (FSC) of Korea in early 2015 is in line of such objectives, being an attempt to guide mortgage borrowers toward fixed rate loans and amortization. In short, the program offered a chance of switching to a fixed rate amortized loan for mortgage borrowers who had variable rate and/or no amortization (interest-only) loan. The offered fixed rate was lower than the prevailing mortgage rate at the time. But since it also required amortization, the short-term payments could be higher for borrowers who had interest-only loans.[6] Although this program was offered as a one-time only event, there is strong possibility for a similar program to be conducted in the near future.[7]

Different borrowers have different preferences over fixed and variable rates. However, *ceteris paribus*, a risk averse borrower would prefer a fixed rate loan. A naive assessment of the situation suggests that many risk averse borrowers should have been happy to switch to the new fixed rate loan. Reaction of borrowers was however a bit short of meeting such expectations.

The first wave of fixed rate loans in the amount of KRW 20 trillion was sold out in four days after its introduction on March 24, 2015. But the demand for the ambitious second wave of KRW 20 trillion a week later fell short. The final amount of switches was KRW 31.7 trillion (out of the supplied 40 trillion). To put this amount in perspective, it is about 10% of all mortgage loan balances as of the end of 2014. When we factor in eligibility requirements (*i.e.*, fixed rate and/or interest only,

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[5] FSC press releases: "Korea's Household Debt and Policy Response," Feb 26, 2015; "Additional KRW 20 trillion To Be Provided for the Mortgage Refinancing Program," Mar 30, 2015, http://www.fsc.go.kr/eng/new_press.

[6] According to FSC, 40% of switchers had variable rate loans with amortization, 47.5% had variable rate and interest-only loans, and 12.5% had fixed rate yet interest-only loans before the switch. Therefore, the switch posed higher short-term payments for 60% of those switchers who did not have amortization before. Statistics for non-switchers are not available.

[7] "FSC's Mortgage Refinancing Program to be extended to non-banking financial sector," *The Asia Business Daily*, May 19, 2017. The FSC had officially denied the claims of this report at the time of its publication. However, the FSC's *Policy Roadmap for 2018* includes a similar plan to be implemented in 2018.

the ceiling on loan size, etc.), the switch rate may be about 20% of all eligible loans in value;[8] significant yet far below the naive assessment.

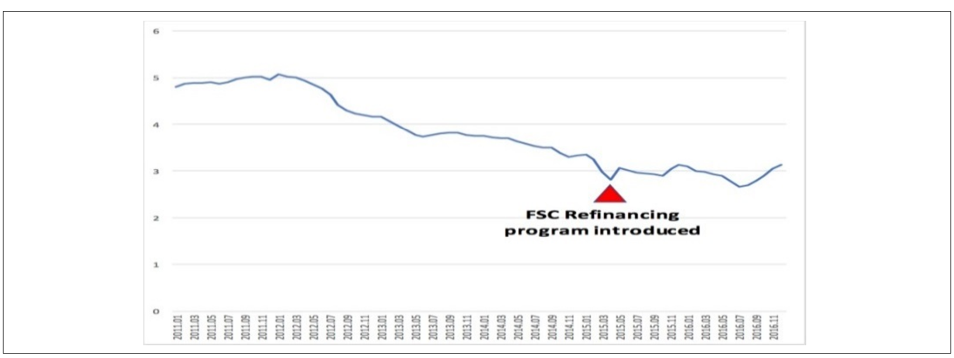
Our analysis hopes to offer some insights into this apparent puzzle: Why didn't most borrowers switch to the fixed rate loan? To add another related puzzle: Why were ARMs so popular in the US when they were introduced as an alternative to FRMs? (Although the US puzzle is not about switches, our analysis still can shed some lights.)

2.3. A Brief Discussion of the Literature

Before we delve into the model, let us discuss some immediate explanations for the puzzles and the related literature. Our analysis and these explanations are complementary rather than competing. First, the "financial (il)literacy" explanation says that borrowers are not sophisticated enough to appreciate the nuances of loan contracts. Conklin (2017) offered indirect support for the existence of financial illiteracy by showing that a face-to-face interaction with brokers lead to lower defaults. Some borrowers are ill-informed and do not make rational choices over fixed versus variable rates. Ignoring the illiteracy issue, our model assumes that all borrowers maximize expected utility.

Another explanation is that borrowers are optimistic, rightly or wrongly, about future fluctuations in the interest rates, *i.e*. they expect the interest rate to fall in the future. This has some empirical relevance for Korean borrowers as they were experiencing downward trends in the market interest rates in early 2015 (Figure 2), though the realized rates afterwards show stagnant and even some upward trends. What we require in our model is that the expected value of the variable rate to be lower than the alternative fixed rate. We will discuss this in Section 5.

[Figure 2] Prevailing mortgage rates, 2011-2016



Source: Bank of Korea, accessed at HOUSTA Housing Stats Portal (http://housta.khug.or.kr/khhi/web/hi/fi/hifi050008.jsp)

[8] See the FSC press releases of March 29, 2015 and March 20, 2016 for these estimates.

The literature on the choice of mortgage borrowers between FRMs and ARMs is rich. Badarinza *et al*. (2015) is a recent nine-country panel study. Ghent and Yao (2016) is a survey on the key issues in mortgage choices, a section of which is on the "choice of fixed and adjustable rate mortgages." Previous research has examined the diverse aspects and determinants of choice between fixed rate and variable rate mortgages, sometimes puzzling over why variable rate mortgages are so popular (*e.g*., Mori *et al.*, 2009; Brueckner, 1993).

Dhillon *et al*. (1987) contrast two views, one focusing on prices and loan contract terms ("pricing view") and the other on observable borrower characteristics ("borrower characteristics view"). Their empirical analysis leans toward supporting the pricing view, while other papers lend support to the borrower characteristics view. For instance, Brueckner (1992) considered the future mobility of borrowers as a determinant, while Koh and Ju (2011) and Kim (2015) empirically examined the mortgage choices of Korean borrowers based on their characteristics. We, on the other hand, focus on unobservable borrower characteristics. Choi (2016) attempted to apply the methodology of Kim (2016) in studying FSC's 2015 program. While this paper shares the basic outlook with them, we correct an error found in both Choi (2016) and Kim (2016), as well as provide a different analysis.

Brueckner (1993) posed the popularity of ARMs as a puzzle because the risk sharing between risk neutral banks and risk averse borrowers is apparently sub-optimal. The resolution suggested by Brueckner (1993) is intertemporal considerations (the borrowers' impatience and the flexibility of interest payment streams enabled by variable rates). Our model abstracts away the intertemporal nature of the problem to focus on two-dimensional selection. In addition, we will argue that the banks need to be risk averse for ARMs to be viable in our framework.

Several papers have examined the selection and screening issues arising from the private information of borrowers. For instance, Brueckner (2000) and Harrison *et al*. (2004) considered the default cost of the borrower as private information and examined how such costs affect mortgage choices and equilibrium contracts (*e.g*., the size of loans). Although our model does not explicitly consider default costs or loan sizes, we do discuss potential extensions in such directions in Section 6.

Our model is not a general model of the mortgage market and uses some strong simplifying assumptions in order to facilitate an intuitive and tractable analysis. Using an essentially single-period model is restrictive, but a pioneering work by Dokko and Edelstein (1991) also used a single-period model; they also explained the co-existence of ARMs and FRMs by assuming that both the borrowers and the bank are risk averse (similar to our work). Posey and Yavas (2001) considered self-selection in the choice between FRM and ARM. However, in contrast to our work, Posey and Yavas (2001) predicted that low-risk borrowers would choose FRM over ARM. But their modeling approaches and specifications of "riskiness" are quite different from ours and are not directly comparable.

III. Overview of the Model

The timeline of our model is given in the following figure.

[Figure 3] The Timeline of the Model

date 0	date 1	(unanticipated) date 1.5	maturity date 2
agent type realized	**variable rate loan**	**fixed rate loan**	cost of funds r_C realized
(θ, y)	random rate r	fixed rate μ^*	r realized
			repayment/default

In date 0, each agent's private type is realized and observed by the agent only. In date 1, a variable rate loan contract is offered and some agents accept it to become borrowers. The size of the loan is uniform across agents and normalized as 1. The loan lasts for 1 period and matures in date 2 when the borrowers pay $(1+r)$, the principal and the realized random interest rate. A borrower defaults if she does not have enough to repay $(1+r)$. This part of the model is analyzed in Section 4.

At one point during the term of the variable rate loan, denoted as date 1.5, a fixed rate loan is introduced. This event is not anticipated by agents and banks at date 1 when they sign the variable rate contract. The borrowers of the variable rate loan are given a chance to switch to this fixed rate loan, and those borrowers who choose to switch must pay the principal and fixed interest rate $(1+\mu^*)$ in date 2. The switching decision of the borrower and the impact of such decision on the riskiness of each borrower pool are examined in Section 5.

This bare-bones structure of the model is simplistic and abstracts away many features of the mortgage market (*e.g.*, long-term multi-period nature). Given that we limit our focus on how heterogeneous borrowers respond to the fixed rate against the variable rate, some important issues, such as the size of the loan and the amortized interest payments, are ignored.

3.1. Demand Side: Agents with Two-dimensional Type

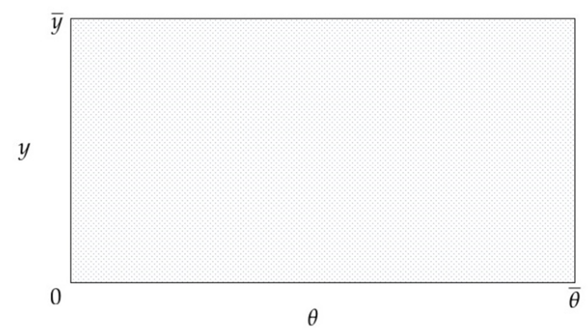
An agent is a potential borrower characterized by a two-dimensional vector of private type $(\theta, y) \in \mathbb{R}_+^2$, where θ represents her degree of risk aversion and y represents her ability to repay the loan, which is inversely related to default risk.

An agent of type $(\theta, \cdot)$ has a CARA von Neumann-Morgenstern utility function $u(w;\theta) = -e^{-\theta w}$ whose Arrow-Pratt measure of risk aversion is θ. An agent of type $(\cdot, y)$ has a project that yields $(1+y)$ in date 2 from investing 1 unit of money in date 1. In other words, y is the date 2 extra income of the agent that can be secured by investing 1 unit in date 1.[9]

We assume that y is exogenously determined and non-random. We also assume that the agent has no existing wealth,[10] so she borrows from the bank and must default when y is lower than the interest owed on the loan. Therefore, a high y denotes a low probability of default (low riskiness). For convenience, we refer to y as "income" although it really means the ability to repay the loan at maturity.

Assumption 1. The population of agents is uniformly distributed over the box $[0,\overline{\theta}]\times[0,\overline{y}]\subset\mathbb{R}_{+}^{2}$. (Figure 4)

[Figure 4] Support of the distribution of (θ, y)



Assumption 1 says that each dimension's marginal distribution is uniform with marginal means of $\frac{1}{2}\overline{\theta}$ and $\frac{1}{2}\overline{y}$ respectively. Furthermore, it implies that two type dimensions are independently distributed. We will later consider relaxing this assumption to include non-uniform distributions and correlated type dimensions (Section 6). The assumption also rules out risk-loving attitudes $(\theta<0)$ and allows risk neutrality $(\theta=0)$ only on the margin, so we focus on risk averse agents. Finally it also rules out negative income $(y<0)$; the agent is assured of earning enough at least to pay back the principal 1 of the loan. This is purely for convenience, and allowing $0>y>\underline{y}$ for some finite $\underline{y}$ should not change our results significantly.

[9] An alternative interpretation, perhaps better suited to the mortgage context, is that 1 represents the value of the real estate property and y is the future income that can be used to pay for the interest on the loan.

[10] If an agent holds non-zero initial wealth that is liquid, then it be used to finance the project (unless the interest rate of the bank is sufficiently low). In addition, any illiquid wealth could be used as collateral and affect the interest rate and size of the loan. However, we ignore these issues to keep our model simple.

3.2. Supply Side: The Bank

The supply side (lenders) of the loan market is competitive so that each bank takes the prevailing market interest rate as well as the prevailing form (variable or fixed rate) of the loan contract as given. (For simplicity, we will refer to "the bank" in the singular.) The only cost that the bank faces is the cost of funds r_C (*e.g.*, interest rate for savings or national treasury bond), which is subject to random fluctuations outside its control. Specifically, we assume that the cost of funds r_C follows a gamma distribution.

Assumption 2. The cost of funds r_C is a random variable following the gamma distribution $\Gamma(\alpha,\beta)$, where its mean is $\mu \equiv \alpha / \beta$ and its variance is $\sigma^2 = \alpha / \beta^2$.

The gamma distribution is useful because it is flexible with semi-infinite support $[0,\infty)$; the normal distribution cannot be used because the cost of funds cannot fall to large negative values. The gamma distribution is widely used in economics and finance to handle such situations.[11] Most of our qualitative results do not depend on specific features of the gamma distribution and its use here is only to facilitate concrete formulaic results.

The factors that determine the random cost may range from general macroeconomic conditions to monetary policies to any shocks to financial and real estate markets. We are assuming that each bank takes the random cost as given in the funds market (as well as taking the interest rate in the consumer loan market).

To accommodate the risk attitude of the bank in our model, we assume that the bank's payoff is also the CARA function of its profit with the risk aversion parameter θ_B. In other words, if we let $\rho = r - r_C$ be the unit profit of the bank from a loan, the bank's payoff per non-defaulting borrower is $v(\rho;\theta_B) = -e^{-\theta_B \rho}$. In our model, the borrowers have non-random future incomes and the bank has random future costs. So variable and fixed rates impose risky and safe payoffs, respectively, to borrowers but conversely impose safe and risky payoffs, respectively, to the bank. Moreover, the choice of the form (variable or fixed) of the loan contract has opposing risk implications to the borrowers and the bank.

Since the market interest rate is determined at the time of signing the loan contract (date 1 for variable rate and date 1.5 for fixed rate) and since the market is competitive, the equilibrium rate is determined by the condition that the bank's expected payoff from the loan is equated with its reservation level (which is normalized to be the payoff from zero profit). We discuss the equilibrium rates for the variable and fixed rate loans in Sections 4 and 5.

[11] See Kleiber and Kotz (2003, Chapter 5) and Kim (2016) for discussion and related references.

IV. Variable Rate Loan: Demand and Default

This section examines what happens in date 1 (loan demand) and date 2 (default) with no consideration for the (unanticipated) date 1.5 event (introduction of the fixed rate loan). The expected loan demand and expected default affect the equilibrium interest rate.

Since the loan applicant's type (θ, y) is private information, the bank cannot observe it and offers an identical loan contract for everyone with repayment $(1+r)$ for the loan of 1 unit of money. We assume that the variable rate r is determined by the *fixed* formula $r = r_C + \rho$, where the unit profit ρ is a constant, announced to the agents as part of the loan contract. Although restrictive, this formulation is not completely unrealistic as the real-life variable rate loans are often structured as adding some fixed rate to a benchmark floating rate.

Supply side competition drives down the *ex ante* expected profit of the bank to zero. However, this does not necessarily mean that $\rho = 0$ because the bank needs to account for defaults. We will discuss the equilibrium level of ρ after examining the default probabilities. Until then, suppose that some ρ is given.

For the agents, the fact that ρ is announced and fixed simply means that the variable rate r moves with the cost of funds, hence also follows r_C's gamma distribution although its mean is "shifted" by ρ, which we can write (with some abuse of notation) as $r \sim \Gamma(\alpha, \beta) + \rho$.

4.1. Effective Interest Rate

If the realized value of r is r_0, then a borrower of type (θ, y) gets the net income $(y - r_0)$ with utility $u(y - r_0) = -e^{-\theta(y - r_0)}$. Technically speaking, this statement is correct only if $y \geq r_0$. If $y < r_0$, then the borrower is unable to pay back in full and must default.

Let us suppose, for the moment, that the agent's net income is $(y - r)$ whether this is positive or negative. Since the value of r is random, the certainty equivalent w_C of the random net income $w \equiv y - r$ is defined by $u(w_C) \equiv E[u(w)]$. We can show (see Appendix A1) that

$$w_C = y - \frac{\alpha}{\beta} \ln\left(\frac{\beta}{\beta - \theta}\right) - \rho = y - \frac{\mu^2}{\theta\sigma^2} \ln\left(\frac{\mu}{\mu - \theta\sigma^2}\right) - \rho$$

for $r \sim \Gamma(\alpha, \beta) + \rho$. In order for the log function to be well-defined, we need to restrict the values of θ to $\theta < \beta = \mu / \sigma^2$, so we assume $\overline{\theta} < \beta$ from now on.

Since the borrower begins with $w = 0$, she takes the loan when $w_C \geq 0$; in other words when y is no less than the effective interest rate $\overline{r}(\theta)$ defined in the following:

$$w_C \geq \frac{\alpha}{\beta}\ln\left(\frac{\beta}{\beta-\theta}\right)+\rho=\frac{\mu^2}{\theta\sigma^2}\ln\left(1+\frac{\theta\sigma^2}{\mu}\right)+\rho\equiv\tilde{r}(\theta), \tag{1}$$

Note that $\tilde{r}(\theta)$ is a non-random interest rate that is considered equivalent in *ex ante* expected utility to the random rate r by an agent of type $(\theta,\cdot)$.

Some remarks are warranted for condition (1). The agent has no wealth and even if she defaults, her final wealth cannot fall below 0. This could create an incentive for the agent to take the loan and default if things turn out bad. But there must be substantial cost (*e.g.*, fall in credit rating, seizure on future income, social stigma, etc.) to the borrower in the event she defaults, so we assume that she does not take a loan if she "expects" to default. By treating the net income of the agent as $(y-r)$ instead of $\max\{y-r,0\}$ in calculating $\tilde{r}(\theta)$, we are actually assuming that the default cost is $(y-r)$ when it is negative. It is plausible that the default cost is proportional to the "size" of the default, and taking $y-r<0$ itself as the default cost is very convenient.[12] We establish some properties of the effective interest rate $\tilde{r}(\theta)$ in the following lemma.

Lemma 1. Let $\overline{r}(\theta)$ be as defined in (1), then we have
(i) $\lim_{\theta\to 0}\tilde{r}(\theta)=\frac{\alpha}{\beta}+\rho=\mu+\rho$; for a risk neutral agent, the effective rate is the mean rate
(ii) $\tilde{r}'(\theta)>0$ for $0<\theta<\beta$; the effective rate is strictly increasing in θ
(iii) $\tilde{r}''(\theta)>0$ for $0<\theta<\beta$; the effective rate is strictly convex in θ.

Proof: See Appendix A2. ■

4.2. Loan Demand

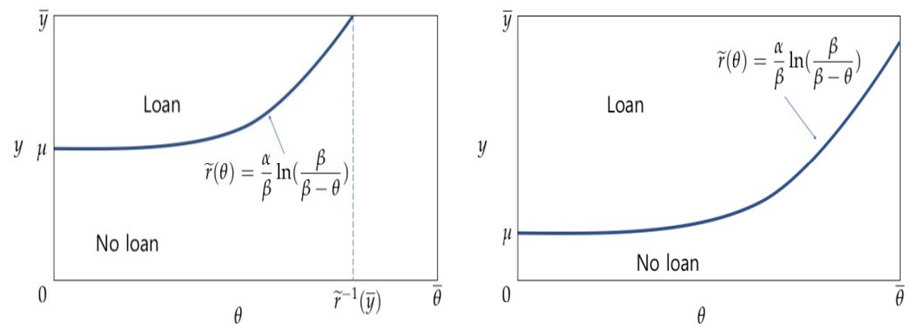
In Sections 4.2 and 4.3 where we discuss loan demand and default, we assume for convenience that $\rho=0$ because ρ can simply added to the resulting formulas later. For example, when we say μ, we actually mean $\mu+\rho$ in this subsection. The presence of ρ will be made explicit in Section 4.4 when we discuss the equilibrium value of ρ.

By Lemma 1, the marginal agents who are indifferent between taking and not taking the loan are represented by a strictly convex increasing curve $\tilde{r}(\theta)$ with the vertical intercept μ in the (θ,y)-plane (Figure 5). Those agents below the curve do not take the loan ("No loan") while those above the curve take the loan ("Loan").

[12] See Section 6 for a discussion of default costs.

For $\theta = 0$ (risk neutral), the agent takes the loan if her income is not below the mean interest rate μ. As θ increases, agents require higher y to accept the same loan. Note that (i) and (ii) in the lemma would apply even if r follows a distribution other than gamma. The use of a gamma distribution places a further restriction of convexity (iii) on $\tilde{r}(\theta)$.

[Figure 5] Borrowing decisions of agents under variable rate contract



(a) $\tilde{r}^{-1}(\overline{y}) < \overline{\theta}$ (b) $\tilde{r}^{-1}(\overline{y}) > \overline{\theta}$

Figure 5 shows two cases. In (a), no agents with $\theta > \tilde{r}^{-1}(\overline{y})$ take the loan since the effective interest rate exceeds the maximum possible income $\overline{y}$. In (b), $\tilde{r}(\overline{\theta}) < \overline{y}$ so that there are agents with income higher than the maximum possible effective interest rate $\tilde{r}(\overline{\theta})$ and would always take the loan. For convenience, we make the following assumption.[13]

Assumption 3. $\tilde{r}(\overline{\theta}) = \overline{y}$ (See Figure 6 below.)

Let the total mass of the agent population be $\overline{\theta}\,\overline{y}$ so that the density is 1 for all (θ, y). Then, the loan demand is

$$L = \overline{\theta}\,\overline{y} - \int_0^{\overline{\theta}} \int_0^{\tilde{r}(\theta)} 1 dy d\theta = \overline{\theta}\,\overline{y} - \int_0^{\overline{\theta}} \tilde{r}(\theta) d\theta . \tag{2}$$

An analytic solution of the above integral is not available, but we can resort to approximate expansions if we need more concrete formulas.

[13] This can be achieved by truncating some portions out of the population box: in (a), remove the box $[\tilde{r}^{-1}(\overline{y}), \overline{\theta}] \times [0, \overline{y}]$ and rename $\tilde{r}^{-1}(\overline{y})$ as $\overline{\theta}$; in (b), remove the box $[0, \overline{\theta}] \times [\tilde{r}(\overline{\theta}), \overline{y}]$ and rename $\tilde{r}(\overline{\theta})$ as $\overline{y}$. Those agents belonging to the removed portions are either never active (in (a)) or always active (in (b)) in the loan market and can therefore be ignored.

Approximations (for θ small and α large)

Since $\tilde{r}(\theta)$ is nonlinear, it may be useful to have simpler approximate formulas. We can apply Taylor series expansion for sufficiently small θ so that a quadratic expression results:

$$\begin{aligned}\tilde{r}(\theta) &= \frac{\alpha}{\theta}\ln\left(\frac{\beta}{\beta-\theta}\right) \approx \frac{\alpha}{\theta}\left(\frac{1}{\beta}\theta + \frac{1}{2\beta^2}\theta^2 + \frac{1}{3\beta^3}\theta^3\right) \\ &= \frac{\alpha}{\beta} + \frac{\alpha}{2\beta^2}\theta + \frac{\alpha}{3\beta^3}\theta^2 = \mu_1 + \frac{1}{2}\mu_2\theta + \frac{1}{6}\mu_3\theta^2 = \mu + \frac{1}{2}\mu^2\theta + \frac{1}{6}\mu_3\theta^2 ,\end{aligned} \tag{3}$$

Here μ_n is the n-th standardized moment, *i.e.* $\mu_n = E[(r-\mu)^n]$, so $\mu_1 = \mu$ (mean), $\mu_2 = \sigma^2$ (variance), and $\mu_3 = \gamma_1\sigma^3$ (where γ_1 is the skewness parameter[14]).

Furthermore, if we assume an almost symmetric gamma distribution (*i.e.*, α very large so γ_1 very small), we may use the linear form

$$\tilde{r}(\theta) \approx \frac{\alpha}{\theta}\left(\frac{1}{\beta}\theta + \frac{1}{2\beta^2}\theta^2\right) = \frac{\alpha}{\beta} + \frac{\alpha}{2\beta^2}\theta = \mu + \frac{1}{2}\sigma^2\theta .$$

This says, quite plausibly, that a risk averse agent perceives a random interest rate as effectively higher than its mean, with the interest "premium" being approximately $\frac{1}{2}\sigma^2\theta$ (proportional to the variance σ^2 and her degree of risk aversion θ) when θ is small. In fact, this approximate relationship is generally applicable (see Pratt, 1964). If r were normally distributed (and with the CARA utility function), it is well-known that the linear formula holds exactly.

Of course, when the interest rate distribution is not symmetric, (3) is a better approximation where the interest premium has an additional term $\frac{1}{6}\mu_3\theta^2 = \frac{1}{6}\gamma_1\sigma^3\theta^2$, proportional to skewness γ_1, the cube of the standard deviation σ^3, and the square of risk aversion θ^2.

Using the approximation (3), we have (for the almost symmetric case, take $\mu_3 = 0$)

$$\int_0^{\bar{\theta}} \tilde{r}(\theta)d\theta \approx \int_0^{\bar{\theta}} \left(\mu + \frac{1}{2}\sigma^2\theta + \frac{1}{2}\mu_3\theta^2\right)d\theta = \mu\bar{\theta} + \frac{1}{4}\sigma^2\bar{\theta}^2 + \frac{1}{18}\mu_3\bar{\theta}^3 . \tag{4}$$

so the loan demand is approximately

[14] For gamma distribution $\Gamma(\alpha,\beta)$, we have $\gamma_1 = 2\alpha^{-1/2}$.

$$L \approx \bar{\theta}\bar{y} - \bar{\theta}\left(\mu + \frac{1}{4}\sigma^2\bar{\theta} + \frac{1}{18}\mu_3\bar{\theta}^2\right) = \bar{\theta}\left(\bar{y} - \left(\mu + \frac{1}{4}\sigma^2\bar{\theta} + \frac{1}{18}\mu_3\bar{\theta}^2\right)\right). \tag{5}$$

4.3. Repayment and Default

In this subsection, we still assume $\rho = 0$ for convenience. For more general formulas, ρ should be added where appropriate.

Upon maturity, the random interest rate r will be realized as some value and those agents whose income y is lower than the realized r must default. How much does a defaulting borrower repay the bank? The maximum repayment for a defaulter is when the bank can seize whatever is available so the interest repayment equals the income y in case $y < r$. Then, the actual interest paid can be taken to be $\min\{r, y\}$. On the other hand, in the worst possible case, the bank cannot claim anything from a defaulter, say because the defaulter has other priority claimants or because substantial costs are involved in claiming y. In Section 4.4, we will assume that the bank claims a pre-specified minimal amount from all defaulters to derive the upper bound for the equilibrium value of ρ.

[Figure 6] Realized interest rate and payments

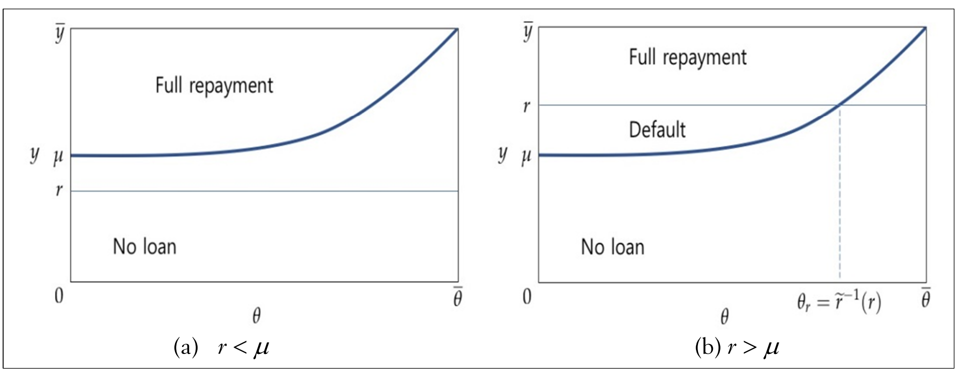


Figure 6(a) shows the case $r < \mu$ so that every agent who took the loan is able to repay in full, while (b) shows the case $r > \mu$ so some borrowers default. Specifically, those agents with $\theta < \tilde{r}^{-1}(r)$ and $\tilde{r}(\theta) \le y < r$ default. For a borrowing agent with type $(\cdot, y)$, the *ex ante* (before the realization of r) probability of her default is $\text{Prob}(r > y) = 1 - \text{Prob}(r \le y) = 1 - F(y)$, where $F(\cdot)$ is the cumulative distribution function for the Gamma random rate r.[15] From the bank's perspective, the *ex ante* probability that none of the borrowers default is

[15] For $r \sim \Gamma(\alpha, \beta)$, the CDF is $F(r) = \frac{1}{\Gamma(\alpha)}\gamma(\alpha, \beta r)$, where $\Gamma(\cdot)$ is the gamma function and $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function. But we only need general properties of CDF in our results.

$\text{Prob}(r < \mu) = F(\mu)$, which is greater than 1/2 since the gamma distribution is skewed so that the median is less than the mean μ.

Figure 6(b) yields our first set of meaningful observations. (Intuitions for most of our conclusions can be grasped using similar figures.) Among the borrowers, the low θ (less risk averse) agents are more likely to default. Given any $r > \mu$, the low θ agents have a greater share among the defaulters. If r is only slightly higher than μ, then only the lowest θ agents default, and as r increases, the successively higher θ agents become defaulters. All of these observations are driven by the simple fact that $\tilde{r}(\theta)$ is an increasing function, or *ceteris paribus*, low θ agents are more willing to take the variable rate loan and many high θ agents stay away from the loan. Self-selection by agents occurs such that high θ (more risk averse) and low y (riskier) agents do not participate in the loan market.

These graphical arguments can be formalized in the following propositions. Both propositions can be interpreted as showing that more risk averse "borrowers" (actual borrowers rather than potential borrowing agents) have lower default risk. The first proposition considers the average *ex ante* probability of default as a function of θ. The second proposition considers the *ex post* distribution of defaulters as a function of θ.

Proposition 1. *Let $\phi(\theta)$ be the average ex ante probability of default among borrowers of type $(\theta,\cdot)$. Then $\phi'(\theta) < 0$ for $\theta > 0$, i.e. more risk averse borrowers have lower average default probability.*

Proof: Since the default probability of type $(\cdot, y)$ is $1 - F(y)$ and since for a given , the borrowers are of the type $y \ge \tilde{r}(\theta)$, we have

$$\phi'(\theta) = \frac{1}{\overline{y} - \tilde{r}(\theta)} \int_{\tilde{r}(\theta)}^{\overline{y}} (1 - F(y))dy .$$

Therefore,

$$\begin{aligned}\phi'(\theta) &= \frac{1}{(\overline{y} - \tilde{r}(\theta))^2}\left(-[1 - F(\tilde{r}(\theta))]\tilde{r}'(\theta)(\overline{y} - \tilde{r}(\theta)) + \tilde{r}'(\theta)\int_{\tilde{r}(\theta)}^{\overline{y}} (1 - F(y))dy\right) \\ &= \frac{\tilde{r}'(\theta)}{(\overline{y} - \tilde{r}(\theta))^2}\left(-[1 - F(\tilde{r}(\theta))](\overline{y} - \tilde{r}(\theta)) + \int_{\tilde{r}(\theta)}^{\overline{y}} (1 - F(y))dy\right) < 0 ,\end{aligned}$$

which is negative because $\tilde{r}'(\cdot) > 0$ by Lemma 1 and $(1 - F(y))$ is strictly decreasing in y. ■

Since $\phi(\theta)$ compresses the y-dimension for each θ, the overall average *ex* *ante* probability of default can be then computed as

$$P = \frac{1}{\bar{\theta}}\int_0^{\bar{\theta}} \phi(\theta)d\theta = \frac{1}{\bar{\theta}}\int_0^{\bar{\theta}} \frac{1}{\bar{y}-\tilde{r}(\theta)}\int_{\tilde{r}(\theta)}^{\bar{y}} 1 - F(y)dyd\theta . \tag{6}$$

For the next proposition, let $\theta_r = \tilde{r}^{-1}(r)$ (note that θ_r depends on r).

Proposition 2. *Fix some $r > \mu$. Let $G(\theta)$ be the cumulative distribution of defaulters on the θ-dimension when the realized interest rate is r. Denote the density as $g(\theta) = G'(\theta)$. Then, $g'(\theta) = G''(\theta) < 0$ for $0 < \theta \le \theta_r$, i.e. the density of defaulters for a given r is the highest at $\theta = 0$ and strictly decreases in θ.*

Proof: For $\theta_0 \in (0, \theta_r]$, $G(\theta_0) = \Pr\text{ob}(\theta \le \theta_0) = \Pr\text{ob}(\tilde{r}(\theta_0) \le y \le r$ and $0 < \theta \le \theta_0)$. So (ignoring the necessary normalizations),

$$G(\theta) \propto \int_0^{\theta}\int_{\tilde{r}(\theta)}^{r} 1dyd\theta = \int_0^{\theta}[r - \tilde{r}(\theta)]d\theta = r\theta - \int_0^{\theta}\tilde{r}(\theta)d\theta ,$$

Therefore,

$$G'(\theta) \propto r - \tilde{r}(\theta) \ge 0 , \quad G''(\theta) \propto -\tilde{r}'(\theta) < 0 .$$

The sign again follows from Lemma 1. ■

For any realized value $r > \mu$, we can also compute the number of defaulters as well as the default rate. The number of defaulters equals the area between the horizontal line $y = r$ and the curve $y = \tilde{r}(\theta)$. To estimate it, we can use the approximation from (4),

$$D(r) = r\theta_r - \int_0^{\theta_r}\tilde{r}(\theta)d\theta \approx \theta_r\left(r - \mu - \frac{1}{4}\sigma^2\theta_r - \frac{1}{18}\mu_3\theta_r^2\right). \tag{7}$$

The exact and approximate formulas for the default rate can be obtained by combining (7) with (2) and (5)

$$\frac{D(r)}{L} = \frac{r\theta_r - \int_0^{\theta_r}\tilde{r}(\theta)d\theta}{\bar{y}\bar{\theta} - \int_0^{\bar{\theta}}\tilde{r}(\theta)d\theta} \approx \frac{\theta_r\left(r - \mu - \frac{1}{4}\sigma^2\theta_r - \frac{1}{18}\mu_3\theta_r^2\right)}{\bar{\theta}\left(\bar{y} - \mu - \frac{1}{4}\sigma^2\bar{\theta} - \frac{1}{18}\mu_3\bar{\theta}^2\right)}.$$

It is obvious that

$$\lim_{r\to\mu}\frac{D(r)}{L}=0\,,\quad \lim_{r\to\bar{y}}\frac{D(r)}{L}=1$$

since $\lim_{r\to\mu}\theta_r=0$ and $\lim_{r\to\bar{y}}\theta_r=\bar{\theta}$. Moreover, we have the following characterizations.

Proposition 3. *Fix some* $r>\mu$. *The number of defaulters* $D(r)$ *and the default rate* $D(r)/L$ *increase at accelerating rates as* r *rises, i.e.*

$$D'(r)>0\,,\quad D''(r)>0$$

Proof: Since

$$\frac{d\theta_r}{dr}=\frac{1}{\tilde{r}'(\theta_r)}>0\,,$$

we have (remembering that θ_r is a function of r such that $\tilde{r}(\theta_r)=r$)

$$D'(r)=\theta_r+r\frac{d\theta_r}{dr}-\overline{r}(\theta_r)\frac{d\theta_r}{dr}=\theta_r>0\,,\quad D''(r)=\frac{d\theta_r}{dr}>0\,.$$

Since L does not depend on the realized value of r, $D(r)/L$ behaves in the same way as $D(r)$ when r changes. ■

If we make a stronger assumption of $\mu_3\approx 0$ (almost symmetric distribution for r), then we have much simpler approximate expressions because $\tilde{r}(\theta)\approx\mu+\frac{1}{2}\sigma^2\theta$ is linear:

$$\bar{\theta}\approx\frac{2(\bar{y}-\mu)}{\sigma^2}\,,\quad \theta_r\approx\frac{2(r-\mu)}{\sigma^2}\,,\quad L\approx\frac{(\bar{y}-\mu)^2}{\sigma^2}\,,\quad D(r)\approx\frac{(r-\mu)^2}{\sigma^2}$$

and

$$\frac{D(r)}{L}\approx\frac{(r-\mu)^2}{(\bar{y}-\mu)^2} \tag{8}$$

For these simpler approximations, we again have $D'(r)>0$, $D''(r)>0$. Moreover,

(8) reveals that the default rate increases approximately quadratically as r rises, if the random rate r is almost symmetrically distributed around its mean μ.

4.4. Equilibrium Level of Unit Profit $\rho^* = r - r_C$

We complete the preliminary discussion of our model by characterizing the equilibrium value of unit profit $\rho^* = r - r_C$ for the bank and by extension the expected value of the variable interest rate $E[r] = E[r_C] + \rho^* = \mu + \rho^*$.

Note that the contract is signed in date 1 before the random cost r_C is realized and default occurs in date 2, so the equilibrium value ρ^* can only incorporate the *ex ante* expectations at the beginning of date 1. For an individual bank, once ρ^* is determined, there is no risk in the value of unit profit. The risk aversion of the bank then does not play a role here. Our equilibrium condition is simply that the expected payoff of the bank be that of zero profit.

Conceptually, determining ρ is straightforward. The bank earns ρ for each borrower who does not default. On the other hand, the bank's earning is less and perhaps negative for defaulters. The value of ρ^* should be such that the expected earning equals the expected loss. To derive ρ^*, we need to specify how much the bank can claim from a defaulter.

Since the actual value of ρ^* does not offer much additional insight, we will only derive an upper bound for ρ^* by making some simplifying assumptions. We can classify the support of r_C into three intervals or *events*, namely, $E_1 = [0, \mu]$, $E_2 = [\mu, \bar{y} - \rho]$, and $E_3 = [\bar{y} - \rho, \infty]$. If $r_C \in E_1$, then $r = r_C + \rho \le \mu + \rho \le \tilde{r}(\theta)$ for all θ so that every borrower is able to repay; E_1 is "no one defaults" event. In E_2 some borrowers default and others do not, while in E_3 every borrower defaults.

First, what happens in the event E_1 is easy to see. For any ρ, the *ex ante* probability that no borrower defaults is $\Pr\text{ob}(E_1) = F(\mu)$. We already noted that this probability is $> 1/2$ by the skewness of the gamma distribution.

Next, for $r_C \in E_2 \cup E_3$, an agent of type $(\cdot, y)$ defaults if $r = r_C + \rho > y$. She is still able to repay up to y so the bank's maximum net earning from her is $y - r_C$, which may be positive or negative (and, of course, $< \rho$). But computing the expected value for every borrower and every $r_C \in E_2 \cup E_3$ is cumbersome.

We assume that the loan contract specifies that all defaulters pay back $\mu + \rho^*$, which is the lowest possible income of a borrower. Then the bank's expected net earning per borrower is

$$\int_0^{\mu} \rho f(r_C) dr_C + \int_{\mu}^{\bar{y}-\rho} \rho \left(1 - \frac{D(r_C)}{L}\right) f(r_C) dr_C + \int_{\mu}^{\bar{y}-\rho} (\mu + \rho - r_C) \frac{D(r_C)}{L} f(r_C) dr_C$$

$$
\begin{aligned}
&+\int_{\overline{y}-\rho}^{\infty}(\mu+\rho-r_C)f(r_C)dr_C \\
&=F(\overline{y}-\rho)\rho+\int_{\mu}^{\overline{y}-\rho}(\mu-r_C)\frac{D(r_C)}{L}f(r_C)dr_C+(1-F(\overline{y}-\rho))(\mu+\rho)-\int_{\overline{y}-\rho}^{\infty}r_C f(r_C)dr_C \\
&=\rho+(1-F(\overline{y}-\rho))\mu+\int_{\mu}^{\overline{y}-\rho}(\mu-r_C)\frac{D(r_C)}{L}f(r_C)dr_C-\int_{\overline{y}-\rho}^{\infty}r_C f(r_C)dr_C \\
&>\rho-F(\overline{y}-\rho)\mu-\int_{\mu}^{\overline{y}-\rho}(r_C-\mu)\frac{D(r_C)}{L}f(r_C)dr_C,
\end{aligned}
$$

where the last line is a very crude lower bound. From this, we see that a crude upper bound for ρ^* is

$$\rho^* \le F(\overline{y}-\rho^*)\mu+\int_{\mu}^{\overline{y}-\rho^*}(r_C-\mu)\frac{D(r_C)}{L}f(r_C)dr_C, \tag{9}$$

where the righthand side also depends on ρ^*. Simple arguments along the line of the intermediate value theorem can be invoked to establish the existence of ρ^* that satisfies the inequality.[16] What is important for our later analysis is that the borrower faces the expected interest $E[r]=\mu+\rho^*$, where ρ^* satisfies the above inequality.

V. Fixed Rate Loan: Switching and Its Impact on Default

This section considers date 1.5 whose event is unanticipated at the time of the contract signing. An alternative loan contract with a fixed interest rate μ^* is offered to the borrowers of the variable rate loan. In our model framework, a borrower only accepts a loan contract if she expects not to default. Then no default occurs for a fixed rate loan, since only those agents with $y \ge \mu^*$ accept the fixed rate loan.[17]

We discuss how μ^* is determined, who switches to it, and the impact of switches on the default probabilities. In this section, the risk attitude of the bank comes into play, because the bank exposes itself to risk by collecting a fixed interest while being subject to a random cost of funds.

---

[16] Rewrite the inequality (9) in the form $\varphi(\rho^*)\ge 0$. Substituting $\rho^*=0$ and $\overline{y}$, we observe $\varphi(0)>0$ and $\varphi(\overline{y})<0$. Since $\varphi(\cdot)$ is continuous (and $\varphi'(\cdot)<0$), there must be a $\overline{\rho}^*$ such that $\varphi(\overline{\rho}^*)=0$ and $\varphi(\rho^*)\ge 0$ for $\rho^*\in[0,\overline{\rho}^*]$.

[17] In real life, some borrowers of a fixed rate loan will default. But it is reasonable to suppose that defaulting is less likely for borrowers of a fixed rate loan because these borrowers are better able to plan ahead. See Koh and Ju (2011) and the references cited therein.

5.1. Equilibrium Level of Fixed Interest Rate μ^*

For a meaningful "switching" model, μ^* should be high enough so that at least some borrowers would prefer to stay with the variable rate loan. If $E[r] = \mu + \rho^* > \mu^*$, then every agent with $y > \mu^*$ will choose the fixed rate loan because risk averse agents perceive the random rate to be higher than its mean $(\tilde{r}(\theta) \geq \mu + \rho^*)$. But all borrowers of the variable rate loan have $y > \mu + \rho^*$, so every borrower would switch. Let us then consider under what conditions we can have $\mu^* > \mu + \rho^*$.

The bank's payoff function per borrower is $v(\rho; \theta_B) = -e^{-\theta_B \rho}$ and for a fixed rate loan, we have $\rho = \mu^* - r_C$ where μ^* is fixed and r_C is random. The bank's expected payoff from the fixed rate loan is $E[v(\mu^* - r_C)]$, which can be expressed as the certainty equivalent

$$\rho \equiv \mu^* - \frac{\alpha}{\theta^B} \ln\left(\frac{\beta}{\beta - \theta^B}\right) = \mu^* - \frac{\mu^2}{\theta_B \sigma^2} \ln\left(\frac{\mu}{\mu - \theta_B \sigma^2}\right).$$

The above formula is immediate because the bank now plays the role of a risk averse borrower who has a certain income μ^* and a random interest $r_C \sim \Gamma(\alpha, \beta)$. So the above formula is analogous to the borrower's $\tilde{r}(\theta)$, where $\theta = \theta_B$. The supply side competition implies that ρ will be driven down to zero. We have thus determined the equilibrium level of μ^*.

Proposition 4. *The market equilibrium level of fixed interest rate* μ^* *is*

$$\mu^* = \frac{\alpha}{\theta^B} \ln\left(\frac{\beta}{\beta - \theta^B}\right) = \frac{\mu^2}{\theta_B \sigma^2} \ln\left(\frac{\mu}{\mu - \theta_B \sigma^2}\right) \approx \mu + \frac{1}{2}\sigma^2 \theta_B + \frac{1}{6}\mu_3 \theta_B^2 \quad ,$$

where the last expression is an approximation as in (3). Therefore, for sufficiently high θ_B, $\mu^* > \mu + \rho^*$.

Assuming an almost symmetric distribution (α large, $\mu_3 \approx 0$), we have $\mu^* > \mu + \rho^*$ if $\theta_B > 2\rho^* / \sigma^2$. From now on, we assume that θ_B is sufficiently high as in the proposition.

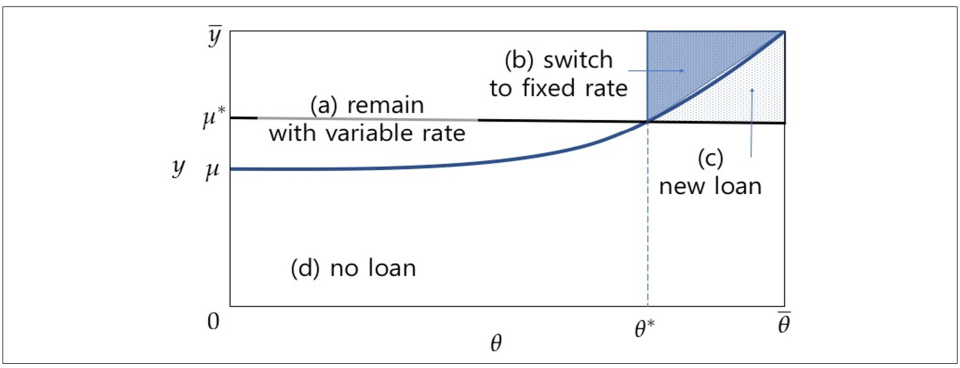
5.2. Decision to Switch to the Fixed Rate Loan

To simplify the notations, we again suppress ρ^*. So we have $\mu^* > \mu$, with the fixed rate loan's interest premium denoted as $\delta \equiv \mu^* - \mu > 0$. When agents are faced with the two loan contracts, there are four possible reactions. As shown in

Figure 7: (a) Some agents remain with the variable rate loan because they find the fixed rate μ^* too high; they have relatively low θ (less risk averse). (b) Some agents switch to the fixed rate loan; they are more risk averse (high θ) as well as have higher income (high y) [the dark shaded triangular region in Figure 7]. This region is the focus of our attention.

In addition, we also have (c) agents who did not take the variable rate loan but now willing to take the fixed rate loan; they are more risk averse (high θ) but have relatively lower y [the light dotted triangular region in Figure 7]. FSC's 2015 Program didn't allow such new loans, but the region (c) warrants some attention as well, as FSC's lead may have encouraged additional fixed rate loans to be introduced in the market. Finally (d) many agents continue not to participate in the loan market.

[Figure 7] Response to the new fixed rate contract



It is obvious from the figure that the switchers have both high y and high θ. Specifically they are characterized by $y \in [\tilde{r}(\theta), \bar{y}]$ and $\theta \in [\theta^*, \bar{\theta}]$, where θ^* is the threshold risk aversion type that is indifferent between the two loan contracts:

$$\tilde{r}(\theta^*) \equiv \mu^* \Rightarrow \theta^* \equiv \tilde{r}^{-1}(\mu^*) .$$

Obviously, θ^* depends on μ^*. In fact, θ^* is simply θ_r defined for Proposition 2 with $r = \mu^*$. From Proposition 3 and Lemma 1, we already know that

$$\frac{d\theta^*}{d\mu^*} = \frac{1}{\tilde{r}'(\theta^*)} > 0 .$$

The key observation on Figure 7 is that the number of switchers in (b) can be quite small compared to the number of those who remain in (a). Selection has occurred so that many agents who would have liked the fixed rate loan do not hold variable rate loan and cannot become switchers. Since Figure 7 only indicates a possibility for some arbitrary parameter values, we develop more formal results in the next subsection.

5.3. Relative Number of Switches

We can compute the relevant areas to determine the size of each agent group. For the switches and new loans, we have

$$\text{(b) switches} = \int_{\theta^*}^{\bar{\theta}} \int_{\tilde{r}}^{\bar{y}(\theta)} 1 dy d\theta = \int_{\theta^*}^{\bar{\theta}} \bar{y} - \tilde{r}(\theta) d\theta = \bar{y}(\bar{\theta} - \theta^*) - \int_{\theta^*}^{\bar{\theta}} \tilde{r}(\theta) d\theta \equiv S$$
$$\text{(c) new loans} = \int_{\theta^*}^{\bar{\theta}} \int_{\mu^*}^{\tilde{r}(\theta)} 1 dy d\theta = \int_{\theta^*}^{\bar{\theta}} \tilde{r}(\theta) - \mu^* d\theta = \int_{\theta^*}^{\bar{\theta}} \tilde{r}(\theta) d\theta - \mu^*(\bar{\theta} - \theta^*) \equiv NL,$$

with the area of rectangle (b) + (c) being $(\bar{\theta} - \theta^*)(\bar{y} - \mu^*)$.

The following proposition examines how the numbers of switches and new loans respond to the changes in the fixed rate μ^*.

Proposition 5. *Let S be the number of switches and NL be the number of new loans as defined above.*

$$\frac{dS}{d\mu^*} < 0, \quad \frac{d^2S}{d\mu^{*2}} > 0, \quad \frac{dNL}{d\mu^*} < 0, \quad \frac{d^2NL}{d\mu^{*2}} > 0,$$

i.e. as the fixed rate increases, the numbers of switches and new loans both decrease at accelerating rates.

Proof: See Appendix A3. ■

Hence, if the fixed rate μ^* is sufficiently high, then the number of switches can be very low. In fact, using the approximations laid out at the end of Section 4.2, we can show that the number of switches falls at least quadratically as the interest premium $\delta = \mu^* - \mu$ increases.

5.4. Comparison of Average Default Probability Before and After the Switch

Let us now examine the average riskiness of the two borrower pools (a) and (b). As we know, the switchers tend to be good risks, so we expect that the remaining borrowers (a) become worse in terms of default risks.

The *ex ante* average default probability for type $(\theta, \cdot)$ agents was computed as

$\phi(\theta)$ in Proposition 1. After introduction of the fixed rate loan, we can recompute the average default probabilities for groups (a) and (b) separately. For the remaining borrowers in (a), it is

$$P_a = \frac{1}{\theta^*}\int_0^{\theta^*} \phi(\theta)d\theta ,$$

and for the switchers in (b), it would have been

$$P_b = \frac{1}{\bar{\theta}-\theta^*}\int_{\theta^*}^{\bar{\theta}} \phi(\theta)d\theta$$

if they did not switch. But they do switch to the fixed rate loan and do not default.

Lemma 2. $P_a > P_b$

Proof (sketch): The key is that $\int_0^{\theta} \phi(\theta)d\theta$ is strictly concave in θ. (Appendix A4.) ■

Finally, let us confirm that $P_a > P$, where P is the overall average default probability when there was only the variable rate loan. Recall from (6) that P has the following form:

$$P = \frac{1}{\bar{\theta}}\int_0^{\bar{\theta}} \phi(\theta)d\theta = \frac{1}{\bar{\theta}}\int_0^{\bar{\theta}} \frac{1}{\bar{y}-\tilde{r}(\theta)}\int_{\tilde{r}(\theta)}^{\bar{y}} (1-F(y))dyd\theta .$$

The following proposition is an immediate corollary of Lemma 2.

Proposition 6. $P_a > P$*, i.e. after switches to the fixed rate loan, the average default probability for the remaining borrowers is greater than the average default probability for the borrowers before the switch.*

Proof: P is the weighted average of P_a and P_b:

$$P = \frac{\theta^*}{\bar{\theta}}P_a + \frac{\bar{\theta}-\theta^*}{\bar{\theta}}P_b .$$

By Lemma 2, $P_a > P_b$ therefore $P_b < P < P_a$. ■

In other words, the introduction of the fixed rate loan by the FSC in 2015 may have exposed the bank (of the remaining variable rate borrowers) to higher default risks. That the fixed rate loans can have the “cream skimming” effect is intuitively plausible. Although we do not have comprehensive data for empirical analysis, FSC’s own evaluation report released a year later[18] has some interesting findings. The FSC compared the micro-characteristics of the switchers in the 2015 program with the new mortgage loan borrowers during March and May, 2015. Quite consistent with our model predictions, the switchers exhibit lower increases in debt balances as well as lower overdue payment rates.

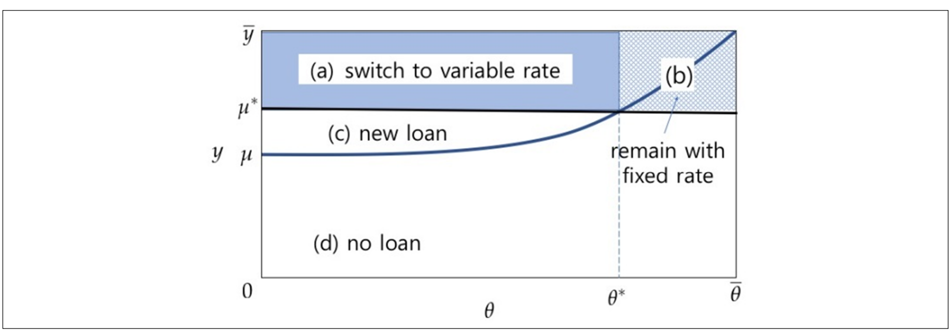
5.5. Alternative Timeline: From Fixed Rate to Variable Rate

Imagine a switching scenario in the opposite direction, inspired by the 1980s US mortgage market. The *status quo* is a fixed rate loan and an alternative variable rate loan is introduced. See the alternative timeline below. What would happen then?

		(unanticipated)	
date 0	date 1	date 1.5	date 2
agent type realized	**fixed rate loan**	**variable rate loan**	maturity

A glance at Figure 8 should convince us that (i) the borrowers of the *status quo* fixed rate loan are in the rectangle (a) + (b); those in (b) remain with the fixed rate loan and those in (a) switch to the new variable rate loan. The additional borrowers in (c) would join in if allowed. For brevity, we will sketch the ideas instead of formally developing results.

[Figure 8] Switch from the fixed rate loan to the variable rate loan



The *status quo* fixed rate loan does not induce two-dimensional selection. High agents and low θ agents are equally likely to find the fixed rate loan attractive

[18] FSC press release dated March 20, 2016.

depending on their income level y. When an alternative variable rate loan is offered, only the relatively low θ agents would find it attractive. The sizes of (a) and (b) depend on $\mu^* - \mu$: If $\theta^* < \frac{1}{2}\overline{\theta}$, then (a) switchers can be fewer than (b) those who remain. On the other hand, if θ^* and μ^* are high as in the figure, there can be a large mass of switches, bolstered by a sizable mass of new loans. The figure's situation may be likened to that of the 1980s US: regulatory changes allowed introduction of new variable rate loans which found much demand.

VI. Extensions of the Model

In this section, we discuss the potential extensions of our model to check the robustness of its conclusions and to explore ways for improvement.

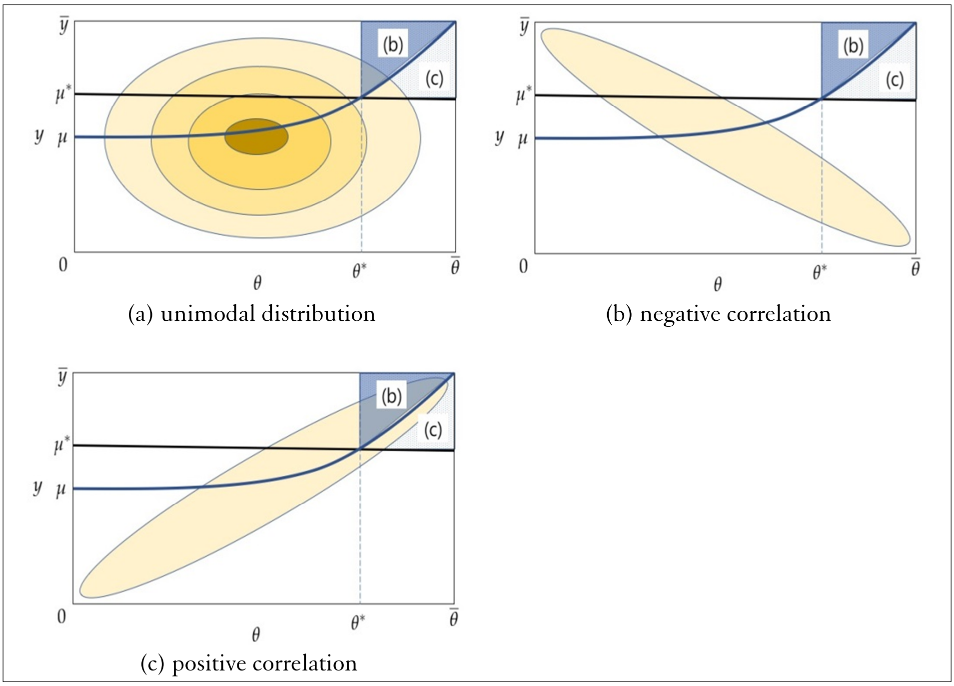
6.1. Type Distribution: Non-Uniform Distributions and Correlation between Dimensions

By Assumption 1, our model posited a jointly independent uniform distribution of types. Without going into detailed analysis, Figure 9 illustrates by showing some examples of non-uniform type distributions. In the figures, the basic ingredients of Figure 7 are reproduced including the effective interest rate curve $\tilde{r}(\theta)$, the fixed rate μ^*, the dark shaded region (b) of switchers, and the light dotted region (c) of new loans.

Figure 9(a) shows the case of a unimodal distribution with the mode occurring around the mean (think of a joint normal distribution). The innermost, darkest-colored oval shape contains the largest mass of agents, while the outer, lighter-colored shapes contain the smaller masses. The region (b) (switchers) falls on the outermost light-colored part of the distribution. In this situation, a very small number of switches will be observed when the new fixed rate loan is introduced because extreme types (which characterize the switchers) are rare in this population.

Figure 9(b) shows the case of a negative correlation between θ and y, *i.e.* when less risk averse agents tend to have higher income and vice versa. Here again, switches will be rarely observed. This is because highly risk averse agents have little income so they are not an active part of the loan market. Figure 9(c) shows the opposite case of a positive correlation, *i.e.* when more risk averse agents tend to have higher income. Among the three scenarios considered here, this is the only one with a possibly large number of switchers. This is because the two-dimensional selection occurs in a similar direction as the correlation between the types.

[Figure 9] Examples of non-uniform type distributions



(a) unimodal distribution
(b) negative correlation
(c) positive correlation

6.2. Random Income

In the model, we assumed that each agent's income y is fixed once realized. We can relax this by positing that both y and r are independent gamma-distributed random variables (remember that y represents the extra profit generated by a funded project, which cannot fall to large negative values). Then we need to handle the "difference" $(y-r)$ or, more generally, the linear combinations of two independent gamma-distributed random variables. An easy preliminary result is the following.

Lemma 3. If two gamma-distributed random variables $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$ are independently distributed, then $X + Y \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.

Proof: See Appendix A5. ■

In other words, if two independent random variables are gamma distributed with the same rate parameter β, then their sum is also a gamma-distributed random variable. While we could look for a more general result for the case with different β's (see, *e.g.* Moschopoulos, 1985), our immediate concern is the difference $X - Y$. The difference $X - Y$ when X, Y are both gamma distributed is not gamma distributed but its distribution can be characterized, dubbed the *Gamma difference distribution* by Mathai (1993) and Klar (2015). Without going into details, we report the following (quite natural) facts on the statistical moments of the distribution.

Lemma 4. *Suppose that two gamma-distributed random variables* $X \sim \Gamma(\alpha_1, \beta_1)$ *and* $Y \sim \Gamma(\alpha_2, \beta_2)$ *are independently distributed. Let* $\mu_i = \alpha_i / \beta_i$ *and* $\sigma_i^2 = \alpha_i / \beta_i^2$ *be each variable's mean and variance. Then the difference* $X - Y$ *has the mean* $(\mu_1 - \mu_2)$ *and the variance* $(\sigma_1^2 + \sigma_2^2)$.

Proof: See Klar (2015). ■

Hence, suppose now that an agent's income y is a gamma-distributed random variable with mean μ_y and variance σ_y^2 and that the variable interest rate r is another gamma random variable with mean μ_r and variance σ_r^2. Also assume that y and r are independent. We can now let μ_y vary over agents and take it as the second dimension of her type vector. Then for an agent of type (θ, μ_y), we could carry out a similar analysis as before. For example, the effective interest rate curve $\tilde{r}(\theta)$ can be generalized to accommodate the randomness of y. The effect would be that a risk averse agent puts an even higher premium on a variable interest rate.

6.3. Distributions Other Than Gamma for r

In fact, there is no reason to insist on r having a gamma distribution. We chose the gamma distribution because it is flexible and tractable enough for a variable having non-negative values. Another popular distribution with the non-negative support is the lognormal distribution. We can use the constant relative risk aversion (CRRA) utility function $u(w) = w^{1-\theta}$ to deal with the lognormal distribution comfortably (see Kim, 2016).

More generally, reflecting on our analysis would reveal that the most critical part of the model that drives the results is that $\tilde{r}(\theta)$ is an increasing function. We can show that the interest rate premium is an increasing function of θ, along the similar lines of Theorem 2 in Pratt (1964). So most of the "qualitative" results should continue to hold. The results restricted to gamma-distributed rates are more concrete quantitative formulas derived.

6.4. Default Cost

In deriving the certainty equivalent of net income and the effective interest rate $\tilde{r}(\theta)$, we assumed that a defaulter's default cost is exactly the negative net income

$y - r < 0$. This assumption eased our analysis immensely. However, it restricts the default cost to be exactly proportional to the size of the "shortage" of income from the full interest rate.

A more reasonable assumption would be that the default cost is convex: a bigger default may be more heavily punished. If we incorporate such convex default costs, an agent's expected utility from the variable rate loan would fall from what we computed in the model. This would presumably lead to an even higher (and more convex) effective interest rate schedule. But the qualitative results would again remain mostly intact.

More generally, if we want to fully account for the default costs in our model, it should perhaps be added as a third private type dimension of the borrower (Brueckner, 2000; Harrison et al., 2004). Such an endeavor is best left for future work.

6.5. Size of the Loan

Our model assumed that every borrower takes the identical loan of 1 unit. Suppose instead that the size $a(y)$ of the loan depends on the agent's income y. There are two questions. First, how would $a(y)$ depend on y? Is it increasing or decreasing? Is it non-monotone and perhaps uni-modal? Second, how would such dependence affect the choice of the borrower?

As for the first question, Brueckner (2000) showed through a signaling model that riskier borrowers borrow more. Harrison *et al*. (2004) revealed *via* a different signaling model that the relationship depends on the size of the borrower's default costs. When default cost is low enough, their model predicts an inverse relationship between riskiness and loan size (also see discussion of empirical studies there). So dealing with this question again requires adding a third type dimension.

On the other hand, we might envision a model where $a(y)$ is increasing in y.[19] This is perhaps because the bank can observe an informative signal about y and adjust the size of its granted loans. Or it may be that the agent derives a higher value from a larger loan (think of the size and/or quality of the house being purchased), but only high y agents can afford it.

Therefore, the first question is essentially an empirical one with some theoretical challenges involved. Furthermore, now that the bank can grant different sizes of loan contracts, the size of the loan can work as a screening device. We should at least consider non-linear interest schedules, rather than a constant interest rate. Therefore, it seems that answering the second question also requires a separate

[19] Related to this point, an anonymous referee noted some interesting figures in a recent survey on Household Finances and Welfare by *Statistics Korea*. For example, the top quintile (in terms of income) households hold 46% of the credit balance, while the bottom quintile households hold only 4%. I thank the referee for pointing out the survey.

investigation. Such an investigation is quite relevant to the 2015 FSC Refinancing Program as well because it also imposed a ceiling of the existing loan size for eligibility of switching. We leave these questions for future research.

VII. Conclusion

For an intuitive recapitulation of our analysis, Table 2 offers a classification of agents into 4 categories according to whether each type dimension is high or low. The first row has high income (y) types whose probability of default is relatively low, so they are "good risk" types from a lender's perspective. In contrast, the second row has low income (y) or "bad risk" types. The first column has low risk aversion (θ) types who are not willing to pay a high premium for insurance, while the second row has high risk aversion (θ) types with a high insurance premium.

From social point of view, it would be better if the bad risk types stay away from the loan market if possible; and if they are in the market, some policy intervention may be warranted. Any policy intervention is going to involve some form of insurance, and the willingness of each agent to pay for such insurance (premium) is inversely related to the cost of policy implementation in the sense that the policy is going to be more costly for the low premium (low θ) types.

[Table 2] Classification of agent types

	θ low	θ high
y high	**HL types** high income, low risk aversion good risk, low insurance premium	**HH types** high income, high risk aversion good risk, high insurance premium
y low	**LL types** low income, low risk aversion bad risk, low insurance premium	**LH types** low income, high risk aversion bad risk, high insurance premium

From these considerations, it appears that LL types are going to be the most problematic as they are bad risks as well as impose high policy costs. The problem with the policy initiative of the Korean FSC in 2015 becomes apparent when we superimpose these (2×2) classification onto the two-dimensional type space. Figure 10 shows the part of the agent population box where $y \geq \mu$ (the rest of the box is not policy relevant) with the $\tilde{r}(\theta)$ curve and the horizontal line representing the fixed rate μ^*. The dark shaded triangular region is the potential switchers from the variable rate loan.

[Figure 10] Overview of self-selection by agents and policy implications

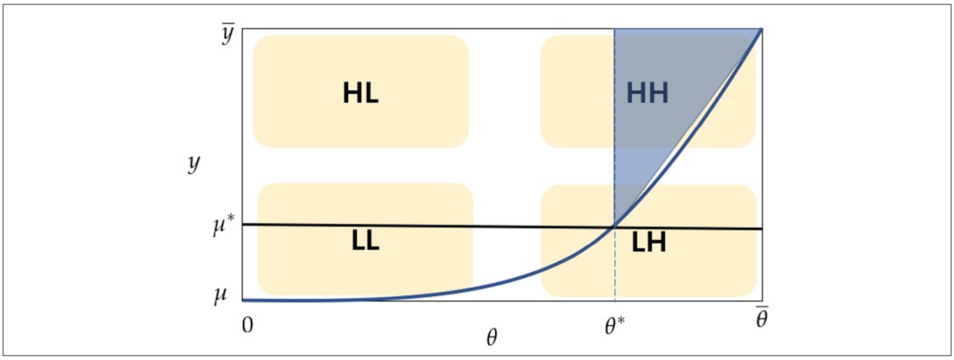


Figure 10 shows why offering a fixed rate loan is ineffective. The switchers come from the HH types, who are good risks anyway. The LH types who are bad risks and impose a low policy cost self-selected out of the loan market before the policy initiative. On the other hand, the most problematic LL types are not affected by the policy at all. The banks that have these agents as borrowers may have been made worse off by the policy initiative. The difficulty is that there are no clear policy tools to separate LL (high cost, bad risks) from HL (good risks).

So what policy lessons can we draw from our two-dimensional selection analysis? It is our claim that the FSC's inducement of switches to fixed rate loans does not achieve much in terms of reducing default risks because the switchers are mostly good risks anyway. On the other hand, those who did not switch to the fixed loan have revealed themselves to be less risk averse and some of them are bad risks and should be monitored in the future.

Our analysis at least highlights how difficult it is to single out the bad risks among existing borrowers. In our simple framework, there appears to be no incentive-compatible screening device for them. We find some interesting clues from Veiga and Weyl (2016), who examined the multi-dimensional selection (called "sorting") occurring in what they call "selection markets" which are prevalent in finance and insurance sectors. They emphasized that the usual selection occurs *via* pricing while the sorting by quality occurs *via* nonprice features such as downpayments. For example, in the setting of Korean mortgage market, perhaps the existing bad risks are a lost cause, but at least for future borrowers, requiring amortization can be a more effective instrument for selecting future bad risks out of the loan market.

Any lessons drawn from our simple model are speculative at best. It is our hope that market players and policy makers see the importance of multi-dimensional selection and they can think about it in an intuitive framework.

Appendix: Proofs

A1. Derivation of $\tilde{r}(\theta)$

For simplicity, suppose $\rho = 0$, then the density function of r is $f(r) = \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}$. Because it is a probability density, $\int_0^\infty f(r)dr = 1$.

$$\begin{aligned}
E[u(y-r)] &= \int_0^\infty -e^{-\theta(y-r)} \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r} dr \\
&= -e^{-\theta y} \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-(\beta-\theta)r} dr \\
&= -e^{-\theta y} \frac{\beta^\alpha}{(\beta-\theta)^\alpha} \int_0^\infty \frac{(\beta-\theta)^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-(\beta-\theta)r} dr \\
&= -e^{-\theta y} \left(\frac{\beta}{\beta-\theta} \right)^\alpha = -e^{-\theta w_C} = u(w_C)
\end{aligned}$$

because the integral in the third line involves integrating a probability density. Taking logs of each side

$$-\theta y + \alpha \ln\left(\frac{\beta}{\beta-\theta} \right) = -\theta w_C \Rightarrow w_C = y - \frac{\alpha}{\theta} \ln\left(\frac{\beta}{\beta-\theta} \right).$$

If $\rho \neq 0$, then we can simply replace "y" with "$y-\rho$" and the result follows.

Remark: Since y is not random, $E[u(y-r)] = E[e^{-\theta(y-r)}] = e^{-\theta y} E[e^{-\theta(-r)}]$. Setting $w^C = y - \tilde{r}(\theta)$, we have $u(w_C) = e^{-\theta y} e^{-\theta(-\tilde{r}(\theta))}$. So determining $\tilde{r}(\theta)$ boils down to determining the certainty equivalent of $(-r)$ when $r \sim \Gamma(\alpha, \beta)$ and u is CARA. Since $(-r)$ is the "negative" of gamma distribution, it has the same variance α / β^2 but its mean is $-\alpha/\beta$. It is as if $(-r)$ follows the gamma distribution with parameters $(\alpha, -\beta)$. Using Kim's (2016) formula but replacing β with $-\beta$, we obtain the same formula.[20]

A2. Proof of Lemma 1

(i) Since $(\frac{\beta}{\beta-\theta}) \to 0$ as $\theta \to 0$, we can apply L'Hopital's rule:

[20] Kim's (2016) discussion in pp. 18-19 is incorrect in omitting the minus sign.

$$\lim_{\theta\to 0}\tilde{r}(\theta)=\lim_{\theta\to 0}\frac{\alpha}{\theta}\ln\left(\frac{\beta}{\beta-\theta}\right)+\rho=\lim_{\theta\to 0}\frac{d}{d\theta}\left[\alpha\ln\left(\frac{\beta}{\beta-\theta}\right)\right]$$
$$+\rho=\lim_{\theta\to 0}\frac{\alpha}{\beta-\theta}+\rho=\frac{\alpha}{\beta}+\rho .$$

(ii) Since $\alpha>0$, let us examine the sign of $\tilde{r}'(\theta)/\alpha$:

$$\frac{1}{\alpha}\tilde{r}'(\theta)=\frac{d}{d\theta}\left[\frac{1}{\theta}\ln\left(\frac{\beta}{\beta-\theta}\right)\right]=\frac{1}{\theta^2}\left(\frac{\theta}{\beta-\theta}-\ln\left(\frac{\beta}{\beta-\theta}\right)\right)\equiv\frac{1}{\theta^2}A(\theta) . \qquad (*)$$

To determine the sign in (*), it is sufficient to determine the sign of the expression enclosed in the parentheses, denoted as $A(\theta)$. First note that $A(0)=0$ and

$$A'(\theta)=c\frac{\beta-\theta+\theta}{(\beta-\theta)^2}-\frac{1}{\beta-\theta}=\frac{\beta-(\beta-\theta)}{(\beta-\theta)^2}=\frac{\theta}{(\beta-\theta)^2}>0 \quad \text{for} \quad \theta>0 .$$

Therefore, we conclude $A(\theta)>0$ for $\theta>0$, and by extension that $\tilde{r}'(\theta)>0$ for $\theta>0$.
(iii) Again consider $\tilde{r}''(\theta)/\alpha$.

$$\frac{1}{\alpha}\tilde{r}''(\theta)=\frac{d}{d\theta}\left[\frac{1}{\theta(\beta-\theta)}-\frac{1}{\theta^2}\ln\left(\frac{\beta}{\beta-\theta}\right)\right]$$
$$=-\frac{\beta-2\theta}{\theta^2(\beta-\theta)^2}-\frac{1}{\theta^3}\left(\frac{\theta}{\beta-\theta}-2\ln\left(\frac{\beta}{\beta-\theta}\right)\right)$$
$$=\frac{1}{\theta^3(\beta-\theta)^2}\left(-\beta\theta+2\theta^2-\theta(\beta-\theta)+2(\beta-\theta)^2\ln\left(\frac{\beta}{\beta-\theta}\right)\right)$$
$$=\frac{1}{\theta^3(\beta-\theta)^2}\left(3\theta^2-2\beta\theta+2(\beta-\theta)^2\ln\left(\frac{\beta}{\beta-\theta}\right)\right)\equiv\frac{1}{\theta^3(\beta-\theta)^2}B(\theta) . \quad (**)$$

To determine the sign in (**), it is sufficient to determine the sign of the expression enclosed in the last parentheses, denoted as $B(\theta)$. Again note that $B(0)=0$ and

$$B'(\theta)=6\theta-2\beta-4(\beta-\theta)\ln\left(\frac{\beta}{\beta-\theta}\right)+2(\beta-\theta)^2\frac{1}{\beta-\theta}$$
$$=4\theta-4(\beta-\theta)\ln\left(\frac{\beta}{\beta-\theta}\right)$$

$$4(\beta-\theta)\left(\frac{\theta}{\beta-\theta}-\ln\left(\frac{\beta}{\beta-\theta}\right)\right)=4(\beta-\theta)A(\theta)>0 \quad \text{for} \quad \theta>0$$

because $\beta-\theta>0$ by domain requirement of log function and $A(\theta)>0$ for $\theta>0$ as determined in (ii). Therefore, we conclude $B(\theta)>0$ for $\theta>0$ and by extension that $\tilde{r}''(\theta)>0$ for $\theta>0$.

A3. Proof of Proposition 5

$$\frac{dS}{d\mu^*}=-\overline{y}\frac{d\theta^*}{d\mu^*}+\tilde{r}(\theta^*)\frac{d\theta^*}{d\mu^*}=(\tilde{r}(\theta^*)-\overline{y})\frac{d\theta^*}{d\mu^*}<0$$

The negative sign follows since $\tilde{r}(\theta^*)<\overline{y}$ and $d\theta^*/d\mu^*=1/\tilde{r}'>0$.

$$\frac{d^2S}{d\mu^{*2}}=\tilde{r}'(\theta^*)\left(\frac{d\theta^*}{d\mu^*}\right)^2+(\tilde{r}(\theta^*)-\overline{y})\frac{d^2\theta^*}{d\mu^{*2}}>0$$

The above is positive because

$$\frac{d^2\theta^*}{d\mu^*}=-\tilde{r}''(\theta^*)\tilde{r}'(\theta^*)\left(\frac{d\theta^*}{d\mu^*}\right)^2<0.$$

In addition,

$$\frac{dNL}{d\mu^*}=-\tilde{r}(\theta^*)\frac{d\theta^*}{d\mu^*}-(\overline{\theta}-\theta^*)+\mu^*\frac{d\theta^*}{d\mu^*}=-(\overline{\theta}-\theta^*)<0$$
$$\frac{d^2NL}{d\mu^{*2}}=\frac{d\theta^*}{d\mu^*}>0.$$

A4. Proof of Lemma 2

It is helpful to introduce a new notation:

$$\Phi(\theta)\equiv\int_0^\theta \phi(\theta)d\theta.$$

Then we can re-write P, P_a, P_b as follows:

$$P_a = \frac{1}{\theta^*}\int_0^{\theta^*} \phi(\theta)d\theta = \frac{1}{\theta^*}\Phi(\theta^*),$$

$$P_b = \frac{1}{\overline{\theta}-\theta^*}\int_{\theta^*}^{\overline{\theta}} \phi(\theta)d\theta = \frac{1}{\overline{\theta}-\theta^*}(\Phi(\overline{\theta})-\Phi(\theta^*)).$$

Claim: $\Phi''(\theta) < 0$

Proof: $\Phi'(\theta) = \phi(\theta)$ and $\Phi''(\theta) = \phi'(\theta)$. But by Proposition 1, we know $\phi'(\theta) < 0$. ■

Knowing that $\Phi(\theta)$ is strictly concave immediately yields that $P_a > P_b$. To see this, note that by the mean value theorem, $P_a = \Phi'(\theta_a)$ for some $\theta_a \in (0, \theta^*)$ and $P_b = \Phi'(\theta_b)$ for some $\theta_b \in (\theta^*, \overline{\theta})$. Since $\theta_a < \theta_b$ and $\Phi'' < 0$, we conclude $P_a = \Phi'(\theta_a) > \Phi'(\theta_b) = P_b$.

A5. Proof of Lemma 3

We can use the moment generating function $m(t) = E[e^{tX}]$. If $X \sim \Gamma(\alpha, \beta)$, then

$$m_X(t) = \int_0^\infty e^{tx}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}dx = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-(\beta-t)x}dx.$$

For independent random variables X and Y, we have $m_{X+Y}(t) = m_X(t)m_Y(t)$. Therefore, when $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$,

$$m_{X+Y}(t) = \frac{\beta^{\alpha_1}}{(\beta-t)^{\alpha_1}} \cdot \frac{\beta^{\alpha_2}}{(\beta-t)^{\alpha_2}} = \beta^{\alpha_1+\alpha_2}(\beta-t)^{\alpha_1+\alpha_2}.$$

It is obvious that $m_{X+Y}(t)$ is the moment generating function of a random variable $\sim \Gamma(\alpha_1 + \alpha_2, \beta)$.

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