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ARBITRAGE AND VALUATION WITH A MINIMUM WEALTH CONSTRAINT*

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This purpose of this paper is to study the relation among "absence of arbitrage", "viability of a price pair", and "existence of a linear pricing rule" in many interesting spaces with a minimum wealth constraint imposed on consumption sets. The study of a relation among these concepts takes roots in Kreps, which provides a rationale of asset pricing theories based on pricing rules by arbitrage. The main contribution of this paper is to establish the equivalence relation with a minimum wealth constraint imposed on a consumption set. Inspired by Mas-Colell, we consider investors with uniformly proper preferences. We generalize the equivalence relation to economies where investors trade contingent claims in markets with transaction costs.

I. INTRODUCTION

The purpose of this paper is to study the relation among "absence of arbitrage", "viability of a price pair", and "existence of a linear pricing rule" in many interesting spaces with a minimum wealth constraint imposed on consumption sets.

We consider an exchange economy. Let X denote a contingent claim space and M denote a marketed contingent claim space. If markets are incomplete, then $X \neq M$. Investors trade contingent claims given a price system π. We call (M, π) a price pair. An arbitrage opportunity is a marketed claim with a non-positive value that increases some investor's utility. A price pair is viable if there exists an investor who finds an optimal choice given the price pair. Finally a linear pricing rule is a strictly positive linear functional which extends π to X.

The study of a relation among these concepts takes roots in Kreps(1981), which provides a rationale of asset pricing theories based on pricing rules by arbitrage. Kreps shows that three concepts are equivalent, taking the whole space X

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as a consumption set, if "absence of arbitrage" is modeled by "no free lunches". As Kreps points out, the equivalence relation fails if a consumption set without interior points is taken instead of the whole space. This means that the positive orthant of any reflexive Banach space cannot be considered as a consumption space. However it is unrealistic to allow unlimited negative wealth even without short-sale restrictions.

The main contribution of this paper is to establish the equivalence relation with a minimum wealth constraint imposed on a consumption set. Inspired by Mas-Colell (1986), we only consider investors with uniformly proper preferences. Furthermore we generalize the equivalence relation where investors trade contingent claims in markets with transaction costs. The organization of the remainder is as follows.

In section Ⅱ, we explain the model used in this paper. We define a contingent claim space and explain how the marketed contingent claim space is constructed given available claims.

In section Ⅲ, we define the viability of the price pair with a minimum wealth constraint on consumption sets. We show that the viability of the price pair is equivalent to the existence of a linear pricing rule in general contingent claim spaces. And we show that "no arbitrage opportunities" is equivalent to the existence of a linear pricing rule in L_p, $1<p<\infty$.

In section Ⅳ, we extend the results in section Ⅲ to the economy with transaction costs. By representing the price system with transaction costs by a sublinear functional, we can extend Theorem 1 in section Ⅲ to the economy with frictions, i.e., transaction costs. Proposition 1 is a generalized form of Theorem 2.1 in Jouini and Kallal (1991).

Section Ⅴ contains remarks about the closedness assumption made on the marketed contingent claim space. We argue that the Back and Pliska's example (1991) does not constitute a counterexample to the equivalence relation, but shows that the equivalence relation fails without the closedness assumption on M.

Ⅱ. MODEL

Suppose we are given a contingent claim space X which is a Hausdorff, locally convex, topological vector space with topology τ. X is equipped with a natural order relation $\geq$ (i.e., $\geq$ is reflexive, antisymmetric, and transitive). The positive orthant of X, denoted by X_+, is given by $X_+=\{x\in X\colon x\geq 0\}$. $>$ denotes asymmetric part of $\geq$ such that $x>y$ if and only if $x\geq y$ and $x\neq y$. X_{++} denotes the strictly positive orthant of X, that is, $X_{++}=\{x\in X\colon x>0\}$. We call $x\in X_{++}$ a strictly positive element in X. We also assume that X is a reflexive Banach lattice.[1)]

Suppose there are markets for some contingent claims. $\{x_i\}_{i\in I}$ denotes the set of marketed contingent claims. The cardinality of I can be either finite or infinite. There are investors who buy and sell marketed contingent claims in frictionless markets. Let λ_i be the number of shares of the traded claim x_i held by some investor. Let M_0 denote the span of marketed contingent claims. Formally,

$$M_0 = \{m \in X; \; m = \sum_{i \in F} \lambda_i x_i \text{ for some } F \subset I, \; \#F < \infty \;\; \forall i \;\; \lambda_i \in R\}.$$

Then any $m \in M_0$ can be interpreted as a payoff from holding λ_i shares of corresponding x_i's. The collection of λ_i's is called a portfolio. We allow unlimited short sales, which implies that λ_i can be negative. If $\#I = \infty$, then M_0 is not necessarily closed. Let M be the closure of M_0. Then M is a closed linear subspace of X.[2] We interpret M as a space of all available marketed contingent claims. Markets are said to be incomplete when $X \neq M$.

Investors are identified by preferences $\succsim$, which are complete and transitive orderings defined on $X_+ \times X_+$. Let A denote the set of $\succsim$'s on $X_+ \times X_+$ which is convex, τ-continuous, and strictly increasing.[3] Note that we take the positive orthant of a contingent claim space as investors' consumption sets. In some financial literature, the whole contingent claim space is often used as a consumption set. It, however, seems unreasonable to allow unlimited negative wealth even with unlimited short sales.

Suppose that each x_i is traded at the price π_i. Let π_0: $M_0 \rightarrow R$ be a linear functional[4] such that for all $m = \sum_{i \in F} \lambda_i x_i \in M_0$, $\pi_0(m) = \sum_{i \in F} \lambda_i \pi_i$. We define a linear functional π: $M \rightarrow R$ such that the graph of π is a closure of the graph of π_0. Now (X, τ, M, π) describes our model. We call the pair (M, π) "a price pair". π will be called "a price system".

Ⅲ. VIABILITY OF THE PRICE PAIR (M, π)

Suppose we are given X, τ, M, π. If the price pair (M, π) is given so that no equilibrium can be found, then the price pair (M, π) does not deserve a serious consideration as an economic equilibrium. Kreps (1981) suggests a criterion such that if some agent with a preference from the class A can find an optimal choice among those which are feasible under the price system π, then the price pair $(M,$

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1 A Banach lattice is a Banach space which is a Riesz space and its norm is a lattice norm. See Aliprantis, Brown, and Burkinshaw (1989) for material on lattices.

2 Notice that M is a Banach space but not necessarily a Banach lattice.

3 This means for all x in X_+ and k in X_{++}, $x + k \succ x$, where $\succ$ is a asymmetric part of $\succsim$.

4 In other words, we take law of one price as a primitive concept. Clark (1993) starts from π_0 which is not a function but a correspondence.

π) could be a model of an economic equilibrium. Such a price pair (M, π) is called "viable". Kreps took the whole space as a consumption space. Then the viable pair (M, π) gives two disjoint convex sets C and D, such that

$$C = \{x \in X;\ x \succ m^*\}$$
$$D = \{m \in M;\ \pi(m) \leq \pi(m^*)\},$$

where m^* is an optimal choice with $\succeq$. Since $\succeq$ is continuous, C is a τ-open set. Therefore the viability of the price pair (M, π) is immediately followed by a necessary condition such as the existence of a continuous linear functional on X which separates C from D due to the well-known Hahn-Banach Theorem. Kreps established an equivalence relation between "the viability of the price pair" and so called "the extension property" (See Kreps 1981). These arguments, however, are no longer valid if a minimum wealth restriction is imposed on a consumption set. Let $C^* \equiv C \cap X_+$ and $D^* \equiv D \cap X_+$. Then C^* and D^* are two disjoint convex sets. If C^* has an interior, then Kreps's results need no modification. Unfortunately, however, the positive orthants of many interesting spaces have no interior points, e.g., every reflexive Banach spaces. Then C^* and D^* are not necessarily separable by a non-trivial continuous linear functional. Thus Kreps's equivalence results do not apply to this general case.

The problem of having consumption sets without interior points can be handled if preferences are restricted to certain class of preferences known as uniformly proper preferences.

Definition 1 (Mas-Colell (1986)): The preference is said to be uniformly proper whenever there exists some $v \in X_+$ and some neighborhood V of zero such that for any arbitrary $x \in X_+$ satisfying $x - \alpha v + z \succeq x$ in X_+ with $\alpha > 0$, we have $z \notin \alpha V$.

We restrict the set of preference defined on the positive orthant of a reflexive Banach lattice to include uniformly proper preferences.

Definition 2: Let $\mathcal{A}$ denote the set of $\succeq$'s on $X_+ \times X_+$ which is convex, τ-continuous, strictly increasing, and uniformly proper.

Now we give the definition of viability which is similar to the definition in Kreps (1981), except that $\succeq$ is defined on $X_+ \times X_+$ and belongs to $\mathcal{A}$.

Definition 3: The price pair (M, π) is said to be viable if there exists some $\succeq \in \mathcal{A}$ and $m^* \in M \cap X_+$ such that $\pi(m^*) \leq 0$ and $m^* \succeq m$ $\forall m \in M \cap X_+$ satisfying $\pi(m) \leq 0$.

The following theorem establishes the equivalence between the viability of a price pair and the existence of a strictly positive linear functional which extends the given price system.

Theorem 1: A pair (M, π) is viable if and only if there exists $P \in \mathcal{P}$ such that $P|_M = \pi$, where $\mathcal{P}$ is the set of τ-continuous and strictly positive linear functionals on X.

Proof: ($\Leftarrow$) Let p be an extension of π. Let e be an endowment. And it is natural to assume that e is traded. Define $x \succeq x'$ if $p(x) \geq p(x')$. Then $\succeq \in \mathcal{A}$[5]. Let $m^* = e$, then $m^* \in M \cap X_+$ and $p(m^*) = p(e) = \pi(m^*)$. Take $m \in M \cap X_+$ such that $\pi(m) \leq \pi(m^*)$, then we must have $p(m) \leq p(m^*) \Rightarrow m^* \succeq m \ \forall m \in M \cap X_+$ such that $\pi(m) \leq \pi(e)$.

($\Rightarrow$) Define the following sets.

$J = \{x \in X_+;\ x \succ m^*\}$

$\overline{F} = \{m \in M \cap X_+;\ \pi(m) \leq 0\}$

$F = \overline{F} + m^*$.

$\succeq$ is uniformly proper. We, first, prove the following Claim 1.

Claim 1: There exists a τ-continuous and linear functional P: $X \rightarrow R$ such that $P(m) \geq P(m^*)\ \forall m \in J$ and $P(m) \leq P(m^*)\ \forall m \in F$.

Proof: By theorem 2 and Theorem 3 in Richard and Zame(1986), there exists a convex cone D containing X_+ and having non-empty interior, and a convex, τ-continuous preference $\succeq^*$ on D that agrees with $\succeq$ on X_+. Let $\succ^*$ be an asymmetric part of $\succeq^*$, and $J^* = \{x \in D;\ x \succ^* m^*\}$. Notice that J^* has non-empty interior. Then it is immediate that $J \subset J^*$ and $J^* \cap F = \phi$. Therefore we can apply Hahn-Banach theorem. Q.E.D.[6]

Now it suffices to show that P can be chosen such that it is strictly positive and extends π.

Claim 2: $\forall x \in X_{++},\ P(x) > 0$.

Proof: Take $k \in J$ such that $P(k) > P(m^*)$. Since P is nontrivial, such k exists. Take $x \in X_{++}$. Then $x + m^* \succ m^*$ by strict monotonicity of $\succeq$. Since preference is continuous, there exists $\lambda > 0$ such that

$$\begin{aligned} x + m^* - \lambda k \succ m^* &\Rightarrow x + m^* - \lambda k \in J \\ &\Rightarrow P(x + m^* - \lambda k) = P(x) + P(m^*) - \lambda P(k) \geq P(m^*) \\ &\Rightarrow P(x) \geq \lambda P(k) > 0. \text{ Q. E. D.} \end{aligned}$$

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[5] The preference constructed from P is linear. It is easy to check that this linear preference is uniformly proper. See Cheng (1991) Proposition 2).

[6] The earlier version of the proof of Claim 1 which did not require a lattice structure was partly incomplete. An anonymous referee pointed out this to me and at the same time introduced the Richard and Zame (1986) which simplified the proof a lot.

Claim 3: $P|_M = \pi$.
Proof: Take $m_0 \in M \cap X_{++}$ such that $m_0 \in J$. Then $\pi(m_0) > \pi(m^*)$. By similar arguments, it is easy to see that $P(m_0) > 0$. We normalize P such that $P(m_0) = \pi(m_0)$. Pick $m \in M \cap X_{++}$ and λ such that $\pi(m) + \lambda\pi(m_0) = \pi(m + \lambda m_0) = 0$, which implies that $m + \lambda\, m_0 \in \overline{F}$ and $-m - \lambda m_0 \in \overline{F}$. Therefore $m + \lambda m_0 + m^* \in F$ and $-m - \lambda m_0 + m^* \in F$. Since $P(m + \lambda m_0) + P(m^*) \leq P(m^*)$ and $-P(m + \lambda\, m_0) + P(m^*) \leq P(m^*)$, then $P(m + \lambda m_0) = P(m) + \lambda P(m_0) = 0$. This leads to $P(m) = -\lambda P(m_0) = -\lambda\pi(m_0) = \pi(m)$, and therefore $P|_M = \pi$. Q.E.D.
This completes the proof of Theorem 1. ■

Theorem 1 is general in that it does not depend on the dimension of either M or X. If X_+ contains an interior, then the set $\mathcal{A}$,the collection of preference orderings, need not be restricted to include uniformly proper preferences. If X_+ is a positive orthant of a reflexive Banach space, then uniform properness is indispensable.

In view of Theorem 1, the price pair (M, π) is viable if there exists a strictly positive continuous linear functional which extends π. We establish another criterion for the viability of a price system. A price system π is said to admit "no arbitrage opportunites" if there does not exist a claim with a non-positive value in $M \cap X_{++}$ under the price system π. Formally we define "no arbitrage opportunites" as follows:

Definition 4 (no arbitrage opportunities): A price system π admits no arbitrage opportunities if for all $m \in M$, $m > 0$ implies $\pi(m) > 0$.

If $M \cap X_{++} = \phi$, then Definition 4 is vacuous. We will assume that $M \cap X_{++} \neq \phi$ throughout this paper, which means that investors can trade a contingent claim which is desirable. Another equivalent criterion for viability is that the price system π does not admit 'arbitrage opportunities' if $X = R^l$. The following lemma shows the above equivalence relation.

Lemma 1: Suppose $X = R^l$ and $l < \infty$. Then there exists $p \in \mathcal{P}$ such that $p|_M = \pi$ if and only if π admits no arbitage opportunities.

Proof: Let $\{x \in M\colon \pi(x) \leq 0\} \equiv G$. Notice that Defnition 4 is equivalent to $X_{++} \cap G = \phi$.
($\Rightarrow$) Straightforward.
($\Leftarrow$) By Separating Hyperplane Theorem, $\psi \in \mathcal{P}$ such that $\psi(x) > 0\, \forall\, x \in X_{++}$ and $\psi(x) \leq 0\, \forall\, x \in G$. Take $m \in M$ and $x \in X_{++} \cap M$, then $\pi(x) > 0$. Take $\lambda \in R$ such that $\pi(m) + \lambda\,\pi(x) = 0$, which implies that $\pi(m + \lambda x) = 0$. Since π is linear $m + \lambda x \in G$ and $-m - \lambda x \in G$. Thus $\psi(m + \lambda x) \leq 0$ and $\psi(-m - \lambda x) \leq 0 \Rightarrow \psi(m + \lambda x) = 0$. Therefore $\psi(m) = -\lambda\psi(x) = -\lambda\pi(x) = \pi(m)$, which implies that $\psi|_M = \pi$. Take $p = \psi$. ■

Corollary 1: Suppose $X = R^l$ and $l < \infty$. Then the price pair (M, π) is viable if and only if π does not admit 'arbitrage opportunities'.

Proof: By Theorem 1 and Lemma 1. ■

Since Theorem 1 does not depend on the dimension of either X or M, it will be interesting to investigate the equivalence relation when X is infinite dimensional but M is still finite. This is the case when we have only finitely many marketed contingent claims with an infinite state-space.

Kreps(1981) showed that if X is infinite dimensional, then "no arbitrage opportunities" is still necessary for the viability but far from sufficient. There might be an opportunity of getting arbitrarily close to an arbitrage opportunity. This opportunity is called a "free lunch". It is intuitive that if M is a finite dimensional subspace of X, then it is enough for viability to eliminate only "arbitrage opportunities". And the following lemma establishes the equivalence relation between "arbitrage opportunities" and "the viability of the price pair (M, π)".

Lemma 2: Suppose X is a separable Banach space[7] and M is a finite dimensional subspace of X. Then there exists $p \in \mathcal{P}$ such that $p|_M = \pi$ if and only if $X_{++} \cap \{x \in M: \pi(x) \leq 0\} = \phi$.

Proof:

($\Rightarrow$) Straight forward

($\Leftarrow$) $X_{++} \cap \{x \in M: \pi(x) \leq 0\} = \phi$. Take $m - x$ such that $m \in G$ and $x \in X_+$. Suppose $m - x \in X_{++}$. Then $m > x \geq 0$ implies $\pi(m) > 0$. This leads to a contradiction to $m \in G$. Hence $m - x \notin X_{++}$. Therefore $X_{++} \cap (G - X_+) = \phi$. By Lemma in Clark(1993) $(G - X_+) = \overline{(G - X_+)}$. Then by Theorem 5 in Clark(1993) there exists a strictly positive continuous linear functional p separating X_{++} and G such that $p|_M = \pi$. ■

Corollary 2: Suppose X is a separable Banach space and M is a finite dimensional subspace of X. Then the price pair (M, π) is viable if and only if (M, π) does not admit "arbitrage opportunities".

Proof: By theorem 1 and Lemma 2. ■

Now we consider general cases. Suppose X is a reflexive Banach space and M is an infinite dimensional closed subspace of X. The example 3 in Kreps(1981) shows that "no arbitrage opportunities" is far from sufficient for the viability of the price pair.

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7 Without separability, this lemma is false. See example 3 in Kreps(1981).

Definition 5 (No free lunch): If there exists a net $\{(m_n, x_n): n \in \Gamma\} \subset M \times X$ such that $x_n \to x$, $m_n \geq x_n$ for all $n \in \Gamma$, and $\pi(m_n) \leq 0$, then $x \notin X_{++}$.

Kreps(1981) use a definition equivalent to Definition 5 to show the equivalence between viability of the price pair and "no free lunches" without minimum wealth constraints on the consumption set. By virtue of Theorem 1 and Clarke (1993, Theorem 7), we show that "no free lunches" is necessary and sufficient for the viability of price pair with a generalized consumption set for all L_p, $1 < p < \infty$.

Corollary 3: Let $X = L_p$, $1 < p < \infty$. Then the price pair (M, π) is viable if and only if the price pair (M, π) does not admit free lunches.

Proof: By Theorem 1, it suffices to show that the price pair (M, π) does not admit free lunches if and only if there exists $p \in \mathcal{P}$ such that $p|_M = \pi$. By Theorem 7 in Clarke(1991), there exists such p which extends π. ■

Ⅳ. SUBLINEAR VIABILITY

In this section, we discuss how to generalize the equivalence relation when there exist transaction costs. We extend Jouini and Kallal(1991) to cases where consumption sets have no interior points.

When we buy a contingent claim m in the market, we pay $\pi(m) \geq 0$. When we sell a contingent claim m in the market, or when we take a short position in the market, then we receive $-\pi(-m) \geq 0$. We assume that investors pay more to buy m than they receive by selling m. This could be justified by assuming that there exist transaction costs. We will make the following assumption for this purpose.

Assumption 1:

1. For all $n, m \in M$, $\quad \pi(n + m) \leq \pi(n) + \pi(m)$,
2. For all $\lambda \in R_+$, $\quad \pi(\lambda m) = \lambda\pi(m)$.

Assumption 1 says that π is a sublinear functional defined on M. The sublinearity of π naturally implies that $\pi(m) \geq -\pi(-m)$ $\forall m$. We assume that there is an asset x_0 in M, which is riskless in the following sense; $x_0(\omega) = 1$ $\forall \omega \in \Omega$. Price system will be normalized such that $\pi(x_0) = \pi_0 = 1$. The following is an analogue of Definition 3 given in section Ⅱ.

Definition 6: A price pair (M, π) is said to be sublinear-viable if there exists some $\succeq \in \mathcal{A}$ and $m^* \in M \cap T$ such that $\pi(m^*) \leq \pi(e)$ and $m^* \succeq m$ for all $m \in M \cap T$

satisfying $\pi(m) \leq \pi(m^*)$.

This definition is the same as Definition 4, except that π is a sublinear functional. Jouini and Kallal(1991) showed that a price pair (M, π) is sublinear-viable if and only if there exists a strictly positive linear functional p on X such that $p|_M \leq \pi$ taking X as a consumption set. The following Theorem extends their Theorem 2.1 to general consumption sets with minimum wealth constraints. We omit the proof, since it only requires to slightly modify the proof of Theorem 1.

Theorem 2: A price system is sublinear-viable if and only if there exists $p \in \mathcal{P}$ such that $p|_M \leq \pi$.

We complete the equivalence relation by showing the following Corollary 4, which shows that "no free lunches" in the sense of Definition 5 is also necessary and sufficient for the sublinear viability. The proof of Corollary 4 can be easily obtained using Theorem 5 in Clark(1993) and Theorem 2 and thus is omitted.

Corollary 4: A price system is sublinear viable if and only if the price system admits no free lunches in the sense of Definition 5.

Corollary 4 together with Theorem 2 enables Jouini and Kallal(1991) to be applicable to cases where consumption sets have no interior points.

V. REMARKS

The assumption of closedness of M deserves comments. With this assumption, we could establish the equivalence relation between "the viability of a price pair" and "no free lunches" when investors are constrained to have minimum wealth.

However this result seems to be at odds with Back and Pliska's example (1991). Although their example is built in the context of dynamic securities models, we can obtain a static version where there are infinite states and countably many traded securities.[8] Furthermore we can obtain the same result as "the viability of a price pair" does not necessarily imply "no free lunches". The example seems to be negative, but it does not constitute a counterexample to our case. The gap between their example and our result is in the topological property of the marketed contingent claim space. The marketed contingent claim space in their example is not closed. This stems from the way it is constructed.

Given a set of traded securities, Back and Pliska takes an alternative way of constructing M instead of taking the closure of M_0. Unlike the method used to construct M in this paper, they restrict a portfolio space by a collection of all bounded infinite sequences. A marketed contingent claim is a payoff generated

[8] The modified static version of example is available from the author upon request.

by a portfolio. The marketed claim space constructed in this way is not necessarily closed. It is not difficult to check that the marketed contingent claim space in Back and Pliska's example is not closed. Let $M_1 = \{m \in X;\ m = \sum_{i \in I} \lambda_i x_i \forall i\, \lambda_i \in R \forall i\}$. If $\#I = \infty$, then certainly $M_0 \subset M_1$. Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ such that $\lambda \in \Lambda$ for some Λ. Then the topological property of M_1 depends on the choice of a portfolio space Λ. If $X = L_2$ and $\Lambda = l_2$, then M_1 is closed. However if $X = L_2$ and $\Lambda = l_\infty$, then M_1 is not closed.

Another point to be made regarding the example is that the investigated price system is not countably additive. But the viability of a price pair together with the closedness of M implies that π should be continuous due to Theorem 2 in Clark(1993), since the viable price pair admits no arbitrage opportunities.

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