

## COMPARATIVE STATICS UNDER UNCERTAINTY WITH THE MONOTONE LIKELIHOOD RATIO ORDER

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*In a simple two asset portfolio problem with one-risky and one-safe asset, Landsberger and Meilijson (1990) have shown that a monotone likelihood ratio (MLR) improvement of random returns of the risky asset increases the demand for the asset for all investors with non-decreasing utilities. However, their comparative static statement is made only for the simplest case where the payoff function is linear in both the choice and the random variable. This paper improves the robustness of their result in two ways. One is that the same comparative static statement can also be made for cases of non-linear payoffs. Another improvement is given by extending the admissible set of changes in randomness with the same utility settings. When the concerned payoff is linear in the choice variable, we show that the MLR order is unduly restrictive for the comparative static result and replace it with a more general type of change in randomness, called a "one-side monotone likelihood ratio with respect to a point."*

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### I. INTRODUCTION

Regarding decision models under uncertainty, it is generally accepted that the first-degree stochastic dominance (FSD) change of the given randomness is not sufficient for determining the direction of change in the optimal value of a

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decision variable for all risk averse decision makers. Since Fishburn and Porter (1976) have shown this indeterminacy in the simplest case of a portfolio problem, researchers have tried to find a particular subset of FSD shifts which is sufficient for making a general comparative static statement.

Recently, Landsberger and Meilijson (1990) (L-M) analyze the comparative static effect for a restrictive set of FSD changes, specified by the concept of a "monotone likelihood ratio" (MLR) order. In a one-risky and one-safe asset portfolio problem, they show that an MLR improvement of the random returns of the risky asset increases the demand for the asset for all investors with non-decreasing utilities. While their analysis is established on the most general utility settings, their result is restricted to decision models in which the payoff functions are linear in both the choice and the random variable.

This paper improves the robustness of L-M's result in two ways. One is that the same comparative static statement can also be made for cases of non-linear payoffs. Doing this, we adopt the standard one-argument decision model that is often used in the literature - Kraus (1979), Katz (1981), Meyer and Ormiston (1985), and Black and Bulkley (1989). The general notation for the economic problem is

$$\max_b E[u(z(x, b))] \quad (1)$$

where  $b$  is a choice variable,  $x$  is an exogenous random variable,  $z$  is the payoff or the outcome which depends on both the choice and the random variable, and  $u$  is a von Neumann-Morgenstern utility function. Though the model (1) is a simple form, a variety of economic decision problems are of this type. Aside from the cited portfolio problem, included are the well-known examples of a competitive firm facing random output price analyzed in Sandmo (1971), and the coinsurance problem in Dionne, Eeckhoudt and Gollier (1993).

Another improvement is given by extending the admissible set of changes in randomness. When the concerned payoff is linear in the choice variable  $b$ , we show that the MLR order is unduly restrictive for the comparative static result and replace it with a more general type of change in randomness, called a "one-side monotone likelihood ratio with respect to a point." When doing this, we employ the following linear form model used in Dionne, Eeckhoudt and Gollier (1993):

$$z(x, b) = z_0 + b(x - c) \quad (2)$$

where  $z_0$  and  $c$  are exogenous parameters. The point  $c$  is the value of  $x$  satisfying the first derivative of the payoff with respect to the choice variable  $b$  to be zero, i.e.,  $z_b(x, b) = 0$ . This means that the value of  $z$  does not depend

on the choice variable when  $x = c$ , and thus the point  $c$  can be understood as the rate of return on a safe asset in the portfolio choice problem, the constant marginal cost in the competitive firm model under price uncertainty, or the marginal cost of insurance rate in the coinsurance problem.

In the following section II, we present a review of previous comparative static results regarding the MLR order. This section also presents three definitions of subsets of the general FSD changes which are used for deriving our comparative static results in section III. Conclusions are presented in section IV.

## II. PRELIMINARY DISCUSSIONS AND SOME DEFINITIONS

Throughout the paper, the random variable  $x$  is assumed to be characterized by its initial and final cumulative distribution functions (CDFs)  $G$  and  $F$ , with their corresponding probability density functions (PDFs)  $g$  and  $f$ , respectively. Both the CDFs are assumed to have their points of increase in bounded intervals. For notational convenience, we assume that the support of  $G$  is a finite interval  $[x_1, x_3]$  and the support of  $F$  is another finite interval  $[x_2, x_4]$  where<sup>1</sup>  $x_1 \leq x_2$  and  $x_3 \leq x_4$ . The MLR order used in L-M's analysis is given by imposing a restriction that the ratio between a pair of PDFs  $f$  and  $g$  should be monotone such as:<sup>2</sup>

**Definition 1.**  $F(x)$  represents a monotone likelihood ratio shift from  $G(x)$  (denoted by  $F$  MLR  $G$ ) if there exists a non-decreasing function  $h: [x_2, x_3] \rightarrow [0, \infty)$  such that  $f(x) = h(x)g(x)$  for all  $x \in [x_2, x_3]$ .

It is easy to show that, if  $F$  MLR  $G$ ,  $G(x) \geq F(x)$  for all  $x \in [x_1, x_4]$ ; that is, the graph of CDF  $F$  is located completely to the right-hand side of the graph of  $G$ .<sup>3</sup> This implies that the MLR order specifies a particular subset of FSD changes. For this MLR changes, L-M derive a general comparative static statement using the simple one-risky and one-safe asset portfolio problem. That is, an MLR shift in the distribution of the random return on the risky asset induces all investors with non-decreasing utilities to increase their demand for the risky asset. While their analysis is restricted to the simplest case of the general decision model (1), there are several points that should be mentioned.

First, their result is based on a very general utility setting - all individuals with non-decreasing utilities. This utility setting is usually treated in the

<sup>1</sup> In this paper, we consider only subsets of FSD shifts such that the final distribution  $F$  always dominates the initial distribution  $G$  in the sense of FSD (denoted by  $F$  FSD  $G$ )

<sup>2</sup> Given the notation for the supports of CDFs  $F$  and  $G$ , Definition 1 is slightly modified from the one used in L-M.

<sup>3</sup> See Property 1 in L-M's paper.

economics literature as the most general set of decision makers. Studying the comparative statics under uncertainty, researchers often impose restrictions on the following three components: (i) the structure of decision model, (ii) the set of decision makers, and (iii) the set of changes in randomness. Then, L-M's analysis can be categorized as the one that uses relatively strong restrictions on the components (i) and (iii), and relatively weak restriction on the component (ii). This paper follows the same utility setting but employs a more general decision model (1) than the one used in L-M. In this sense, our study improves the efficiency in the component (i).<sup>4</sup>

Second remarkable point in L-M's analysis is concerned with the assumptions adopted for the optimal solution of the maximization problem (1). In many other studies, it is often assumed that the first and the second-order condition for the maximization problem are satisfied. In addition to guaranteeing an interior bounded solution, these conditions are used to prove comparative static results, along with the continuity and the differentiability assumptions on the utility function  $u$  and the payoff function  $z$ . That is, to determine that  $b_F - b_G \geq 0$  where  $b_G$  and  $b_F$  denote the optimal choices under an initial CDF  $G$  and a final CDF  $F$ , respectively, it is sufficient to show that

$$\int_{x_1}^{x_2} u'[z(x, b_G)z_b(x, b_G)[g(x) - f(x)]]dx \leq 0. \quad (3)$$

However, according to Dionne, Eeckhoudt and Gollier (1993), this approach can be quite restrictive and may exclude some interesting cases such as a corner or an unbounded solution. In particular, when decision makers are risk-neutral and the payoff function  $z(x, b)$  is linear in the choice variable, the second-order condition (sufficient for interior solution) is not met and the expression (3) is not useful for obtaining a general comparative static statement.

L-M's study relies on a very weak optimality condition for the decision problem. The only required condition is that the optimal solution  $b_G$  is determinable for a given initial CDF  $G$ , regardless of an unbounded or a corner solution.<sup>5</sup> In this reason, their analysis gives a very strong result and thus it is interesting to examine the method used in the proof of their comparative static result. For later use in our analysis, we describe the method using the general decision model (1). Let  $EU_F$  and  $EU_G$  be the expected utility calculated with

<sup>4</sup> Another comparative static analysis on the MLR shifts is given by Ormiston and Schlee (1993) who adopt a further generalized decision problem,  $\max_b E[u(x, b)]$ , where a utility function  $u$  depends on both the choice variable  $b$  and the random exogenous variable  $x$ . In particular, this decision problem includes two or more outcome values, while our decision problem (1) contains only one outcome value  $z$ .

<sup>5</sup> Furthermore it does not require the continuity and the differentiability assumptions on the utility function and the payoff function in (1).

respect to  $F$  and  $G$ , which can be expressed as functions of the choice variable  $b$  as,

$$EU_F(b) = \int_{x_2}^{x_4} u[z(x, b)]f(x)dx \text{ and } EU_G(b) = \int_{x_1}^{x_3} u[z(x, b)]g(x)dx, \quad (4)$$

respectively. To prove  $b_G \leq b_F$ , it suffices to show that for any pair of choices  $b_1$  and  $b_2$  such that  $b_1 < b_2$ ,

$$\text{if } EU_G(b_1) \leq EU_G(b_2), \text{ then } EU_F(b_1) \leq EU_F(b_2). \quad (5)$$

It is because (5) implies that  $EU_F(b) \leq EU_F(b_G)$  for every  $b < b_G$  which in turn guarantees the value of  $b$  maximizing  $EU_F(b)$  to be at least equal or larger than  $b_G$ , i.e.,  $b_G \leq b_F$ . We follow the same method to prove the main results in section III, and thus our comparative static results can also be applied to the cases of a corner or an unbounded solution.

The rest of this section devotes to introduce two definitions of CDF orders which are more general than the MLR.

An MLR shift from  $g$  to  $f$  restricts the two PDFs to cross only once, and requires the ratio of  $f$  to  $g$  to be non-decreasing for both side of the crossing point. Relaxing some of these restrictions, we introduce two more general subsets of FSD shifts. One, called a “left-side monotone likelihood ratio” (L-MLR) with respect to a point, is obtained from relaxing the monotonicity requirement for points to the right side of the crossing point. The other, called a “right-side monotone likelihood ratio” (R-MLR) with respect to a point, drops the monotonicity requirement to the left side of the crossing point. Formally, these changes are defined as:

**Definition 2.** Given a point  $p \in [x_2, x_3]$ ,  $F(x)$  represents a ‘left-side monotone likelihood ratio shift with respect to a point  $p$ ’ from  $G(x)$  (denoted by  $F$  L-MLR( $p$ )  $G$ ) if there exist a point  $k \in [p, x_3]$  and a non-decreasing function  $h: [x_2, k] \rightarrow [0, 1]$  such that  $f(x) = h(x)g(x)$  for all  $x \in [x_2, k]$  and  $g(x) \leq f(x)$  for all  $x \in [k, x_3]$ .

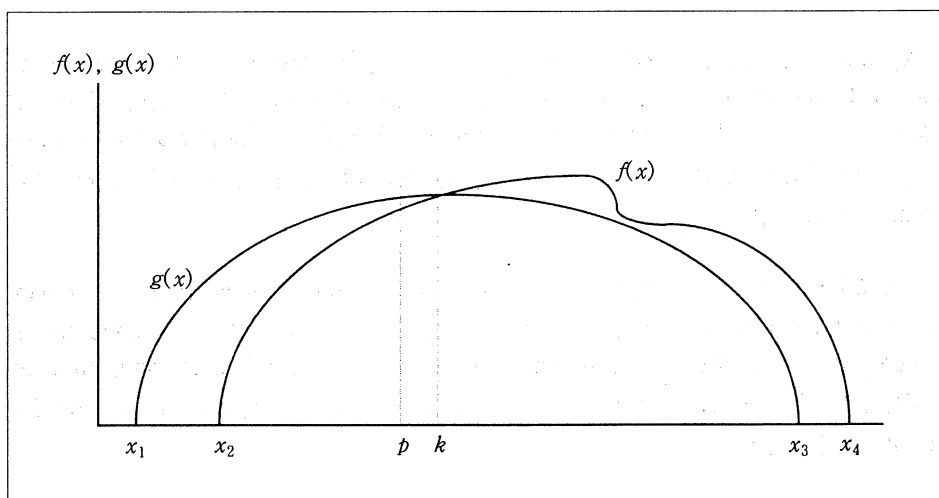
**Definition 3.** Given a point  $p \in [x_2, x_3]$ ,  $F(x)$  represents a ‘right-side monotone likelihood ratio shift with respect to a point  $p$ ’ from  $G(x)$  (denoted by  $F$  R-MLR( $p$ )  $G$ ) if there exist a point  $k \in [x_2, p]$  and a non-decreasing function  $h: [k, x_3] \rightarrow [1, \infty)$  such that  $f(x) = h(x)g(x)$  for all  $x \in [k, x_3]$  and  $g(x) \geq f(x)$  for all  $x \in [x_2, k]$ .

Both the L-MLR( $p$ ) and the R-MLR( $p$ ) conditions require that the PDFs  $f$

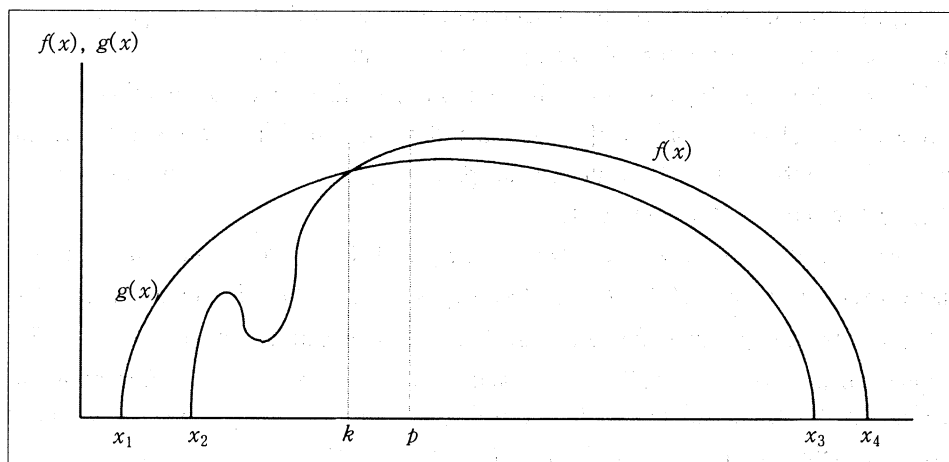
and  $g$  cross only once at the point  $k$  and that  $g(x) \geq f(x)$  for all points to the left side of  $k$  and  $g(x) \leq f(x)$  for all points to the right side of  $k$ . This implies that each of the two orders gives a special class of the general FSD changes. Since an L-MLR( $p$ ) requires the condition of monotone likelihood ratio only for the left side of the point  $k$  and an R-MLR( $p$ ) requires the monotonicity only for the right side of the point  $k$ , both the orders are more general than the MLR order. Examples of these types of shifts are illustrated in Figures 1 and 2, representing  $F$  L-MLR( $p$ )  $G$  and  $F$  R-MLR( $p$ )  $G$ , respectively. An L-MLR( $p$ ) shift specifies a probability transformation such that a decreasing proportion of probability mass of the left side of the point  $k$  is transferred to the right side of the point  $k$ . An R-MLR( $p$ ) shift gives a transformation such that some probability mass taken from the left side of  $k$  is transferred to the right side of  $k$ , keeping the ratio of the final PDF to the initial PDF to be non-decreasing for the right side of  $k$ .

In order to give special examples of these two orders, consider an initial random variable which has only two outcomes. If the lower (higher) outcome becomes larger than before, then for an appropriately given point  $p$  it is the case of an L-MLR( $p$ ) shift where  $k = x_2$  (an R-MLR( $p$ ) shift where  $k = x_3$ ). Generally, given an initial PDF  $g$  with its support  $[x_1, x_3]$ , let's consider a conditional PDF  $g^l$ , given by  $g^l(t) = g(t)/G(T)$  with its support  $[x_1, T]$  where  $T \leq x_3$ . Then an MLR shift in  $g^l$  gives an L-MLR( $p$ ) shift in the initial  $g$ . If we define a conditional PDF  $g^r(t) = g(t)/[1 - G(T)]$  with its support  $[T, x_3]$  where  $x_1 \leq T$ , an MLR shift in  $g^r$  gives an R-MLR( $p$ ) shift in the initial  $g$ .

[Figure 1]  $F$  L-MLR( $p$ )  $G$  with  $k \in [p, x_3]$



[Figure 2]  $F$  R-MLR( $p$ )  $G$  with  $k \in [x_2, p]$



### III. COMPARATIVE STATIC RESULTS

Using the general one-argument decision model (1), this section provides general comparative static statements concerning the subsets of FSD shifts defined in section II. First we examine the MLR case.

**Proposition 1.** For all decision makers with non-decreasing utilities,  $b_F \geq b_G$  if

(a)  $F$  MLR  $G$

(b)  $z_{bx} \geq 0$

**Proof:** Remember that the support of  $G$  is a finite interval  $[x_1, x_3]$  and the support of  $F$  is another finite interval  $[x_2, x_4]$ . As we noted before, it suffices to show that, for a pair of points  $b_1$  and  $b_2$  ( $b_1 < b_2$ ), (5) is satisfied. Assume that  $\Delta_G = EU_G(b_2) - EU_G(b_1) \geq 0$  where  $b_1 < b_2$ , then it is sufficient to show that the following is non-negative,

$$\Delta_F = EU_F(b_2) - EU_F(b_1) = \int_{x_2}^{x_4} A(x)f(x)dx \quad (6)$$

where  $A(x) = u[z(x, b_2)] - u[z(x, b_1)]$ . Note that the assumption of  $z_{bx} \geq 0$  implies that  $z_b(x, b)$  is non-decreasing in  $x$  and the sign of  $z_b(x, b)$  must change from negative to positive. The assumption  $\Delta_G = \int_{x_1}^{x_3} A(x)g(x)dx \geq 0$  excludes the case where  $z(x, b_2) - z(x, b_1) \leq 0$  for all  $x \in [x_2, x_3]$  because the

difference  $z(x, b_2) - z(x, b_1)$  is non-decreasing in  $x$  by the assumption of  $z_b(x, b)$ , and it contradicts the assumption, that is, with the assumption  $u' \geq 0$ , the case implies  $A \leq 0$  for all  $x \in [x_1, x_3]$  and thus  $\Delta_G \leq 0$ .<sup>6</sup>

If  $z(x, b_2) - z(x, b_1) \geq 0$  for all  $x \in [x_2, x_3]$  then the assumption  $u' \geq 0$  implies that  $A \geq 0$  for all  $x \in [x_2, x_4]$  and thus  $\Delta_F \geq 0$ . These cases are true for any FSD shift from  $G$  with its support  $[x_1, x_3]$  to  $F$  with its support  $[x_2, x_4]$  where  $x_1 \leq x_2$  and  $x_3 \leq x_4$ .<sup>7</sup>

Now consider the case that, with  $b_1, b_2$  and the payoff function  $z$  given, there exists a point  $x^*(b_1, b_2, z) \in [x_2, x_3]$  such that the difference  $z(x, b_2) - z(x, b_1)$  is non-positive for all  $x \leq x^*$  and non-negative for all  $x \geq x^*$ . This implies that  $A \leq 0$  for all  $x \leq x^*$  and  $A \geq 0$  for all  $x \geq x^*$ . According to Definition 1, the condition  $h: [x_2, x_3] \rightarrow [0, \infty)$  non-decreasing implies that there exists a point  $k \in [x_2, x_3]$  such that  $0 \leq h \leq 1$  for all  $x \in [x_2, k)$  and  $1 \leq h < \infty$  for all  $x \in [k, x_3]$ . Without loss of generality, the non-decreasing function  $h$  can be written as,

$$h(x) = \begin{cases} 1 - \delta(x), & \text{for } x \in [x_1, k) \\ 1 + \eta(x), & \text{for } x \in [k, x_3] \end{cases}$$

where  $\delta = 1$  for  $x \in [x_1, x_2)$ ,  $0 \leq \delta \leq 1$  for  $x \in [x_2, k)$  and  $\eta \geq 0$ .

**Case (i):** when  $x^* \leq k$ .

Since  $f(x) = 0$  for  $x \in [x_1, x_2)$ , (6) can be rewritten as,

$$\begin{aligned} \Delta_F &= \int_{x_1}^{x_4} A(x)f(x)dx \\ &= \int_{x_1}^k A(x)[1 - \delta(x)]g(x)dx + \int_k^{x_3} A(x)[1 + \eta(x)]g(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx. \end{aligned} \quad (7)$$

Rearranging (7),

$$\Delta_F = \Delta_G + \int_{x_1}^k A(x)[- \delta(x)]g(x)dx + \int_k^{x_3} A(x)\eta(x)g(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx.$$

Since  $A(x)$  changes its sign from negative to positive at  $x = x^*$  and  $\delta(x)$  is

<sup>6</sup> If  $u' > 0$  and  $z_{bx} > 0$ , then  $\Delta_G < 0$  is satisfied in the strong sense. However, if  $u' = 0$ , then  $\Delta_G = 0$ . Therefore  $\Delta_G \leq 0$  is satisfied in the weak sense.

<sup>7</sup> We do not exclude the case  $x_3 \leq x_2$  where all the possible outcome values of  $x$  under  $F$  are higher than under  $G$ . For any shift of this type, it is easy to see that  $\Delta_F \geq 0$ .



non-increasing,

$$\Delta_F \geq \Delta_G - \delta(x^*) \int_{x_1}^k A(x)g(x)dx + \int_k^{x_3} A(x)\eta(x)g(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx.$$

Since  $\Delta_G \geq 0$  by assumption,  $0 \leq \delta(x^*) \leq 1$  and  $\Delta_G \geq \int_{x_1}^k A(x)g(x)dx$ , we have  $\Delta_F \geq 0$ .

**Case (ii):** when  $k \leq x^*$ .

For the first term in the right-hand side of (7), since  $A(x) \leq 0$  and  $1 - \delta(x) \leq 1$  for all  $x \in [x_1, k]$ ,

$$\int_{x_1}^k A(x)[1 - \delta(x)]g(x)dx \geq [1 + \eta(x^*)] \int_{x_1}^k A(x)g(x)dx$$

and for the second term, since  $A(x)$  changes its sign from negative to positive at  $x = x^*$  and  $\eta(x)$  is non-decreasing,

$$\int_k^{x_3} A(x)[1 + \eta(x)]g(x)dx \geq [1 + \eta(x^*)] \int_k^{x_3} A(x)g(x)dx.$$

Thus from (7), we have

$$\begin{aligned} \Delta_F &\geq [1 + \eta(x^*)] \int_{x_1}^{x_3} A(x)g(x)dx + \int_{x_3}^{x_4} A(x)f(x)dx = [1 + \eta(x^*)]\Delta_G \\ &\quad + \int_{x_3}^{x_4} A(x)f(x)dx \end{aligned}$$

and hence the assumption  $\Delta_G \geq 0$  implies that  $\Delta_F \geq 0$ .

Q.E.D.

Proposition 1 is a direct extension of L-M's study, showing that the same comparative static result holds for decision models with non-linear payoff functions. Proving our result, we follow the same technique used in their study. Thus it improves the robustness of their result without any additional cost of assumptions.

In addition, if we closely examine the proof of Proposition 1, we see some notable things. For the comparative static result, when  $x^* \leq k$  (the point of single crossing between the PDFs  $f$  and  $g$  is larger than the value of  $x^*$ ), the MLR required for the right-side interval of  $x$  such that  $g(x) \leq f(x)$  is unduly restrictive, and when  $x^* \geq k$  (the point of single crossing is smaller than the value of  $x^*$ ), the MLR required for the left-side interval of  $x$  such that  $g(x) \geq f(x)$  is unduly restrictive. These findings allow us to make another

comparative static statement. In particular, if the concerned payoff is restricted to be linear in the choice variable, a further generalized result is possible.

**Proposition 2.** For all decision makers with non-decreasing utilities,  $b_F \geq b_G$  if

- (a)  $z(x, b)$  is linear in  $b$
- (b)  $F$  L-MLR( $c$ )  $G$  or  $F$  R-MLR( $c$ )  $G$
- (c)  $z_{bx} \geq 0$ ,  $z_{bb} = 0$  and  $z_b(c) = 0$ .

**Proof:** Since the payoff function  $z$  is restricted to be linear in the choice variable  $b$ , the point  $x^*(b_1, b_2, z)$  defined in the proof of Proposition 1 is independent of the choice values  $b_1$  and  $b_2$ . This implies that  $x^*$  is the value of  $x$  satisfying  $z_b(x) = 0$  and thus  $x^* = c$ . Hence the proof of Proposition 1 completes the proof. Q.E.D.

The trade-off used between Propositions 1 and 2 is that a larger class of changes in CDF is allowable at the cost of the linearity assumption on the payoff function. If the payoff function is linear in the choice variable, then, according as the point of crossing between the pair of PDFs is smaller or larger than the point  $c$ , the restriction of monotone likelihood ratio on either one of the left- or the right-side (of the point of crossing) is not necessary for the result.

Consider a linear payoff function given in (2), in which  $x^* = c$ . As we noted in section I, the point  $c$  can be understood as the rate of return on a safe asset in a portfolio problem, the constant marginal cost in a firm theory, or the marginal cost of insurance rate in an insurance model. Then Proposition 2 implies that, for both the shifts given in Figures 1 and 2, we can say that  $b_F \geq b_G$  for all the individuals with non-decreasing utility functions. Therefore the MLR condition for the comparative static result in L-M's study is unduly restrictive for the case of linear payoff and Proposition 2 improves the robustness of their result without additional cost of assumptions.

#### IV. CONCLUSIONS

Based on the MLR changes, L-M's comparative static result is restricted to decision models with linear payoffs. Our paper improves the efficiency of their results in two ways. First, we generalize their comparative static result to decision models which allow non-linear payoffs as the argument of utility function. Second and more importantly, we show that their result is also obtainable for a more general class of FSD shifts than the set of MLR shifts.

There is a trade-off between the two results in this paper. A larger class of changes in CDF is allowable at the cost of the linearity assumption on the payoff function. That is, if the payoff in the decision model is restricted to be linear in the choice variable, the size of the admissible set of changes in randomness can be enlarged from MLR to MLR with respect to a point.

When we develop our comparative static results, we do not require any additional assumptions, compared to the L-M's study. We use the same set of decision makers, that is, all individuals with non-decreasing utilities, and allow both the cases of an interior or a corner solution. Therefore our study improves the robustness of the results in L-M.

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