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HETEROGENEOUS AGENTS AND ECONOMIC GROWTH

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I study the effects of the distribution of individual abilities on economic growth using a Lucas (1988) type two-sector growth model. The relation between investment in education and individual education efficiency is positive but less than one-to-one, so the model predicts a negative relation between the variance of ability and aggregate investment in education via Jensen's inequality theorem. As a result, a more homogeneous country is predicted to achieve a higher growth rate than a more heterogeneous country within finite time. This result obtains in the absence of any political mechanism commonly used in the inequality-growth literature. This novel feature remains robust to a convex education technology of Rebelo (1991).

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I. INTRODUCTION

Researchers have documented various possible causes for cross-country differences in growth rates with models of heterogeneous agents.

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Why should countries of similar income and average education levels grow at different rates? Many authors have introduced credit market imperfections or a political mechanism with income or ability inequality to explain this. Most of them adopt an overlapping generations (OLG) model with altruistic motives.

Galor and Zeira (1993) show that credit market imperfections have a short-run effect on the amount investment in human capital and indivisibility in investment in human capital has a long-run effect on its amount. They show that only an economy which is initially rich and of which wealth is widely distributed continues to be rich. Persson and Tabellini (1994) study the effect of income inequality on growth where individuals vote for the optimal redistributive policy. They argue that inequality induces more redistribution by the government, hence discourages investment and growth. Bénabou (1996a) also shows that, in the absence of credit constraints, the greater the initial income inequality is, the larger the redistribution threat and hence the lower the investment and long-run growth will be. In addition, when credit constraints exist, the poor cannot undertake the efficient amount of investment. Hence, with diminishing marginal productivity from investment, redistributions can increase total production and growth. Bénabou (1996b) studies the effects of stratification and education financing on growth. His main findings are that stratification of families by economic status tends to minimize the costs of existing heterogeneity, but mixing reduces heterogeneity faster and therefore integration tends to slow down growth in the short run yet raise it in the long run. Castelló and Doménech (2002) show in cross-section regressions that human capital inequality has a significantly negative effect on economic growth rates because education inequality lowers investment rates and, consequently, lowers income growth. Besley and Burgess (2003) show for 60 low and middle income countries a positive and significant impact of income inequality on the level of poverty within a country. They use panel regressions of the log of the poverty rate on the log of real per capita national income and a measure of income inequality using the standard deviation of the log of the income distribution after controlling for a country fixed effect. According to their study, the world poverty would decline by 67% if we could lower the level of inequality in each region of the world by one standard deviation.

As far as I understand, Galor and Tsiddon (1997) are the only authors who directly deal with individual ability differences unlike my model, they study the interactions between technological progress and two components of individual earnings - parental human capital and individual ability - in an OLG setup, and thus the effects on income mobility, income inequality and growth. They distinguish between new technologies and diffusion of existing technologies. By assuming perfect international capital mobility, they predict a stationary rental rate $\overline{r}$ and further assume a stationary wage rate $\overline{w}$ (pp. 366-377). Galor and Tsiddon (1997) and my model are the same in that individual agents have the same preferences (willingness to pay) as consumers but different abilities (productivities) as producers. Yet in my model, agents are assumed to optimize investment expenditure in physical and human capital accumulation to maximize lifetime utility, unlike in Galor and Tsiddon (1997). More importantly, the shares of individual's time endowment, θ^j, invested in human capital, and $1-\theta^j$, invested in the production of consumption goods, are exogenously fixed in Galor and Tsiddon while agents optimally allocate their human capital (i.e., time) between goods production, $\theta_t(i)$, and education, $1-\theta_t(i)$, in my model. Essentially, Galor and Tsiddon (1997) focus on the endogenous choice of production sectors j, a technologically advanced and an old sector, i on the ability of individual agents, a_t^i and their parental human capital externality, and on the change of distribution of wealth over time, given exogenous time allocation rules ($\theta^j, 1-\theta^j$). They assume that individual ability a_t^i is identically and independently distributed from a uniform distribution each period, and they consider a possibility of intergenerational correlation of abilities.

In this paper, I study the dispersion of individual education efficiency $e(i)$ on economic growth in a two-sector growth model similar to Uzawa (1965) and Lucas (1988). The model uses infinitely-lived agents but does not rely on indivisibility in investment in human capital or a political mechanism to redistribute wealth. In my model, the cross-country differences in investment in human capital and economic growth originate from distributional differences in individual abilities (productivities).

When the individual return to investment in education is always increasing but has diminishing marginal productivity, the model predicts a less than *one-to-one* relation between the investment in education and individual ability. This concave relation leads to a negative effect from the dispersion of individual abilities on the aggregate amount of human capital formation via Jensen's inequality, and thus a negative effect on the long-run growth. In this analysis, I model constant-returns-to-scale and diminishing marginal productivity production processes and a linear education productivity function. There are two input factors, physical capital k and human capital h, to produce a quantity of final goods and services, y. Human capital is produced as a factor of production from only human capital as an input in the education sector. Consumer agents are identical in their preferences but different in their ability to produce according to their education efficiency, $e(i)$, where $i \in [0,1]$. I also consider a case of log-normally distributed education efficiency. As an extension, I study a convex education productivity function similar to Rebelo (1991) with both physical and human capital in the education sector, allowing general factor intensity in human capital production. I could confirm that the main results of the basic model remain robust under the convex education productivity function.

Section II shows the model describing firms' and consumers' optimization problems. Section III summarizes optimal investment expenditure on education. Section IV identifies system dynamics and a steady state. Section V investigates the effects of the education efficiency distribution on growth. Section VI illustrates an application to a log-normal distribution. Section VII extends the base model to a convex education technology and checks the robustness of the model. Section VIII concludes.

II. MODEL

I model two production sectors: one for final goods and services, y, one for a factor of production, human capital h, the latter being equivalent to the education sector. Both sectors are assumed to be competitive and autarkic. Final goods and services, y, are produced using physical capital k and human capital h. Time, t, is continuous and infinite.

1. Final Good y Manufacturer's Problem

In the final goods and services market, manufacturers are assumed to maximize profit given a physical capital payment/cost rate r_t and the human capital wage rate w_t. The production function for final goods and services is given as follows.

$$\begin{aligned} Y_t &= A\left(\int_0^1 k_t(i)di\right)^{\alpha}\left(\int_0^1 \theta_t(i)h_t(i)di\right)^{1-\alpha} = AK_t^{\alpha}(\theta_t H_t)^{1-\alpha} \\ &= C_t + Q_t = C_t + \dot{K}_t + \delta_k K_t, \end{aligned} \tag{1}$$

where Y_t is the aggregate production of final goods and services, y, at time t, $k_t(i)$ is the input of physical capital, $h_t(i)$ is the input of human capital, $\theta_t(i)$ is the share of human capital invested in the production of final goods and services by agent i, $i \in [0,1]$ at time t, and A is a constant productivity factor. Note that physical capital and human capital inputs are symmetrically aggregated in the production process. Hence, what matters in the production of final goods and services is the aggregate amount of physical capital or human capital and not where the two inputs come from across agents. Good y is used as a numeraire and its price is normalized to unity ($p_{y_t} = 1$). Aggregate variable K_t denotes aggregate physical capital, H_t aggregate human capital, C_t aggregate consumption, and Q_t aggregate investment in physical capital within a country, respectively. Investment in physical capital is divided into net physical capital change, $\dot{K}_t$, and replacement of depreciated physical capital, $\delta_k K_t$. As indicated in (1), final goods and services are perfect substitutes for consumption or physical capital accumulation. Aggregate physical and human capital are defined as follows.

$$K_t = \int_0^1 k_t(i)di, \tag{2}$$

$$H_t = \int_0^1 h_t(i)di. \tag{3}$$

We also define the index of the aggregate share of human capital used in the production of final goods and services, θ_t, as follows.

$$\theta_t \equiv \frac{\int_0^1 \theta_t(i) h_t(i) di}{H_t}. \tag{4}$$

That is to say, θ_t is a weighted average of individual agents' shares of their human capital used in the production of final goods and services, weighted by their relative human capital, $h_t(i) / H_t$.
The representative manufacturer of final goods and services maximizes its profit each period by choosing inputs of $k_t(i)$ and $\theta_t(i)h_t(i)$ at time t.

$$\begin{aligned}\max_{k_t(i),\theta_t(i)} \Pi_t &= Y_t - r_t \int_0^1 k_t(i)di - w_t \int_0^1 \theta_t(i)h_t(i)di \\ &= A\left(\int_0^1 k_t(i)di\right)^{\alpha} \left(\int_0^1 \theta_t(i)h_t(i)di\right)^{1-\alpha} \\ &\quad - r_t \int_0^1 k_t(i)di - w_t \int_0^1 \theta_t(i)h_t(i)di\end{aligned}$$

Solving the first order conditions, we get the following equations for the physical capital rental rate r_t and human capital wage rate w_t.

$$\begin{aligned} r_t &= \frac{\partial Y_t}{\partial k_t(i)} = \alpha A\left(\int_0^1 k_t(i)di\right)^{\alpha-1} \left(\int_0^1 \theta_t(i)h_t(i)di\right)^{1-\alpha} \\ &= A\alpha \left(\frac{K_t}{\theta_t H_t}\right)^{-(1-\alpha)}, \end{aligned} \tag{5}$$

$$\begin{aligned} w_t &= \left[\frac{\partial Y_t}{\partial \theta_t(i)}\right] \frac{1}{h_t(i)} = (1-\alpha)A\left(\int_0^1 k_t(i)di\right)^{\alpha} \left(\int_0^1 \theta_t(i)h_t(i)di\right)^{-\alpha} \\ &= A(1-\alpha)\left(\frac{K_t}{\theta_t H_t}\right)^{\alpha}. \end{aligned} \tag{6}$$

We can see here that rental rate and wage rate are respectively the same for all agents within a country regardless of their educational efficiency

$e(i)$, but the national rental rate and wage rate may differ across countries depending on the distribution of agents' efficiencies $e(i)$ and thus physical and human capital holdings.

With the definition of $\omega_t \equiv K_t / H_t$, the aggregate physical capital/ aggregate human capital ratio, the equilibrium factor prices are written from (5) and (6) as follows.

$$r_t = A\alpha\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)}, \tag{7}$$
$$w_t = A(1-\alpha)\theta_t^{-\alpha}\omega_t^{\alpha}. \tag{8}$$

2. Agent i's Problem

All agents have identical preferences as consumers but different education efficiency $e(i)$ as producers, which is assumed to be constant over time. We assume that agents own all the physical and human capital. For simplicity, we assume that the initial physical and human capital holdings are the same at k_0 and h_0 respectively for all agents. The human capital levels of agents at $t > 0$ will be different due to different investment patterns for different education efficiencies. Agent i is modelled to maximize lifetime utility. according to

$$\max_{c_t(i), q_t(i), \theta_t(i)} U(i) = \int_0^\infty e^{-\rho t}\left\{\frac{c_t(i)^{1-\sigma}-1}{1-\sigma}\right\}dt, \quad 0<\rho<1, \sigma>0, \sigma\neq 1, \tag{9}$$

subject to the budget constraint

$$c_t(i) + q_t(i) \le r_t k_t(i) + w_t\theta_t(i)h_t(i),$$

where ρ is a constant time preference rate and σ is the coefficient of relative risk aversion. We also call $\dfrac{1}{\sigma}$ the elasticity of intertemporal substitution because it represents the willingness to save or borrow resources. Both ρ and σ are assumed to be the same across countries and consumers. Individual variables $c_t(i)$, $k_t(i)$, and $h_t(i)$ are agent i's consumption of the final goods and sevices y, physical capital, and human capital, respectively. Furthermore, $q_t(i)$ is investment in physical capital and $\theta_t(i)$ is the share of human capital invested in good y production by agent i. In general, lower case variables other than r_t and w_t represent per capita variables. We also have the following transition equations of physical and human capital. For the average agent i.

$$\dot{k}_t(i) = q_t(i) - \delta_k k_t(i)\,, \tag{10}$$

$$\dot{h}_t(i) = e(i)[1-\theta_t(i)]h_t(i) - \delta_h h_t(i)\,, \tag{11}$$

where $e(i)$ is represents the efficiency or productivity of education for the average agent i. $1-\theta_t(i)$ is the share of human capital invested in education by agent i, and δ_k and δ_h are constant depreciation rates of physical and human capital that are the same across countries and agents. As shown by (11), additional human capital is generated via a linear education productivity function using the existing stock of human capital as the only input. We relax this assumption of linear education productivity function in an extension in Section VII. Hence, the maximum human capital growth rate is $e(i)-\delta_h$. The higher $e(i)$ is, the faster the human capital grows given the share of human capital invested in education, $1-\theta_t(i)$, at any level of human capital holdings, $h_t(i)$. We will see that the distribution of $e(i)$ affects the growth rate of production of final goods and services. We will study the details of the effects of the distribution of $e(i)$ on the steady state in Section V.

Using the Hamiltonian to solve the optimization problem, we get the following conditions in equilibrium.[1]

$$\frac{\dot{\nu}_t(i)}{\nu_t(i)} = \delta_k - r_t\,, \tag{12}$$

$$\frac{\dot{\mu}_t(i)}{\mu_t(i)} = -e(i) + \delta_h\,, \tag{13}$$

where $\nu_t(i)$ and $\mu_t(i)$ are costate variables representing marginal values at the optimal levels of the state variables (“shadow values”)

[1] ref. Appendix A for details.

associated with changes in physical and human capital for agent i respectively. The growth rate of aggregate consumption expenditure and per capita consumption expenditure is given as

$$\gamma_C \equiv \frac{\dot{C}_t}{C_t} = \frac{\dot{c}_t(i)}{c_t(i)} = \frac{1}{\sigma}(r_t - \delta_k - \rho) \equiv \gamma_c . \tag{14}$$

In particular, the growth rate of per capita (individual) consumption expenditure does not depend on the efficiency of education $e(i)$, and is the same across all agents. This result is reasonable when all the agents have the berrowing/saving/rental rate, r_t, and have the same preferences as consumers, represented by ρ and σ.

III. OPTIMAL INVESTMENT IN EDUCATION

We can see from (10) and (11) that the return to investment in physical capital does not depend on the efficiency of education $e(i)$ but the return to investment in human capital depends positively on $e(i)$. From (A1) of Appendix A, we know that $\nu_t(i)$ and $\mu_t(i)$ are respectively the shadow values of marginal increases in physical capital and in human capital in terms of present utility. From (12) and (13), the rate of change in the shadow value of investment in physical capital, $\dot{\nu}_t(i)/\nu_t(i)$, is also independent of $e(i)$ but the rate of change in the shadow value of investment in human capital, $\dot{\mu}_t(i)/\mu_t(i)$, is negatively proportional to $e(i)$. Because everyone can receive the same rates of return to physical and human capital when producing goods and services, as shown in (7) and (8), an individual who is more efficient or productive in education can earn a relatively higher return (in the future) from investing in education than from producing more goods and services at the present time compared to a less efficient agent, but at a diminishing rate. Because $0 \le 1-\theta_t(i) \le 1$, $0 \le \theta_t(i) \le 1$, and because we assume $0 \le e(i) < \infty$. An agent with more productive or efficient education (higher $e(i)$) is predicted to invest a higher proportion (higher $1-\theta_t(i)$) of his time in more education and a lower proportion (lower $\theta_t(i)$) in the production of final goods and services at the current time than a less efficient agent (lower $e(i)$), but will do so at a diminishing rate. To put it differently yet, comparing the proportions of human capital holdings invested in education of two agents, one with higher $e(i)=e^H(i)$ and the other with lower $e(i)=e^L(i)$, the proportion will be higher for the agent with $e(i)=e^H(i)$ but the proportional increase of $1-\theta_t(i)$ will be smaller than the proportional increase of $e(i)$. This may be expressed as follows.

$$[1-\theta_t(i)]^H > [1-\theta_t(i)]^L \tag{15}$$

and

$$\frac{[1-\theta_t(i)]^H - [1-\theta_t(i)]^L}{[1-\theta_t(i)]^L} < \frac{e^H(i)-e^L(i)}{e^L(i)} \tag{16}$$

for $e^H(i) > e^L(i)$. This is because the time endowment is bounded from above by 1.

The results in equations (15) and (16) show that Gorman aggregation does not occur as implied by the positive but diminishing productivity of individual and aggregate human capital in equation (1). With identical levels of initial physical and human capital, a more efficient agent is predicted to accumulate more human capital than a less efficient agent, and so he will eventually have more human capital for investment in both in further education and the production of final goods and services. This will lead to higher real income and higher consumption for the more efficient agent in the long run. This is summarized as a result.

Result 1. *The model shows a positive relation between the individual efficiency of education,* $e(i)$*, and the individual level of human capital,* $h_t(i)$*, at all time and also a positive relation between* $e(i)$ *and individual consumption,* $c_t(i)$*, provided that initial holdings of physical and human capital are identical across agents.*

IV. SYSTEM DYNAMICS AND STEADY STATE

To begin with, I define two stationary variables $z_t \equiv Y_t / K_t$, the gross average product of physical capital, and $\chi_t \equiv C_t / K_t$, the aggregate consumption/aggregate physical capital ratio as in Barro and Sala-i-Martin (1999, pp.183-184). We can then write z_t using production function (1), equation (7), and the definition $\omega_t \equiv K_t / H_t$ as follows.

$$z_t \equiv \frac{Y_t}{K_t} = A\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} = \frac{r_t}{\alpha}. \tag{17}$$

The ultimate goal of this section is to find the dynamics of z_t, χ_t, and $\theta_t \equiv \left(\int_0^1 \theta_t(i)h_t(i)di\right)/H_t$. However, we need the dynamics of ω_t and θ_t as a preliminary step towards the main dynamics of z_t and χ_t.[2]

First of all,

$$\gamma_\omega = \frac{\dot{\omega}_t}{\omega_t} = \gamma_K - \gamma_H. \tag{18}$$

Aggregate physical capital growth rate is obtained from (1) by dividing both sides by K_t with the definition of $\omega_t \equiv K_t / H_t$, $\chi_t = C_t / K_t$, and (17).

$$\gamma_K = \frac{\dot{K}_t}{K_t} = A\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} - \chi_t - \delta_k = z_t - \chi_t - \delta_k. \tag{19}$$

Aggregate human capital growth rate is obtained from (11).

$$\gamma_H = \frac{\dot{H}_t}{H_t} = \frac{\int_0^1 \dot{h}_t(i)di}{H_t} = \int_0^1 \{e(i)[1-\theta_t(i)] - \delta_h\}\left(\frac{h_t(i)}{H_t}\right)di. \tag{20}$$

To derive the dynamics of θ_t, we define a variable $p_t(i) \equiv \mu_t(i) / \lambda_t(i)$.

Definition 1. ($p_t(i) \equiv \mu_t(i) / \lambda_t(i)$, shadow price of human capital in

[2] The differential equation for θ_t need to be further rearranged using the result for z_t.

terms of final good y for agent i): *We define the ratio* $p_t(i) \equiv \mu_t(i) / \lambda_t(i)$, *where* $\mu_t(i)$ *is the costate variable for human capital transition equation (11) and* $\lambda_t(i)$ *is the Lagrange multiplier for the budget constraint of consumer agent* i *'s optimization problem. Then, we get from (A4) of Appendix A,* $p_t(i) = \frac{w_t}{e(i)}$ *for* $h_t(i) \neq 0$.[3] □

The price, $p_t(i)$, equals the ratio of the marginal product of human capital in the final good y sector (the wage rate) to its marginal product in the education sector.[4] As we have from (8) $w_t = A(1-\alpha)\theta_t^{-\alpha}\omega_t^{\alpha}$, hence $p_t(i)$ is a function of the ratio of aggregate physical capital to aggregate human capital invested in the final good sector, ($K_t / (\theta_t H_t)$), and individual education efficiency $e(i)$.

We integrate $p_t(i) = w_t / e(i)$ over $i \in [0,1]$ and differentiate with respect to time. Rearranging the result, we get[5]

$$\gamma_\theta = \frac{\dot{\theta}_t}{\theta_t} = -\left\{ z_t - \frac{1}{\alpha}\left[\left(\int_0^1 [e(i)]^{-1} di \right)^{-1} - \delta_h + \delta_k \right] \right\} + \gamma_\omega . \tag{21}$$

As a second step, we find the main dynamics of the system. From (17) and (21),

$$\begin{aligned} \gamma_z = \frac{\dot{z}_t}{z_t} &= (1-\alpha)(\gamma_\theta - \gamma_\omega) \\ &= -(1-\alpha)\left\{ z_t - \frac{1}{\alpha}\left[\left(\int_0^1 [e(i)]^{-1} di \right)^{-1} - \delta_h + \delta_k \right] \right\} \\ &= -(1-\alpha)(z_t - z^*) , \end{aligned} \tag{22}$$

and[6]

$$\gamma_\chi = \frac{\dot{\chi}_t}{\chi_t} = \left(\frac{\alpha - \sigma}{\sigma} \right) z_t + \chi_t - \frac{1}{\sigma}(\delta_k + \rho) + \delta_k$$

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[3] ref. the Hamiltonian in equation (A1) of Appendix A of the consumer agent i 's optimization problem. See also Barro and Sala-i-Martin (1999), p.181.

[4] ref. Barro and Sala-i-Martin (1999), p.181.

[5] ref. Appendix B for details.

[6] ref. Appendix C for details.

$$=\left(\frac{\alpha-\sigma}{\sigma}\right)(z_t-z^*)+(\chi_t-\chi^*)\,, \tag{23}$$

where the steady state values z^* and χ^* are given as follows.[7]

$$z^*=\frac{1}{\alpha}\left[\left(\int_0^1[e(i)]^{-1}di\right)^{-1}-\delta_h+\delta_k\right], \tag{24}$$

and

$$\chi^*=\left(\frac{\sigma-\alpha}{\sigma}\right)z^*+\frac{1}{\sigma}(\delta_k+\rho)-\delta_k\,. \tag{25}$$

Equations (22) and (23) are two main equations of dynamics for $z_t\equiv Y_t/K_t$ and $\chi_t\equiv C_t/K_t$. Together with (22) and (23), a differential equation for $\theta_t\equiv\left(\int_0^1\theta_t(i)h_t(i)di\right)/H_t$, aggregate share of human capital invested in output production, will completely describe the dynamics of the entire system.

We can see here from (24) and (25) that the steady state values of the key variables of the economy critically depend on the distribution of individual education efficiency $e(i)$ among agents. In the first instance, $[e(i)]^{-1}$ in (24) is a decreasing convex function of $e(i)$, hence, via Jensen's inequality theorem, z^* will be negatively related to the dispersion of $e(i)$. The more details will be explained in Section V.

Let's further analyse the dynamics of θ_t, aggregate human capital share invested in output production. The growth rate of θ_t in equation (21) can be rewritten as follows using equations (24) and (18).

$$\gamma_\theta=\frac{\dot{\theta}_t}{\theta_t}=-(z_t-z^*)+(\gamma_K-\gamma_H)\,.$$

Further using (17) here, we get

$$\gamma_\theta=-(z_t-z^*)+(z_t-\chi_t-\delta_k-\gamma_H)=z^*-\chi_t-\delta_k-\gamma_H\,. \tag{26}$$

[7] For the details of existence of a steady state (balanced growth path), see Appendix D.

Finally applying $\gamma_H{}^* = \gamma_K{}^* = z^* - \chi^* - \delta_k$ from equations (D5) and (19) at steady state,

$$\gamma_\theta = -(\chi_t - \chi^*) - (\gamma_H - \gamma_H{}^*) \tag{27}$$

The dynamics of z_t, χ_t, and θ_t are described by the three differential equations (22), (23), and (27). These are the main dynamics showing how the aggregate output, aggregate consumption, and the allocation of human capital between production and education are determined within the economy. These equations (22), (23), and (27) together with the steady state values of z^* and χ^* in (24) and (25) define the transition dynamics and the steady state of the economy completely.

Meanwhile, from the definition of aggregate share of human capital invested in final good y production in (4), we get

$$1-\theta_t = \int_0^1 [1-\theta_t(i)]\left(\frac{h_t(i)}{H_t}\right) di .$$

We know from equations (15) and (16) and Result 1 in Section III that the relations between $e(i)$ and $[1-\theta_t(i)]$ and between $e(i)$ and $h_t(i)$ are both positive. Hence, the right hand-side of the above identity increases monotonically with $\int_0^1 e(i)[1-\theta_t(i)]\left(\frac{h_t(i)}{H_t}\right) di = \gamma_H + \delta_h$ (ref. equation (20)).[8] Hence, we get a positive monotonic relation between $[1-\theta_t]$ and γ_H. Consequently, at steady state ($\dot{\theta}_t = 0$), we have from (26) and (25) $\gamma_H{}^* = z^* - \chi^* - \delta_k = \left(\frac{\alpha}{\sigma}\right) z^* - \frac{1}{\sigma}(\delta_k + \rho)$, and the relation between $(1-\theta^*)$ and z^* must be positive. Together with this, we also have a positive (negative) relation between z^* and χ^* if $\sigma > \alpha$ ($\sigma < \alpha$) from (25). This is summarized as a result.[9]

[8] Note here that we are considering a particular fixed distribution of $e(i)$ and not comparing between two different distributions of $e(i)$.

[9] This is the same result as in Barro and Sala-i-Martin (1999), pp.184-186.

Result 2. *For our model economy, we always get a positive relation between* $z^* = Y^* / K^*$ *and* $(1-\theta^*)$*. We also get a positive (negative) relation between* z^* *and* $\chi^* = C^* / K^*$ *if* $\sigma > \alpha$ $(\sigma < \alpha)$*.*

Result 2 states that, at steady state, the greater the aggregate investment in education is, the higher the output per physical capital for any values of α and σ and the higher (lower) consumption per physical capital for $\sigma > \alpha$ ($\sigma < \alpha$) will be. Here, σ is the coefficient of relative risk aversion of consumer agents in equation (9) and α is the share of physical capital in output production in equation (1) (i.e., $\alpha \approx 1/3$). First of all, human capital accumulation does not depend on consumers' preferences but positively depends on the share $1-\theta_t(i)$ from (11) and output per physical capital positively depends on the aggregate human capital from (17) ceteris paribus. Hence, it is natural that we get a positive relation between $z^* = Y^* / K^*$ and ($1-\theta^*$) irrespective of preference parameter σ. In contrast, consumption depends on individual preferences represented by σ from (9) and returns to saving and work depend on the production technology parameter α from (5) and (6). Consumers are risk-neutral, risk-averse, or risk-loving if $\sigma = 0$, $\sigma > 0$, or $\sigma < 0$. In general, if consumers are sufficiently risk-averse, i.e. if they strongly prefer consumption smoothing over time (strongly care about their remote offspring's welfare), they will save more (invest more in physical capital) early in their life-time and consequently achieve higher level of consumption later (steady state consumption).

Barro and Sala-i-Martin (1999) tabulate the empirical values of the physical capital share, α, in output production for different groups of countries using various works of other researchers.[10] For the group of 7 countries (Canada, France, Germany, Italy, Japan, UK, and the US), physical capital share α was on average between 0.38 (Italy) and 0.45 (Canada) for 1960~1990. For the 4 East Asian countries (Hong Kong, Singapore, South Korea, and Taiwan), physical capital share was on average between 0.29 (Taiwan) and 0.53 (Singapore) for 1966~1990. The share was relatively higher at between 0.45 (Brazil) and 0.69 (Mexico) for 7 Latin American countries (Argentina, Brazil, Chile, Colombia, Mexico,

[10] Data from Barro and Sala-i-Martin (1999), TABLE 10.8, pp.380-381.

Peru, and Venezuela) for 1940~1980. Labour input in the final good production in our model amounts to human capital augmented labour, so the true value of α would not be higher than these figures at best. Barro and Sala-i-Martin[11] also derive the minimum value of σ that is consistent with empirically relevant observation that the saving rate increases over time during the transition from initially low physical capital level towards the steady state level of capital per efficient unit of labour with a Cobb-Douglas production technology. For appropriate parameter values of $\rho = 0.02$, $\delta = 0.05$, $n = 0.01$, and $x = 0.02$, Barro and Sala-i-Martin (pp.78-79) show that the minimum value of σ is 17.5 for $\alpha = 0.3$ and 1.75 for $\alpha = 0.75$.[12] The figures would be $\sigma \geq 4.67$ for $\alpha = 0.3$ and $\sigma \geq 1.87$ for $\alpha = 0.75$ for non-decreasing savings rate during the transition for our model with $n = x = 0$. The latter case of $\alpha = 0.75$ pertains to the broad concept of physical capital including human capital. Lucas (1988, p.29 and p.33) also mentions that $\sigma > 1$ is the interesting case, with $\sigma < 1$ being an exception. Overall, it is likely that $\sigma > \alpha$ and the correlation between $z^* = Y^* / K^*$ and $\chi^* = C^* / K^*$ will be positive.

Furthermore, from equations (14), (D4), (D5), (D1), and (24) with $r = \alpha z$, we get the following steady state growth rates.

$$\begin{aligned} \gamma_c^* = \gamma_C^* = \gamma_K^* = \gamma_H^* = \gamma_Y^* = \kappa &= \frac{1}{\sigma}(\alpha z^* - \delta_k - \rho) \\ &= \frac{1}{\sigma}\left\{ \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} - \delta_h - \rho \right\} \equiv \gamma^* . \end{aligned} \tag{28}$$

Equation (28) shows that consumption, physical capital, human capital, and output all grow at the same positive rate at steady state in the absence of any externality and the growth rate critically depends on the distribution of the education efficiency $e(i)$ across agents.[13] This

[11] Barro and Sala-i-Martin (1999), Chapter 2, especially pp.74-87 and pp.89-90. Barro and Sala-i-Martin use θ instead of σ of our model.

[12] In Barro and Sala-i-Martin (1999), ρ and $\delta = \delta_k$ have the same definitions as in our model and n is exogenous population growth rate and $x = \dot{A}_t / A_t$ is exogenous technology growth rate.

[13] We assume that the average education efficiency is sufficiently high to overcome human capital depreciation, δ_h, and agents' impatience, ρ, for the growth rate to be positive.

common growth rate of C_t, K_t, H_t, and Y_t at steady state apart from the distributional effect is a standard outcome in the endogenous growth literature, e.g. in Barro and Sala-i-Martin (1999, pp.179-180, also see pp.198-200 for general returns to scale cases). If we had a positive externality of human capital in output production as in Lucas (1988, pp.22-23), then we would have obtained $\gamma_K{}^* = \gamma^* > \gamma_H{}^*$ instead.

V. THE EFFECTS OF EDUCATION EFFICIENCY DISTRIBUTION ON GROWTH

Let's study the effects of the education efficiency distribution among agents on steady state growth. We apply Jensen's inequality theorem to compare two countries with different education efficiency distribution.[14] We assume that the average values of $e(i)$ are the same between country A and country B but the variance of $e(i)$ is larger in country B. We call the former a more *homogeneous* country and the latter a more *heterogeneous* country. This may also be compared to a mean preserving spread. Then, we claim the following result as in Proposition 1.

Proposition 1. *If the average values of $e(i)$ are the same between country A and country B but the variance of $e(i)$ is smaller in country A, then country A will achieve a higher steady state gross average product of physical capital, $z^* = Y^* / K^*$, than country B, for any strictly convex preferences. Under the same condition, country A will achieve a higher (lower) steady state aggregate consumption per physical capital ratio, $\chi^* = C^* / K^*$, than country B if $\sigma > \alpha$ ($\sigma < \alpha$).*[15]

Proof: We know that $[e(i)]^{-1}$ is a convex function in $e(i)$. Hence, we get a greater value of $\int_0^1 [e(i)]^{-1} di$ for a larger variance of $e(i)$ given the same average value of $e(i)$ via Jensen's inequality theorem. Consequently, we get a smaller value of $\left(\int_0^1 [e(i)]^{-1} di\right)^{-1}$ for a larger

[14] For Jensen's inequality theorem, refer to Theorem 4.8 in Greene (1997), 3rd ed., p.119 and Fishburn and Vickson (1978), p.54.

[15] ref. the discussion after Result 2 in Section IV about the parameter values of α and σ.

variance of the distribution of $e(i)$. That is, the term $\left(\int_0^1 [e(i)]^{-1} di\right)^{-1}$ varies inversely with the variance $\sigma_e{}^2$ of $e(i)$, given the average value of $e(i)$. In other words, we get the following inequality.

$$E([e(i)]^{-1} \mid A) = \int_{\underline{e}^A}^{\overline{e}^A} [e(i)]^{-1} dF(e(i) \mid A) < E([e(i)]^{-1} \mid B)$$
$$= \int_{\underline{e}^B}^{\overline{e}^B} [e(i)]^{-1} dF(e(i) \mid B),$$

where $F(e(i) \mid j)$ is distribution function of $e(i)$ and $[\underline{e}^j, \overline{e}^j]$ is the support of the education efficiency $e(i)$ with $0 = \underline{e}^j = e(0)^j < \overline{e}^j = e(1)^j < \infty$ of country $j = A, B$, with agent 0 being the least bright and agent 1 the brightest.
Therefore, we get

$$\left(E([e(i)]^{-1} \mid A)\right)^{-1} > \left(E([e(i)]^{-1} \mid B)\right)^{-1}.$$

Consequently, we get higher steady state values of $z^* = Y^* / K^*$ and $\chi^* = C^* / K^*$ (assuming the usual inequality $\sigma > \alpha$) from (24) and (25) for a country whose education efficiency is more densely distributed.

Q.E.D.

Roughly speaking, a country with more evenly distributed education efficiency $e(i)$ among agents with reasonably high mean value of $e(i)$ will achieve higher z^* and χ^* than a country with more widely dispersed education efficiency. The dynamics of χ_t is not affected by the distribution of the educational efficiency, $e(i)$, (ref. equation (23)). That is, the $\dot{\chi}_t = 0$ locus in the $z - \chi$ plane does not change (same slope and the same intercept) depending on the distribution of $e(i)$ although the steady state values z^* and χ^* will change.

From Proposition 1 above and Result 2 of Section IV, we get the following proposition.

Proposition 2. *Under the same condition as in Proposition 1, the more* *homogeneous country A achieves a higher aggregate human capital share in education, $1-\theta^*$, and a lower aggregate human capital share in output production, θ^*, than the more heterogeneous country B, at steady state.*

We also get the following Corollary.

Corollary 1. *Under the same condition as in Proposition 1, the steady state growth rates of individual and aggregate consumption, $\gamma_c^* = \gamma_C^*$, of aggregate physical capital, γ_K^*, of aggregate human capital, γ_H^*, and of aggregate output, γ_Y^*, will be higher in the more homogeneous country A than those in the more heterogeneous country B.*

Proof: Steady state growth rates of individual and aggregate consumption, aggregate physical capital, aggregate human capital, and aggregate output are given by $\gamma^* = \frac{1}{\sigma}(\alpha z^* - \delta_k - \rho)$ by equation (28). We have higher value of z^* for the more homogeneous country via Proposition 1. Hence, we get higher growth rates $\gamma_c^* = \gamma_C^* = \gamma_K^* = \gamma_H^* = \gamma_Y^*$ for the more homogeneous country at steady state. *Q.E.D.*

VI. APPLICATION TO A LOG-NORMAL DISTRIBUTION

We show here an application using a specific distribution function. Let's assume a log-normal distribution for $e(i)$.[16]

$$e(i)^j \sim LN(\mu^j, (\Delta_e^{\,2})^j) \quad for\ j = A, B, \tag{29}$$

where μ^j is the mean and $(\Delta_e^{\,2})^j$ is the variance of $\log e(i)^j$ for country $j = A, B$, respectively.[17]

Then the mean and variance of $e(i)$ itself are given as follows.

[16] ref. Aitchison and Brown (1963), pp.7-11 and William H. Greene (1997), p.71. For a general reference to log-normal distribution, see Aitchison and Brown (1963), Chapters 1 and 2.

[17] This is a natural log. Superscript *A* (resp. *B*) represents a more homogeneous country (resp. a more heterogeneous country).

$$E[e(i)] = \exp\left(\mu + \frac{1}{2}\Delta_e^{\ 2}\right) \equiv \overline{e}\,, \tag{30}$$

$$\begin{aligned} \operatorname{var}[e(i)] &= \exp(2\mu + \Delta_e^{\ 2})[\exp(\Delta_e^{\ 2}) - 1] \\ &= (E[e(i)])^2[\exp(\Delta_e^{\ 2}) - 1] \equiv \sigma_e^{\ 2}\,, \end{aligned} \tag{31}$$

where $\eta = [\exp(\Delta_e^{\ 2}) - 1]^{1/2} > 0$ is called the coefficient of variation.
We further assume the followings for a fair comparison.

Assumption 1. (Education efficiency distribution): *We assume that the means of education efficiency are the same between two countries but the variance is larger for the more heterogeneous country B than for the more homogeneous country A. That is, we assume that $E[e(i) \mid A] = E[e(i) \mid B]$ but $\operatorname{var}[e(i) \mid A] < \operatorname{var}[e(i) \mid B]$.*

Assumption 1 implies $(\Delta_e^{\ 2})^A < (\Delta_e^{\ 2})^B$ and $\mu^A > \mu^B$ by (30) and (31).
We define here a marginal agent and so on for our analysis.

Definition 2. (Marginal agent, Regenerative agents and Degenerative agents): *An agent with an index $\hat{i}$ whose investment in education just compensates his human capital depreciation is called a marginal agent. That is, we get $\gamma_{h(\hat{i})} = e(\hat{i})[1 - \theta_t(\hat{i})] - \delta_h = 0$. Furthermore, as we have a positive relation between $e(i)$ and $[1 - \theta_t(i)]$ from equations (15) and (16) of Section III, it must be true that $\gamma_{h(i)} < 0$ for $0 \le i < \hat{i}$ and $\gamma_{h(i)} > 0$ for $\hat{i} < i \le 1$, with $e(i)$ increasing in $i \in [0,1]$. We call the former degenerative agents and the latter regenerative agents.*[18] □

Let $\hat{e}$ be the value of education efficiency, $e(i)$, of the marginal agent. Hence, we have $\hat{e} = e(\hat{i})$. We also define the matching agents, $\tilde{i}^A = \tilde{i}^B$, as follows.

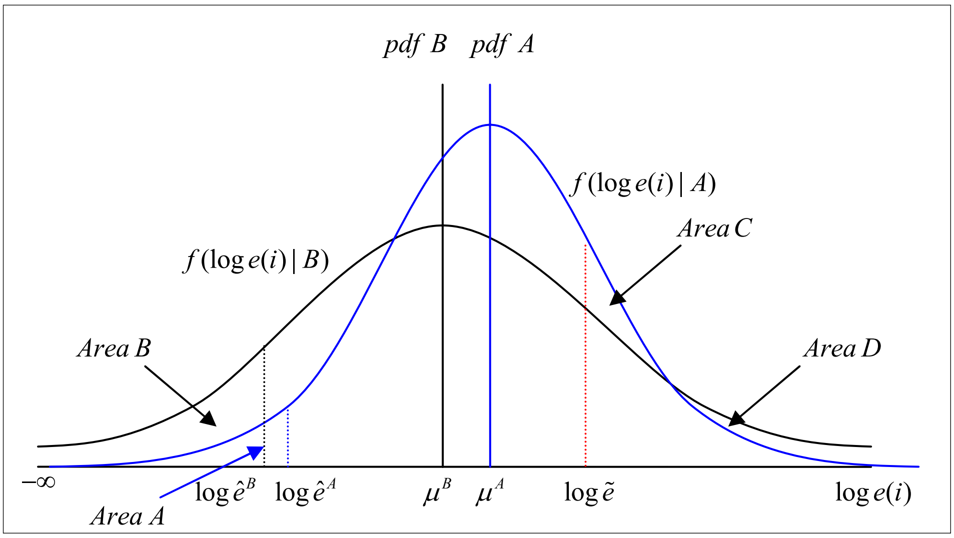
Definition 3. (Matching agents, $\tilde{i}^A = \tilde{i}^B$): *Two agents, $\tilde{i}^A$ of country A (the more homogeneous country) and $\tilde{i}^B$ of country B (the more*

[18] The marginal agent $\hat{i}$ may change during transition period towards steady state as the wage rate, w_t, changes during transition. At steady state, however, $\hat{i}$ will remain the same agent.

heterogeneous country), are called matching agents if both their indices and education efficiencies coincide, i.e. $e(\tilde{i}^A) = e(\tilde{i}^B) \equiv \tilde{e}$ *for* $\tilde{i}^A = \tilde{i}^B \in [0,1]$. □

We show the probability density functions (pdfs) of $\log e(i)$ in Figure 1. The more concentrated curve $f(\log e(i) \mid A)$ is the pdf of $\log e(i)$ of the more homogeneous country A and the more widely spread $f(\log e(i) \mid B)$ is the pdf of the more heterogeneous country B. Country B pdf has thicker tails.

[**Figure 1**] Probability Density Functions of Natural Log of Log-normal Education Efficiency, $e(i)$, for Two Countries A and B (We have $Area\ A < Area\ B$ and $Area\ C = Area\ D$.)



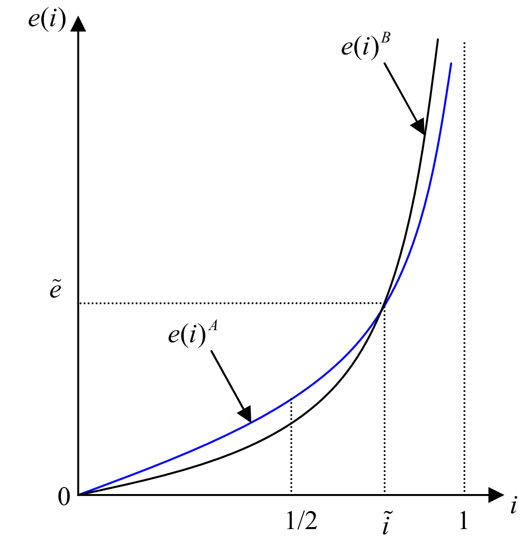
In Figure 1, we must have $Area\ C = Area\ D$ via the definition of the matching agents. We must also have $\log \tilde{e} > \mu^A > \mu^B$ because to the left of μ^A we cannot find the matching agents $\tilde{i}^A = \tilde{i}^B$ with the common educational efficiency $\tilde{e}$. Note here that the area under the pdf to the right of a particular value of $\log e(i)$ is the proportion of the population with $\log e(i)$ greater than that value for each country. Therefore, the matching agents must lie between $1/2 < \tilde{i} < 1$.

Secondly, the area $(= 1 - Area\ A)$ under the pdf $f(\log e(i) \mid A)$ to the right of $\log \hat{e}^A$ is greater than the area ($= 1 - Area\ B$) under

$f(\log e(i)\,|\,B)$ to the right of $\log \hat{e}^B$ where $\hat{e}^A$ and $\hat{e}^B$ are the values of $e(i)$ of the marginal agents of countries A and B, provided that at least more than half of the population achieve a net positive human capital increase ($\gamma_{h(i)} > 0$) in each country.

Now we look at the distribution of $e(i)$ more closely. We know that the median value of $e(i)$ is given by $\exp(\mu)$ for log-normally distributed $e(i)$ where μ is the mean of log of $e(i)$.[19]

[**Figure 2**] Education Efficiency $e(i)$ Curves (The more convex curve $e(i)^B$ is for the more heterogeneous country B.)



Hence, the median agent's efficiency is higher in the more homogeneous country because $\mu^A > \mu^B$. That is, we get $e(i=1/2)^A >$ $e(i=1/2)^B$. We also know that there are relatively greater mass of population near the tails $e(i=0)=0$ and $e(i=1)\approx +\infty$ for the more heterogeneous country B and greater mass in the middle range of $e(i)$ for the more homogeneous country A. Hence, we can draw convex, upward sloping $e(i)$ curves as in Figure 2.

Given the same average values of the education efficiency $e(i)$

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[19] ref. Aitchison and Brown (1963), p.9. Mean is given by $\exp\left(\mu+\frac{1}{2}\Delta_e^{\,2}\right)$ and mode by $\exp(\mu-\Delta_e^{\,2})$.

between the two countries, $E[e(i)\,|\,A] = \exp\left(\mu^A + \frac{1}{2}(\Delta_e^{\,2})^A\right) = E[e(i)\,|\,B]$ $= \exp\left(\mu^B + \frac{1}{2}(\Delta_e^{\,2})^B\right)$, increasing the variance of $e(i)$, i.e. mean preserving spread of $e(i)$ is equivalent to bending the $e(i)$ curve further inwards (increasing curvature). Two curves cross only once at $i = \tilde{i}$ other than at the origin $i = 0$ in Figure 2.

We can see in Figure 2 that education efficiency is higher in the more homogeneous country A for relatively low efficient agents, $e(i)^A > e(i)^B$ for $0 \le i < \tilde{i}$ and is higher in the more heterogeneous country B for relatively high efficient agents, $e(i)^A < e(i)^B$ for $\tilde{i} < i \le 1$. That is to say, more than half of the total population of the more homogeneous country have higher $e(i)$ than their counterparts of the more heterogeneous country, at the low end of $e(i)$ spectrum. In contrast, less than half of the total population of the more heterogeneous country have higher education efficiency $e(i)$ than their counterparts of the more homogeneous country, at the high end of $e(i)$ spectrum. Overall, proportionately higher amounts of investment in education of the low efficiency agents in the more homogeneous country outweigh the absolutely larger but proportionately smaller amounts of investment in education of the high efficiency agents in the more heterogeneous country.[20]

Let us now compare explicitly the steady state values between the two countries for the log-normal distribution. From the properties of a log-normal distribution, we know that if $e(i) \sim LN(\mu, \Delta_e^{\,2})$, then

$$e(i)^{-1} \sim LN(-\mu, \Delta_e^{\,2}). \tag{32}$$

Hence,

$$\int_0^1 [e(i)]^{-1} di = E[e(i)^{-1}] = \exp\left(-\mu + \frac{1}{2}\Delta_e^{\,2}\right).$$

We can then calculate steady state values of z^*, χ^*, and γ^* from equations (24), (25) and (28).

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[20] ref. equations (15) and (16) of Section III.

$$z^* = \frac{1}{\alpha}\left[\exp\left(\mu - \frac{1}{2}\Delta_e^{\ 2}\right) - \delta_h + \delta_k\right], \tag{33}$$

$$\chi^* = \left(\frac{\sigma - \alpha}{\sigma}\right) z^* + \frac{1}{\sigma}(\delta_k + \rho) - \delta_k, \tag{34}$$

and

$$\gamma^* = \gamma_c^{\ *} = \gamma_C^{\ *} = \gamma_K^{\ *} = \gamma_H^{\ *} = \gamma_Y^{\ *} = \frac{1}{\sigma}\left\{\exp\left(\mu - \frac{1}{2}\Delta_e^{\ 2}\right) - \delta_h - \rho\right\}. \tag{35}$$

We know that $\mu^A > \mu^B$ and $(\Delta_e^{\ 2})^A < (\Delta_e^{\ 2})^B$, so we get $(z^*)^A > (z^*)^B$, $(\gamma^*)^A > (\gamma^*)^B$ for any σ and α, and $(\chi^*)^A > (\chi^*)^B$ if $\sigma > \alpha$ (alternatively, $(\chi^*)^A < (\chi^*)^B$ if $\sigma < \alpha$), as predicted by our model.

VII. EXTENSION OF THE MODEL TO A CONVEX EDUCATION TECHNOLOGY

We extend the main model of a linear education technology of Sections II~V to a convex education technology growth model of Rebelo (1991) with both physical and human capital in the education sector as well as in the final good production sector to check the robustness of our model. All other settings are the same as before. We try to keep the original numbering either by keeping the original equation number like (10) or by adding a suffix _a like (1a) for a slight change for continuity. We show here only the main results and the details are provided in Appendix F.

1. Final Good y Manufacturer's Problem

The production function for the final good y is given as follows.

$$\begin{aligned} Y_t &= A\left(\int_0^1 \zeta_t(i)k_t(i)di\right)^{\alpha}\left(\int_0^1 \theta_t(i)h_t(i)di\right)^{1-\alpha} = A(\zeta_t K_t)^{\alpha}(\theta_t H_t)^{1-\alpha} \\ &= C_t + Q_t = C_t + \dot{K}_t + \delta_k K_t \end{aligned} \tag{1a}$$

where $k_t(i)$ is the physical capital stock, and $\zeta_t(i)$ and $\theta_t(i)$ are the shares of physical and human capital invested in the production of final good y by consumer agent i at time t.

We also define the indices of aggregate physical and human capital shares in final good y production, ζ_t and θ_t, as follows. The definition of θ_t is given in equation (4).

$$\zeta_t \equiv \frac{\int_0^1 \zeta_t(i)k_t(i)di}{K_t} \tag{4a}$$

The representative final good y manufacturer maximizes its period profit by choosing inputs of $\zeta_t(i)k_t(i)$ and $\theta_t(i)h_t(i)$ at time t.

$$\begin{aligned}\max_{\zeta_t(i),\theta_t(i)} \Pi_t &= Y_t - r_t\int_0^1 \zeta_t(i)k_t(i)di - w_t\int_0^1 \theta_t(i)h_t(i)di \\ &= A\left(\int_0^1 \zeta_t(i)k_t(i)di\right)^{\alpha}\left(\int_0^1 \theta_t(i)h_t(i)di\right)^{1-\alpha} \\ &\quad - r_t\int_0^1 \zeta_t(i)k_t(i)di - w_t\int_0^1 \theta_t(i)h_t(i)di\end{aligned}$$

Solving the first order conditions, we get the following equations for the physical capital rental rate r_t and human capital wage rate w_t, with the definition of $\omega_t \equiv K_t / H_t$.

$$r_t = A\alpha\left(\frac{\zeta_t K_t}{\theta_t H_t}\right)^{-(1-\alpha)} = A\alpha(\zeta_t/\theta_t)^{-(1-\alpha)}\omega_t^{-(1-\alpha)}, \tag{7a}$$

$$w_t = A(1-\alpha)\left(\frac{\zeta_t K_t}{\theta_t H_t}\right)^{\alpha} = A(1-\alpha)(\zeta_t/\theta_t)^{\alpha}\omega_t^{\alpha}. \tag{8a}$$

2. Consumer Agent i's Problem

Consumer agent i maximizes his lifetime utility

$$\max_{c_t(i),q_t(i),\zeta_t(i),\theta_t(i)} U(i) = \int_0^{\infty} e^{-\rho t}\left\{\frac{c_t(i)^{1-\sigma}-1}{1-\sigma}\right\}dt, 0<\rho<1, \sigma>0, \sigma\neq 1, \tag{9a}$$

subject to the budget constraint

$$c_t(i)+q_t(i)\le r_t\zeta_t(i)k_t(i)+w_t\theta_t(i)h_t(i)\,.$$

We also have the following transition equations of physical and human capital.

$$\dot{k}_t(i)=q_t(i)-\delta_k k_t(i)\,, \tag{10}$$

$$\dot{h}_t(i)=e(i)\left[\left(1-\zeta_t(i)\right)k_t(i)\right]^{\beta}\left[\left(1-\theta_t(i)\right)h_t(i)\right]^{1-\beta}-\delta_h h_t(i)\,, \tag{11a}$$

In (11a), we allow β to be different from α in (1a) so that the education technology can be different from the final good production technology. We assume $\alpha>\beta$ so that the final good production sector is *physical-capital-intensive* and the education sector is *human-capital-intensive*.

Using the Hamiltonian to solve the agent i's optimization, we get the following conditions among others at equilibrium.[21]

$$\frac{\dot{\nu}_t(i)}{\nu_t(i)}=\delta_k-r_t \tag{12}$$

$$\frac{\dot{\mu}_t(i)}{\mu_t(i)}=\delta_h-(1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta}e(i) \tag{13a}$$

where $\nu_t(i)$ and $\mu_t(i)$ are costate variables associated with physical and human capital changes for agent i respectively. And the consumption growth rate is given as

$$\gamma_C=\frac{\dot{C}_t}{C_t}=\gamma_{c(i)}=\frac{\dot{c}_t(i)}{c_t(i)}=\frac{1}{\sigma}(r_t-\delta_k-\rho)\equiv\gamma_c\,. \tag{14}$$

3. Optimal Investment in Education

To begin with, we identify the condition that the rate of return to physical capital must be the same in the two sectors at equilibrium. The

[21] ref. Appendix E for details.

same condition holds for human capital as well. From these conditions, we get the following relation between $\zeta_t(i)$ and $\theta_t(i)$.

$$\left(\frac{\beta}{1-\beta}\right)\frac{\zeta_t K_t}{1-\zeta_t(i)}=\left(\frac{\alpha}{1-\alpha}\right)\frac{\theta_t H_t}{1-\theta_t(i)} \tag{36}$$

where ζ_t and θ_t are defined in (4a) and (4). Equation (36) implies that $1-\zeta_t(i)$ and $1-\theta_t(i)$ are positively related, with $\zeta_t(i)=1$ when $\theta_t(i)=1$.

We can see from (10) and (11a) that the return to investment in physical capital does not depend on the education efficiency $e(i)$ but the return to investment in human capital depends positively on $e(i)$. Furthermore, from (12) and (13a), the change rate of valuation (shadow price) of investment in physical capital, $\dot{v}_t(i)/v_t(i)$, is again independent of $e(i)$ but the change rate of valuation of investment in human capital, $\dot{\mu}_t(i)/\mu_t(i)$, is negatively proportional to $e(i)$, as we have $0<\beta<1$, $r_t>0$ and $w_t>0$. Meanwhile, all the agents face common rates of return to physical and human capital in output production as shown in (7a) and (8a). That is to say, a more efficient agent can get a relatively higher reward from investing in education than from investing in output production of his resources, physical and human capital holdings, compared to a less efficient agent, but at a decelerating rate. In other words, a more efficient agent (with higher $e(i)$) will invest higher proportions of his physical and human capitals in education and lower proportions in output production than a less efficient agent (with lower $e(i)$), but will do so at a decelerating rate.[22] That is, we get the same results as the earlier case of the linear education technology of Section II (i.e. Lucas style linear education technology (11), ref. equations (15) and (16) of Section III). Hence, we can conclude that the positive but less than *one-to-one* relation between the investment in education and the individual education ability $e(i)$ remains robust for general case of factor intensity in the education sector, i.e. a convex education technology

[22] Recall that $1-\zeta_t(i)$ and $1-\theta_t(i)$ are positively related in (36).

in physical and human capitals. This result may be described as follows.

$$\left[1-x_t(i)\right]^H > \left[1-x_t(i)\right]^L \tag{37}$$

and

$$\frac{\left[1-x_t(i)\right]^H - \left[1-x_t(i)\right]^L}{\left[1-x_t(i)\right]^L} < \frac{e^H(i)-e^L(i)}{e^L(i)} \tag{38}$$

for $e^H(i) > e^L(i)$ and $x = \zeta, \theta$.

Equations (37) and (38) show that the positive but less than *one-to-one* relation between the investment in education and the individual education ability $e(i)$ remains robust for wide range of model specification including both Uzawa (1965)-Lucas (1988) model and Rebelo model (1991). For the same reason following equations (15) and (16) of Section III, Result 1 remains valid here for the Rebelo-style convex education technology (11a).

4. System Dynamics and Steady State

As before, we define two stationary variables $z_t \equiv Y_t / K_t$ and $\chi_t \equiv C_t / K_t$. We can then write z_t using production function (1a), physical capital rental rate (7a), and the definition $\omega_t \equiv K_t / H_t$ as follows.

$$z_t \equiv \frac{Y_t}{K_t} = A\zeta_t^{\alpha}\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} = \left(\frac{\zeta_t}{\alpha}\right)r_t. \tag{17a}$$

We also define here $\phi_t \equiv \dfrac{\zeta_t K_t}{\theta_t H_t} = \dfrac{\zeta_t \omega_t}{\theta_t}$, physical capital intensity relative to human capital in the final good production sector.

We derive the dynamics of ϕ_t, defining a variable $p_t(i) \equiv \mu_t(i) / \lambda_t(i)$, *shadow price of human capital in terms of final good* y *for agent* i as before. We integrate $p_t(i)$ over $i \in [0,1]$ and differentiate with respect to time. Rearranging the result, we get with $\phi_t \equiv \frac{\zeta_t K_t}{\theta_t H_t} = \frac{\zeta_t \omega_t}{\theta_t}$,[23]

$$\gamma_\phi = \left(\frac{\dot{\zeta}_t}{\zeta_t} + \frac{\dot{\omega}_t}{\omega_t} - \frac{\dot{\theta}_t}{\theta_t} \right) = \frac{1}{\alpha - \beta}(\delta_h - \delta_k + A\alpha(\phi_t)^{-(1-\alpha)})$$
$$- \frac{1-\beta}{\alpha-\beta}\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha}\right)\phi_t \right]^\beta \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} \tag{39}$$

From here, we can solve for the steady state value ϕ^* of ϕ_t when $\gamma_\phi = 0$ for $\delta_k = \delta_h$.

$$\phi^* = \left\{ \frac{(1-\beta)}{A\alpha}\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha}\right) \right]^\beta \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} \right\}^{\frac{-1}{1-\alpha+\beta}} \tag{40}$$

Here, we can see that a more heterogeneous country with larger spread of $e(i)$ distribution with the same mean value $\bar{e}$ will get higher value of ϕ^*. Note here that $0 < \alpha < 1$ and $0 < \beta < 1$ and the power index of ϕ^* is negative. In other words, a more heterogeneous country (country B) will have higher physical capital intensity relative to human capital in the final good production sector than a more homogeneous country (country A) at steady state.

$$\phi^{A^*} = \frac{\zeta^{A^*} K^{A^*}}{\theta^{A^*} H^{A^*}} = \frac{\zeta^{A^*}\omega^{A^*}}{\theta^{A^*}} < \frac{\zeta^{B^*} K^{B^*}}{\theta^{B^*} H^{B^*}} = \frac{\zeta^{B^*}\omega^{B^*}}{\theta^{B^*}} = \phi^{B^*} \tag{41}$$

This implies in turn that a more homogeneous country employs a more human capital intensive combination of factor inputs in final good production than a more heterogeneous country and thus enjoys higher productivity of physical capital in goods production at steady state.

As a second step, we get the main dynamics of the system, γ_z and γ_χ.[24]

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[23] ref. Appendix F for details.
[24] ref. Appendix F for details.

$$\gamma_z = \frac{\alpha}{\alpha-\beta}\left(\delta_h - \delta_k + A\alpha\left(\frac{z_t}{A\zeta_t}\right)\right)$$

$$-\frac{\alpha(1-\beta)}{\alpha-\beta}\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{z_t}{A\zeta_t}\right)^{\frac{-1}{1-\alpha}}\right]^{\beta}\left(\int_0^1 [e(i)]^{-1}di\right)^{-1} + \gamma_\theta - \gamma_\omega \quad \text{(22a)}$$

and

$$\gamma_\chi = \frac{\dot{\chi}_t}{\chi_t} = \left(\frac{\alpha - \sigma\zeta_t}{\sigma\zeta_t}\right)z_t + \chi_t - \frac{1}{\sigma}(\delta_k + \rho) + \delta_k \quad \text{(23a)}$$

We can solve (22a) and (23a) for the steady state values of $z^* / \zeta^* = Y^* / (\zeta^* K^*)$ and $\chi^* = C^* / K^*$ with $\gamma_z = \gamma_\chi = \gamma_\theta = \gamma_\omega = 0$ for $\delta_k = \delta_h$ as below.

$$\frac{z^*}{\zeta^*} = \frac{Y^*}{\zeta^* K^*} = A\left\{\frac{(1-\beta)}{A\alpha}\left[\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\right]^{\beta}\left(\int_0^1 [e(i)]^{-1}di\right)^{-1}\right\}^{\frac{1-\alpha}{1-\alpha+\beta}} \quad \text{(24a)}$$

and

$$\chi^* = \left(\frac{\sigma\zeta^* - \alpha}{\sigma}\right)A\left\{\frac{(1-\beta)}{A\alpha}\left[\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\right]^{\beta}\left(\int_0^1 [e(i)]^{-1}di\right)^{-1}\right\}^{\frac{1-\alpha}{1-\alpha+\beta}}$$

$$+\frac{1}{\sigma}(\delta_k + \rho) - \delta_k \quad \text{(25a)}$$

First of all, formula (24a) implies that a more homogeneous country will have higher value of $z^* / \zeta^* = Y^* / (\zeta^* K^*)$, average product of physical capital in the final good production sector at steady state, than a more heterogeneous country. Secondly, formula (25a) shows that a more homogeneous country will also get higher value of $\chi^* = C^* / K^*$, aggregate consumption per aggregate physical capital at steady state, if the condition $\sigma > \alpha / \zeta^*$ holds. Note here that α / ζ^* is the share of aggregate output allocated to aggregate physical capital K_t at steady state from (1a) with a Cobb-Douglas production function. This result is reminiscent of Proposition 1 of Section V. Then again, this condition means that if individuals are sufficiently risk-averse with relatively high value of σ compared to α/ζ^* to care well about their remote descendants' welfare, then they will save more early in their life-time and achieve higher steady state consumption per physical capital later. These results are summarized as a proposition.

Proposition 3. *For our extended model economy with a convex education technology as per Rebelo (1991), a more homogeneous country A will achieve a higher steady state gross average product of aggregate physical capital in the final good production sector, $z^*/\zeta^* = Y^*/(\zeta^* K^*)$, than a more heterogeneous country B. Similarly, country A will achieve a higher (lower) steady state aggregate consumption per physical capital ratio, $\chi^* = C^*/K^*$, than country B if $\sigma > \alpha/\zeta^*$ $(\sigma < \alpha/\zeta^*)$.*

Following the similar procedures as in Appendix D, we can get the steady state growth rate of the economy as below.

$$\gamma_c^* = \gamma_C^* = \gamma_K^* = \gamma_H^* = \gamma_Y^* = \frac{1}{\sigma}\left(\frac{\alpha z^*}{\zeta^*} - \delta_k - \rho\right)$$

$$= \frac{1}{\sigma}\left\{ A\alpha \left\{ \frac{(1-\beta)}{A\alpha}\left[\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\right]^{\beta} \left(\int_0^1 [e(i)]^{-1} di\right)^{-1} \right\}^{\frac{1-\alpha}{1-\alpha+\beta}} - \delta_k - \rho \right\} \quad (28a)$$

We can see from (28a) that the steady state growth rates are higher for the more homogeneous country A than the more heterogeneous country B as in the main model.

In summary, we can confirm the robustness of our main model for wide range of education technology from Lucas (1988)-style linear technology to Rebelo (1991)-style convex technology with both physical and human capital in human capital production.

VIII. CONCLUSION

In our two-sector growth model with heterogeneity in individual education efficiency $e(i)$, we get a positive but less than *one-to-one* relation between $1-\theta_t(i)$, the share of agent i's own human capital invested in education, and $e(i)$. Hence, the more homogeneous country invests, in aggregate, proportionately more in education than the more heterogeneous country via Jensen's inequality theorem and thus enhances the former's long term growth potential. As a result, the more homogeneous country achieves, in aggregate, higher steady state output per physical capital ($z^* = Y^* / K^*$), higher steady state consumption per physical capital ($\chi^* = C^* / K^*$), and higher steady state growth rates (γ^*) of consumption and output than the more heterogeneous country. Consequently, the steady state rental rate r^* of physical capital is higher and the wage rate w^* of human capital is lower in the more homogeneous country.

The results of our base model remain robust in the extended model of a convex education technology with both physical and human capital in the human capital production. The extended model with general factor intensity in the education sector reinforces the main findings; the more homogeneous country adopts higher human capital intensity in the final good production and thereby enhances physical capital productivity, output per physical capital and economic growth rate at steady state. This outcome also leads to higher steady state consumption per physical capital if the empirically plausible condition $\sigma > \alpha / \zeta^*$ holds.

In getting the above results, we have adopted a different mechanism from other studies in the literature. First of all, we focus on differences in *abilities*, i.e. differences in education efficiencies $e(i)$, among infinitely-lived individuals rather than differences in *opportunities*, i.e. inherited wealth x_t, in an overlapping generations model of Galor and Zeira (1993, p.36). Secondly, we do not need a political process of redistribution as in Persson and Tabellini (1994) to get cross-country differences in growth rates. Only the differences in the distribution of individual education abilities suffice to get differential growth rates. Thirdly, we employ standard constant returns to scale technologies both in output production and in education, and all the dynamics and steady state solutions are explicitly identified. In summary, our model has showed a novel mechanism of explaining the effects of the inequality in innate individual abilities on the differential economic growth. As a result, we get a simple testable hypothesis that a more homogeneous country will get a higher growth rate and higher physical capital productivity than a more heterogeneous country.

We may further modify the model with overlapping generations, e.g. with agents living for two periods.[25] In the first period, agents decide how much of their time to invest in education and how much in output production depending on their educational efficiency. In the second period, agents consume what they have earned and die. There may be a group of agents whose educational efficiency is too low to invest in education at all and another group of agents whose ability is so high that they completely specialize in education. These comparative advantages in production and education among agents may work to enhance overall efficiency of the economy.[26] However, whether the actual outcome will reverse our findings or not will depend on the specific technologies and more importantly on whether the complementarity between the two groups of agents in acquiring human capital and output production will sufficiently compensate the negative effects of educational efficiency inequality on human capital accumulation, especially when Gorman aggregation does not obtain. We may also endogeneize the individual education efficiency $e(i)$ depending on human capital accumulation or education level, but this will be quite a new study. As a priority, we may rather need to implement an empirical analysis using e.g. PISA (Programme for International Student Assessment)[27] data of OECD (Organisation for Economic Co-operation and Development) to test the practical robustness of the model.

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[25] I am greatly indebted to an anonymous referee for addressing this point.

[26] ref. Grossman and Helpman (1990) for analysis of comparative advantages in R&D and manufacturing between two countries.

[27] PISA conducts internationally standardised test of 15-year olds of mainly industrialized countries every three years in the domains of reading, mathematical and scientific literacy. The first test was taken in 2000 for 43 countries and subsequently in 2003 and 2006, and the next test will be in 2009. Refer to the official website of PISA, http://www.pisa.oecd.org.

Appendix A. Consumer Agent i 's Optimization

The present-value Hamiltonian of the agent i becomes

$$J(i)=e^{-\rho t}\left(\frac{c_t(i)^{1-\sigma}-1}{1-\sigma}\right)+\lambda_t(i)\{r_t k_t(i)+w_t\theta_t(i)h_t(i)-c_t(i)-q_t(i)\}$$
$$+\nu_t(i)\{q_t(i)-\delta_k k_t(i)\}+\mu_t(i)\{e(i)[1-\theta_t(i)]-\delta_h\}h_t(i) \tag{A1}$$

where $\lambda_t(t)$ is the Lagrange multiplier associated with the budget constraint and $\nu_t(i)$ and $\mu_t(i)$ are costate variables for physical and human capital changes for agent i. First order conditions are obtained as follows. Control variables are $c_t(i)$, $q_t(i)$ and $\theta_t(i)$, and state variables are $k_t(i)$ and $h_t(i)$.

$$\frac{\partial J(i)}{\partial c_t(i)}=e^{-\rho t}c_t(i)^{-\sigma}-\lambda_t(i)=0, \tag{A2}$$
$$\frac{\partial J(i)}{\partial q_t(i)}=-\lambda_t(i)+\nu_t(i)=0, \tag{A3}$$
$$\frac{\partial J(i)}{\partial \theta_t(i)}=\lambda_t(i)w_t h_t(i)-\mu_t(i)e(i)h_t(i)=0, \tag{A4}$$
$$\frac{\partial J(i)}{\partial k_t(i)}=\lambda_t(i)r_t-\nu_t(i)\delta_k=-\dot{\nu}_t(i), \tag{A5}$$
$$\frac{\partial J(i)}{\partial h_t(i)}=\lambda_t(i)w_t\theta_t(i)+\mu_t(i)\{e(i)[1-\theta_t(i)]-\delta_h\}=-\dot{\mu}_t(i). \tag{A6}$$

Taking log of (A2) and differentiating with respect to time,

$$\frac{\dot{\lambda}_t(i)}{\lambda_t(i)}=-\rho-\sigma\left(\frac{\dot{c}_t(i)}{c_t(i)}\right).$$

From (A3) and (A5),

$$\frac{\dot{\lambda}_t(i)}{\lambda_t(i)}=\frac{\dot{\nu}_t(i)}{\nu_t(i)}=\delta_k-r_t. \tag{A7}$$

Hence, we get the growth rate of consumption of agent i as follows.

$$\gamma_c(i) = \frac{\dot{c}_t(i)}{c_t(i)} = \frac{1}{\sigma}(r_t - \delta_k - \rho) \equiv \gamma_c . \tag{A8}$$

Consequently, we get the same growth rate of aggregate consumption as of individual consumption.

$$\gamma_C = \frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}(r_t - \delta_k - \rho) = \gamma_c . \tag{A9}$$

Furthermore, we get from (A4) and (A6) for $h_t(i) \neq 0$,

$$\frac{\dot{\mu}_t(i)}{\mu_t(i)} = -e(i) + \delta_h . \tag{A10}$$

Appendix B. Derivation of $\gamma_\theta = \frac{\dot{\theta}_t}{\theta_t}$

We get $p_t(i) \equiv \mu_t(i)/\lambda_t(i)$ from (A4) for $h_t(i) \neq 0$ with $w_t = A(1-\alpha)\theta_t^{-\alpha}\omega_t^{\alpha}$ from (8) as follows.

$$p_t(i) \equiv \frac{\mu_t(i)}{\lambda_t(i)} = \frac{w_t}{e(i)} = \frac{A(1-\alpha)\theta_t^{-\alpha}\omega_t^{\alpha}}{e(i)} . \tag{B1}$$

We integrate this over $i \in [0,1]$ first.

$$\int_0^1 p_t(i)di = \int_0^1 \frac{\mu_t(i)}{\lambda_t(i)} di = \int_0^1 \frac{A(1-\alpha)\theta_t^{-\alpha}\omega_t^{\alpha}}{e(i)} di .$$

Let's differentiate this with respect to time.

$$
\begin{aligned}
\int_0^1 \dot{p}_t(i)di &= \int_0^1 \left\{ \frac{\dot{\mu}_t(i)\lambda_t(i) - \mu_t(i)\dot{\lambda}_t(i)}{\left[\lambda_t(i)\right]^2} \right\} di \\
&= \int_0^1 \left\{ \frac{\dot{\mu}_t(i)}{\mu_t(i)} p_t(i) - p_t(i) \frac{\dot{\lambda}_t(i)}{\lambda_t(i)} \right\} di \\
&= \int_0^1 p_t(i) \left\{ \frac{\dot{\mu}_t(i)}{\mu_t(i)} - \frac{\dot{\lambda}_t(i)}{\lambda_t(i)} \right\} di \\
&= \int_0^1 \frac{A(1-\alpha)}{e(i)} \left\{ -\alpha\theta_t^{-\alpha-1}\omega_t^{\alpha}\dot{\theta}_t + \alpha\theta_t^{-\alpha}\omega_t^{\alpha-1}\dot{\omega}_t \right\} di \\
&= \int_0^1 \frac{A(1-\alpha)}{e(i)} \alpha\theta_t^{-\alpha}\omega_t^{\alpha}(\gamma_\omega - \gamma_\theta)di \\
&= \int_0^1 \alpha p_t(i)(\gamma_\omega - \gamma_\theta)di.
\end{aligned}
$$

Rearranging this, we get

$$\int_0^1 p_t(i) \left\{ \frac{\dot{\mu}_t(i)}{\mu_t(i)} - \frac{\dot{\lambda}_t(i)}{\lambda_t(i)} - \alpha(\gamma_\omega - \gamma_\theta) \right\} di = 0 .$$

Substituting here for $\dot{\mu}_t(i)/\mu_t(i)$ from (A10) and for $\dot{\lambda}_t(i)/\lambda_t(i)$ from (A7),

$$\int_0^1 p_t(i)\left\{-e(i)+\delta_h-\delta_k+r_t-\alpha(\gamma_\omega-\gamma_\theta)\right\} di = 0 .$$

Rearranging this with $p_t(i)e(i) = w_t$ using (B1),

$$\int_0^1 \left[-w_t + p_t(i)\left\{\delta_h-\delta_k+r_t-\alpha(\gamma_\omega-\gamma_\theta)\right\}\right] di = 0 .$$

Solving this for γ_θ,

$$\gamma_\theta = \frac{1}{\alpha}\left\{ \left(\int_0^1 p_t(i)di\right)^{-1} w_t - r_t - \delta_h + \delta_k \right\} + \gamma_\omega$$

$$= \frac{1}{\alpha}\left\{ \left(\int_0^1 \frac{w_t}{e(i)} di \right)^{-1} w_t - r_t - \delta_h + \delta_k \right\} + \gamma_\omega$$
$$= \frac{1}{\alpha}\left\{ \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} - r_t - \delta_h + \delta_k \right\} + \gamma_\omega .$$

Substituting here for $r_t = \alpha z_t$ from (17), we get

$$\gamma_\theta = \frac{1}{\alpha}\left\{ \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} - \alpha z_t - \delta_h + \delta_k \right\} + \gamma_\omega$$
$$= -\left\{ z_t - \frac{1}{\alpha}\left[\left(\int_0^1 [e(i)]^{-1} di \right)^{-1} - \delta_h + \delta_k \right] \right\} + \gamma_\omega . \qquad \text{(B2)}$$

Appendix C. Derivation of $\gamma_\chi = \dot{\chi}_t / \chi_t$

From the definition $\chi_t \equiv C_t / K_t$ and equations (14) and (19), we get

$$\gamma_\chi = \frac{\dot{\chi}_t}{\chi_t} = \gamma_C - \gamma_K = \frac{1}{\sigma}(r_t - \delta_k - \rho) - A\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} + \chi_t + \delta_k .$$

Substituting for $r_t = A\alpha\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)}$ from (7) and rearranging,

$$\gamma_\chi = \frac{1}{\sigma}(A\alpha\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} - \delta_k - \rho) - A\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} + \chi_t + \delta_k$$
$$= \left(\frac{\alpha - \sigma}{\sigma} \right) A\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} + \chi_t - \frac{1}{\sigma}(\delta_k + \rho) + \delta_k . \qquad \text{(C1)}$$

Using the definition $z_t \equiv Y_t / K_t = A\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)}$ here again,

$$\gamma_\chi = \left(\frac{\alpha - \sigma}{\sigma} \right) z_t + \chi_t - \frac{1}{\sigma}(\delta_k + \rho) + \delta_k$$
$$= \left(\frac{\alpha - \sigma}{\sigma} \right)(z_t - z^*) + (\chi_t - \chi^*) , \qquad \text{(C2)}$$

where $z^* = \frac{1}{\alpha}\left[\left(\int_0^1 [e(i)]^{-1} di\right)^{-1} - \delta_h + \delta_k\right]$ and $\chi^* = \left(\frac{\sigma - \alpha}{\sigma}\right) z^* + \frac{1}{\sigma}(\delta_k + \rho) - \delta_k$ are steady state values of z_t and χ_t.

Appendix D. Balanced Growth Path

We show here the existence of a balanced growth path.[28]

Theorem 1. (Existence of a balanced growth path): *For the economy of Sections II and III, there exists a balanced growth path along which* K_t, H_t, Y_t, $c_t(i)$ *and* C_t *all grow at the same constant rate. That is,* $\frac{\dot{K}_t}{K_t} = \frac{\dot{H}_t}{H_t} = \frac{\dot{Y}_t}{Y_t} = \frac{\dot{c}_t(i)}{c_t(i)} = \frac{\dot{C}_t}{C_t} = \kappa$ *along a balanced growth path where* κ *is a positive constant. However, the growth rate of individual human capital holdings,* $\gamma_{h(i)} = \frac{\dot{h}_t(i)}{h_t(i)}$, *is different across agents and different from that of* H_t *even along a balanced growth path.*

Proof; Let $\kappa = \dot{c}_t(i) / c_t(i)$ along a balanced growth path. We know from (14) that the aggregate consumption growth rate is the same as the individual consumption growth rate, so we have $\dot{C}_t / C_t = \kappa$ along a balanced growth path. From (A2), we get $\dot{\lambda}_t(i) / \lambda_t(i) = -\rho - \sigma\kappa$ along a balanced growth path. Then, from (A7), we must have along a balanced growth path

$$r_t = \rho + \sigma\kappa + \delta_k . \tag{D1}$$

Meanwhile, dividing the production function (1) by K_t, we get

$$z_t = \frac{Y_t}{K_t} = \frac{C_t}{K_t} + \frac{\dot{K}_t}{K_t} + \delta_k . \tag{D2}$$

We know $z_t = r_t / \alpha$ from (17), so using (D1) and (D2)

28 ref. Lucas (1988), p.9. We use the term *balanced growth path* interchangeably with *steady state* throughout this paper. The term balanced growth path may be more suitable for our endogenous growth model though.

$$z_t = \frac{Y_t}{K_t} = \frac{C_t}{K_t} + \frac{\dot{K}_t}{K_t} + \delta_k = \frac{\rho + \sigma\kappa + \delta_k}{\alpha}. \tag{D3}$$

By definition, $\dot{K}_t / K_t$ is constant along a balanced growth path and the term $(\rho + \sigma\kappa + \delta_k)/\alpha$ is also constant, so we must have $Y_t / K_t =$ constant and $C_t / K_t =$ constant as well. Hence, we get along a balanced growth path

$$\gamma_Y^{\,*} = \left(\frac{\dot{Y}_t}{Y_t}\right)^* = \left(\frac{\dot{K}_t}{K_t}\right)^* = \gamma_K^{\,*} = \left(\frac{\dot{C}_t}{C_t}\right)^* = \kappa. \qquad \text{(D4)[29]}$$

Log-differentiating (5) with respect to time using (D1), we get $(1-\alpha)\dot{K}_t / K_t = (1-\alpha)\dot{H}_t / H_t$ with θ_t = constant, so we have the following growth rate of aggregate human capital along a balanced growth path.

$$\gamma_H^{\,*} = \left(\frac{\dot{H}_t}{H_t}\right)^* = \left(\frac{\dot{K}_t}{K_t}\right)^* = \kappa. \tag{D5}$$

For the individual human capital growth rate, we have from (11) $\gamma_{h(i)} = \dot{h}_t(i) / h_t(i) = e(i)[1-\theta_t(i)] - \delta_h$. We have different education efficiency $e(i)$ for each agent i and we also have a positive relation between $e(i)$ and $[1-\theta_t(i)]$, the share of human capital invested in education, at equilibrium via equation (15) of Section III, so $\gamma_{h(i)}$ will be different across agents $i \in [0,1]$. This should hold along a balanced growth path as well. Individual human capital growth rate is clearly different from that of aggregate human capital, $\gamma_H = \int_0^1 \left[\left\{e(i)[1-\theta_t(i)] - \delta_h\right\}\left(h_t(i)/H_t\right)\right] di$ except for a coincidental equality of $\gamma_{h(i)}$ with γ_H for some i. *Q.E.D.*

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[29] Superscript * denotes balanced growth path values.

Appendix E. Consumer Agent i's Optimization under Rebelo (1991)-style Convex Education Technology

Introducing Rebelo (1991)-style convex education technology, the present-value Hamiltonian of the agent i becomes

$$J(i)=e^{-\rho t}\left(\frac{c_t(i)^{1-\sigma}-1}{1-\sigma}\right)+\lambda_t(i)\{r_t\zeta_t(i)k_t(i)+w_t\theta_t(i)h_t(i)-c_t(i)-q_t(i)\}$$
$$+\nu_t(i)\{q_t(i)-\delta_k k_t(i)\}+\mu_t(i)\{e(i)[(1-\zeta_t(i))k_t(i)]^{\beta}$$
$$[(1-\theta_t(i))h_t(i)]^{1-\beta}-\delta_h h_t(i)\} \tag{E1}$$

First order conditions are obtained as follows. Control variables are $c_t(i)$, $q_t(i)$, $\zeta_t(i)$ and $\theta_t(i)$, and state variables are $k_t(i)$ and $h_t(i)$.

$$\frac{\partial J(i)}{\partial c_t(i)}=e^{-\rho t}c_t(i)^{-\sigma}-\lambda_t(i)=0\,, \tag{E2}$$
$$\frac{\partial J(i)}{\partial q_t(i)}=-\lambda_t(i)+\nu_t(i)=0\,, \tag{E3}$$
$$\frac{\partial J(i)}{\partial \zeta_t(i)}=\lambda_t(i)r_t k_t(i)-\beta\mu_t(i)e(i)(1-\zeta_t(i))^{\beta-1}(k_t(i))^{\beta}$$
$$[(1-\theta_t(i))h_t(i)]^{1-\beta}=0\,, \tag{E4}$$
$$\frac{\partial J(i)}{\partial \theta_t(i)}=\lambda_t(i)w_t h_t(i)-(1-\beta)\mu_t(i)e(i)[(1-\zeta_t(i))k_t(i)]^{\beta}$$
$$(1-\theta_t(i))^{-\beta}(h_t(i))^{1-\beta}=0\,, \tag{E5}$$
$$\frac{\partial J(i)}{\partial k_t(i)}=\lambda_t(i)r_t\zeta_t(i)-\nu_t(i)\delta_k+\beta\mu_t(i)e(i)(1-\zeta_t(i))^{\beta}(k_t(i))^{\beta-1}$$
$$[(1-\theta_t(i))h_t(i)]^{1-\beta}=-\dot{\nu}_t(i)\,, \tag{E6}$$
$$\frac{\partial J(i)}{\partial h_t(i)}=\lambda_t(i)w_t\theta_t(i)+\mu_t(i)\{(1-\beta)e(i)[(1-\zeta_t(i))k_t(i)]^{\beta}$$
$$(1-\theta_t(i))^{1-\beta}(h_t(i))^{-\beta}-\delta_h\}=-\dot{\mu}_t(i)\,. \tag{E7}$$

Taking log of (E2) and differentiating with respect to time,

$$\frac{\dot{\lambda}_t(i)}{\lambda_t(i)} = -\rho - \sigma\left(\frac{\dot{c}_t(i)}{c_t(i)}\right).$$

From (E3), (E4) and (E6) for $k_t(i) \neq 0$,

$$\frac{\dot{\lambda}_t(i)}{\lambda_t(i)} = \frac{\dot{\nu}_t(i)}{\nu_t(i)} = \delta_k - r_t\zeta_t(i) - (1-\zeta_t(i))r_t = \delta_k - r_t. \tag{E8}$$

Hence, we get the growth rate of consumption of agent i as follows.

$$\gamma_c(i) = \frac{\dot{c}_t(i)}{c_t(i)} = \frac{1}{\sigma}(r_t - \delta_k - \rho) \equiv \gamma_c. \tag{E9}$$

Consequently, we get the same growth rate of aggregate consumption as of individual consumption.

$$\gamma_C = \frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}(r_t - \delta_k - \rho) = \gamma_c.$$

Furthermore, we get from (E5) and (E7) for $h_t(i) \neq 0$,

$$\frac{\dot{\mu}_t(i)}{\mu_t(i)} = \delta_h - (1-\beta)e(i)[(1-\zeta_t(i))k_t(i)]^{\beta}[(1-\theta_t(i))h_t(i)]^{-\beta}. \tag{E10}$$

Meanwhile, dividing (E4) by (E5),

$$\frac{(1-\zeta_t(i))k_t(i)}{(1-\theta_t(i))h_t(i)} = \frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right). \tag{E11}$$

Substituting $(1-\zeta_t(i))k_t(i)/[(1-\theta_t(i))h_t(i)] = (\beta/(1-\beta))(w_t/r_t)$ from (E11) into (E10), we get

$$\frac{\dot{\mu}_t(i)}{\mu_t(i)} = \delta_h - (1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta} e(i). \tag{E12}$$

Appendix F. Solutions to the Extension of the Main Model to a Convex Education Technology

♦ **Derivation of** $\gamma_\phi = \dfrac{\dot{\phi}_t}{\phi_t}$

We define here $\phi_t \equiv \dfrac{\zeta_t K_t}{\theta_t H_t} = \dfrac{\zeta_t \omega_t}{\theta_t}$, physical capital intensity relative to human capital in the final good production sector.

Definition 4. ($\phi_t \equiv \dfrac{\zeta_t K_t}{\theta_t H_t} = \dfrac{\zeta_t \omega_t}{\theta_t}$**, physical capital intensity relative to human capital in the final good production sector):** *We define the ratio* $\phi_t \equiv \dfrac{\zeta_t K_t}{\theta_t H_t} = \dfrac{\zeta_t \omega_t}{\theta_t}$, *physical capital intensity relative to human capital in the final good y production sector, which is equivalent to the aggregate physical capital/aggregate human capital ratio invested in output production.*

$$\phi_t \equiv \frac{\zeta_t K_t}{\theta_t H_t} = \frac{\zeta_t \omega_t}{\theta_t} \tag{F1}$$ □

We derive the dynamics of ϕ_t, defining a variable $p_t(i) \equiv \mu_t(i) / \lambda_t(i)$, *shadow price of human capital in terms of final good* y *for agent* i as before. We get from (E5) of Appendix E,

$$\begin{aligned} p_t(i) &\equiv \frac{\mu_t(i)}{\lambda_t(i)} = \frac{w_t h_t(i)}{(1-\beta)e(i)[(1-\zeta_t(i))k_t(i)]^{\beta}(1-\theta_t(i))^{-\beta}(h_t(i))^{1-\beta}} \\ &= \frac{w_t}{(1-\beta)e(i)}\left[\frac{(1-\zeta_t(i))k_t(i)}{(1-\theta_t(i))h_t(i)}\right]^{-\beta} \end{aligned}$$

Substituting here for $\dfrac{(1-\zeta_t(i))k_t(i)}{(1-\theta_t(i))h_t(i)} = \dfrac{\beta}{1-\beta}\left(\dfrac{w_t}{r_t}\right)$ from (E11), we get

$$p_t(i) = \frac{w_t}{(1-\beta)e(i)}\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{-\beta}.$$

Using (7a) and (8a) here,

$$p_t(i)=\frac{w_t}{(1-\beta)e(i)}\left[\frac{\beta}{1-\beta}\frac{1-\alpha}{\alpha}\left(\frac{w_t}{A(1-\alpha)}\right)^{\frac{1}{\alpha}}\right]^{-\beta}=\psi\frac{w_t^{(\alpha-\beta)/\alpha}}{e(i)} \tag{F2}$$

where $\psi\equiv\frac{1}{1-\beta}\left[\left(A(1-\alpha)\right)^{1/\alpha}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\beta}{\beta}\right)\right]^{\beta}$ is a positive constant.

Using (8a), $w_t=A(1-\alpha)(\zeta_t/\theta_t)^{\alpha}\omega_t^{\alpha}$

$$\begin{aligned}p_t(i)&=\psi\frac{\left[A(1-\alpha)(\zeta_t/\theta_t)^{\alpha}\omega_t^{\alpha}\right]^{(\alpha-\beta)/\alpha}}{e(i)}\\&=\psi\left[A(1-\alpha)\right]^{(\alpha-\beta)/\alpha}\frac{(\zeta_t/\theta_t)^{\alpha-\beta}\omega_t^{\alpha-\beta}}{e(i)}\equiv\tilde{\psi}\frac{(\zeta_t\omega_t/\theta_t)^{\alpha-\beta}}{e(i)}\end{aligned} \tag{F3}$$

where $\tilde{\psi}\equiv[A(1-\alpha)]^{(\alpha-\beta)/\alpha}\psi=\frac{[A(1-\alpha)]^{(\alpha-\beta)/\alpha}}{1-\beta}\left[(A(1-\alpha))^{1/\alpha}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\beta}{\beta}\right)\right]^{\beta}$

$=\frac{A(1-\alpha)}{1-\beta}\left[\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\beta}{\beta}\right)\right]^{\beta}$ is a positive constant.

We integrate $p_t(i)\equiv\mu_t(i)/\lambda_t(i)$ in (F3) over $i\in[0,1]$ first.

$$\int_0^1 p_t(i)di=\int_0^1\frac{\mu_t(i)}{\lambda_t(i)}di=\int_0^1\tilde{\psi}\frac{(\zeta_t\omega_t/\theta_t)^{\alpha-\beta}}{e(i)}di\,.$$

Let's differentiate this with respect to time.

$$\begin{aligned}\int_0^1\dot{p}_t(i)di&=\int_0^1 p_t(i)\left\{\frac{\dot{\mu}_t(i)}{\mu_t(i)}-\frac{\dot{\lambda}_t(i)}{\lambda_t(i)}\right\}di\\&=\tilde{\psi}\int_0^1[e(i)]^{-1}(\alpha-\beta)\left(\frac{\zeta_t\omega_t}{\theta_t}\right)^{\alpha-\beta}\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)di\,.\end{aligned}$$

Using the value of $p_t(i)=\tilde{\psi}\left(\zeta_t\omega_t/\theta_t\right)^{\alpha-\beta}/e(i)$ from (F3),

$$\int_0^1 \dot{p}_t(i)di=\int_0^1 p_t(i)\left\{\frac{\dot{\mu}_t(i)}{\mu_t(i)}-\frac{\dot{\lambda}_t(i)}{\lambda_t(i)}\right\}di=\int_0^1(\alpha-\beta)p_t(i)\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)di\,.$$

Substituting here for $\dot{\mu}_t(i)/\mu_t(i)$ from (E12) and for $\dot{\lambda}_t(i)/\lambda_t(i)$ from (E8),

$$\int_0^1 p_t(i)\left\{\delta_h-(1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta}e(i)-\delta_k+r_t\right\}di$$
$$=\int_0^1(\alpha-\beta)p_t(i)\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)di\,.$$

Rearranging this with $p_t(i)=\tilde{\psi}\left(\zeta_t\omega_t/\theta_t\right)^{\alpha-\beta}/e(i)$ from (F3),

$$\int_0^1\frac{\tilde{\psi}(\zeta_t\omega_t/\theta_t)^{\alpha-\beta}}{e(i)}\left\{\delta_h-(1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta}e(i)-\delta_k+r_t\right\}di$$
$$=\int_0^1\frac{(\alpha-\beta)\tilde{\psi}(\zeta_t\omega_t/\theta_t)^{\alpha-\beta}}{e(i)}\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)di\,.$$

Simplifying this,

$$\int_0^1[e(i)]^{-1}\left\{\delta_h-(1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta}e(i)-\delta_k+r_t\right\}di$$
$$=\int_0^1[e(i)]^{-1}\{\delta_h-\delta_k+r_t\}di-(1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta}$$
$$=\int_0^1(\alpha-\beta)[e(i)]^{-1}\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)di\,.$$

Solving this, we get

$$\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)=\frac{1}{\alpha-\beta}(\delta_h-\delta_k+r_t)$$
$$-\frac{1-\beta}{\alpha-\beta}\left[\frac{\beta}{1-\beta}\left(\frac{w_t}{r_t}\right)\right]^{\beta}\left(\int_0^1[e(i)]^{-1}di\right)^{-1}. \quad \text{(F4)}$$

Meanwhile, we have from (7a) and (8a),

$$\frac{w_t}{r_t}=\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\zeta_t\omega_t}{\theta_t}\right). \quad \text{(F5)}$$

Substituting (7a) and (F5) into (F4),

$$\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)=\frac{1}{\alpha-\beta}\left(\delta_h-\delta_k+A\alpha\left(\frac{\zeta_t\omega_t}{\theta_t}\right)^{-(1-\alpha)}\right)$$
$$-\frac{1-\beta}{\alpha-\beta}\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\zeta_t\omega_t}{\theta_t}\right)\right]^{\beta}\left(\int_0^1[e(i)]^{-1}di\right)^{-1}. \quad \text{(F6)}$$

Rewriting this with $\phi_t\equiv\frac{\zeta_tK_t}{\theta_tH_t}=\frac{\zeta_t\omega_t}{\theta_t}$, we get

$$\gamma_\phi=\left(\frac{\dot{\zeta}_t}{\zeta_t}+\frac{\dot{\omega}_t}{\omega_t}-\frac{\dot{\theta}_t}{\theta_t}\right)=\frac{1}{\alpha-\beta}\left(\delta_h-\delta_k+A\alpha(\phi_t)^{-(1-\alpha)}\right)$$
$$-\frac{1-\beta}{\alpha-\beta}\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha}\right)\phi_t\right]^{\beta}\left(\int_0^1[e(i)]^{-1}di\right)^{-1}. \quad \text{(F7)}$$

From here, we can solve for the steady state value of ϕ_t when $\gamma_\phi=0$.

$$(\delta_h-\delta_k+A\alpha(\phi^*)^{-(1-\alpha)})=(1-\beta)\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha}\right)\phi^*\right]^{\beta}\left(\int_0^1[e(i)]^{-1}di\right)^{-1}$$

Solving this for ϕ^* with $\delta_k=\delta_h$, we get

$$\phi^* = \left\{ \frac{(1-\beta)}{A\alpha} \left[\frac{\beta}{1-\beta} \left(\frac{1-\alpha}{\alpha} \right) \right]^\beta \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} \right\}^{\frac{-1}{1-\alpha+\beta}} .$$

♦ **Derivation of γ_z and γ_χ**

Aggregate physical capital growth rate is obtained from (1a) by dividing both sides by K_t with the definition of $\omega_t \equiv K_t / H_t$, $\chi_t = C_t / K_t$, and (17a).

$$\gamma_K = \frac{\dot{K}_t}{K_t} = A\zeta_t^{\alpha} \theta_t^{1-\alpha} \omega_t^{-(1-\alpha)} - \chi_t - \delta_k = z_t - \chi_t - \delta_k . \quad \text{(F8)}$$

As a second step, we find the main dynamics of the system. From (17a),

$$\gamma_z = \frac{\dot{z}_t}{z_t} = \alpha\gamma_\zeta + (1-\alpha)(\gamma_\theta - \gamma_\omega) = \alpha(\gamma_\zeta + \gamma_\omega - \gamma_\theta) + \gamma_\theta - \gamma_\omega .$$

Using (F6),

$$\gamma_z = \frac{\alpha}{\alpha-\beta} \left(\delta_h - \delta_k + A\alpha \left(\frac{\zeta_t \omega_t}{\theta_t} \right)^{-(1-\alpha)} \right)$$
$$- \frac{\alpha(1-\beta)}{\alpha-\beta} \left[\frac{\beta}{1-\beta} \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{\zeta_t \omega_t}{\theta_t} \right) \right]^\beta \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} + \gamma_\theta - \gamma_\omega . \quad \text{(F9)}$$

We have from (7a) and (8a),

$$\phi_t = \frac{\zeta_t \omega_t}{\theta_t} = \left(\frac{r_t}{A\alpha} \right)^{\frac{-1}{1-\alpha}} = \left(\frac{w_t}{A(1-\alpha)} \right)^{\frac{1}{\alpha}} .$$

Using $r_t = \alpha z_t / \zeta_t$ from (17a),

$$\phi_t = \frac{\zeta_t \omega_t}{\theta_t} = \left(\frac{z_t}{A\zeta_t} \right)^{\frac{-1}{1-\alpha}}.$$

Then, rewriting (F9),

$$\gamma_z = \frac{\alpha}{\alpha-\beta}\left(\delta_h - \delta_k + A\alpha\left(\frac{z_t}{A\zeta_t} \right) \right)$$
$$-\frac{\alpha(1-\beta)}{\alpha-\beta}\left[\frac{\beta}{1-\beta}\left(\frac{1-\alpha}{\alpha} \right)\left(\frac{z_t}{A\zeta_t} \right)^{\frac{-1}{1-\alpha}} \right]^{\beta} \left(\int_0^1 [e(i)]^{-1} di \right)^{-1} + \gamma_\theta - \gamma_\omega . \quad \text{(22a)}$$

From the definition $\chi_t \equiv C_t / K_t$ and equations (14) and (F8), we get

$$\gamma_\chi = \frac{\dot{\chi}_t}{\chi_t} = \gamma_C - \gamma_K = \frac{1}{\sigma}(r_t - \delta_k - \rho) - A\zeta_t^{\alpha}\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} + \chi_t + \delta_k .$$

Substituting here for $r_t = A\alpha\left(\zeta_t\omega_t / \theta_t\right)^{-(1-\alpha)}$ from (7a) and rearranging,

$$\begin{aligned}\gamma_\chi &= \frac{1}{\sigma}(A\alpha(\zeta_t\omega_t/\theta_t)^{-(1-\alpha)} - \delta_k - \rho) - A\zeta_t^{\alpha}\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)} + \chi_t + \delta_k \\ &= \left(\frac{\alpha - \sigma\zeta_t}{\sigma} \right) A(\zeta_t\omega_t/\theta_t)^{-(1-\alpha)} + \chi_t - \frac{1}{\sigma}(\delta_k + \rho) + \delta_k . \qquad \text{(F10)}\end{aligned}$$

Using the definition $z_t \equiv \frac{Y_t}{K_t} = A\zeta_t^{\alpha}\theta_t^{1-\alpha}\omega_t^{-(1-\alpha)}$ from (17a) here again,

$$\gamma_\chi = \left(\frac{\alpha - \sigma\zeta_t}{\sigma\zeta_t} \right) z_t + \chi_t - \frac{1}{\sigma}(\delta_k + \rho) + \delta_k . \qquad \text{(23a)}$$

We can solve (22a) and (23a) for the steady state values of $z^* / \zeta^* = Y^* / (\zeta^* K^*)$ and $\chi^* = C^* / K^*$ with $\gamma_z = \gamma_\chi = \gamma_\theta = \gamma_\omega = 0$ for $\delta_k = \delta_h$ as below.

$$\frac{z^*}{\zeta^*} = \frac{Y^*}{\zeta^* K^*} = A\left\{\frac{(1-\beta)}{A\alpha}\left[\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\right]^{\beta}\left(\int_0^1 [e(i)]^{-1}\,di\right)^{-1}\right\}^{\frac{1-\alpha}{1-\alpha+\beta}}$$

and

$$\chi^* = \left(\frac{\sigma\zeta^* - \alpha}{\sigma}\right) A\left\{\frac{(1-\beta)}{A\alpha}\left[\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\alpha}{\alpha}\right)\right]^{\beta}\left(\int_0^1 [e(i)]^{-1}\,di\right)^{-1}\right\}^{\frac{1-\alpha}{1-\alpha+\beta}}$$
$$+\frac{1}{\sigma}(\delta_k + \rho) - \delta_k\,.$$

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