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SATIATION AND EQUILIBRIUM IN UNBOUNDED EXCHANGE ECONOMIES*

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We show the existence of equilibrium in an economy where preferences may be non-ordered, satiated, and consumption sets need not be bounded from below. Satiation is allowed to occur inside the set of attainable and individually rational allocations. The paper provides a unified approach to the classical general equilibrium theory with non-ordered preferences, the arbitrage-based literature with unbounded choice sets, and the asset pricing models with satiated preferences.

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I. INTRODUCTION

Shafer (1976) proves the existence of a Walrasian equilibrium in economies where agents' preferences need not be either complete or transitive and also no free disposal is assumed. Although he assumes that consumption sets are compact, his proof readily covers the case where consumption sets are only bounded from below. The Shafer existence

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theorem is not directly applicable to economies where consumption (or choice) sets are not necessarily bounded from below. For example, the set of portfolios is unbounded from below in asset markets without short-selling restrictions. In this case, equilibrium may not exist.

Hart (1974) introduces interesting conditions on preferences and feasible portfolios to investigate the existence of equilibrium in asset markets without short-selling restrictions. His idea is developed into the notions of arbitrage by Hammond (1983), Page (1987),Werner (1987), Chichilnisky (1995), Dana et al. (1999), Page et al. (2000), Allouch (2002), among others. The literature shows that the absence of arbitrage is sufficient for the existence of equilibrium. The arbitrage-based equilibrium theorems are useful in addressing the existence issue with asset markets. All the above papers assume that preferences are complete and transitive[1] and do not cover the case where agents are allowed to reach satiation at attainable and individually rational allocations. In this case, equilibrium may fail to exist. In particular, no equilibrium exists if some agent reaches satiation inside the budget set at every conceivable price. Thus, the existence of equilibrium in economies with possibly satiated preferences necessitates a condition on the primitives of the economy under which satiation is kept from occurring inside the budget set at certain prices.

The purpose of this paper is to show the existence of equilibrium in economies with non-ordered and possibly satiated preferences in consumption sets that may be unbounded from below. Thus, the present paper encompasses the classical general equilibrium theory with non-ordered preferences, the arbitrage-based literature with unbounded choice sets, and asset pricing models with possibly satiated preferences into a unified framework.[2]

Satiation can occur in many interesting cases. Satiation is a rule in an economy with compact choice sets. Another example comes from asset pricing models. As indicated in Nielsen (1987, 1990) and Allingham (1991), mean-variance utility functions may have satiation in the capital asset pricing model (CAPM) without riskless assets. They also show the

[1] Allouch (2002) allows preferences to be incomplete.
[2] As discussed later, an exception is Martins-da-Rocha and Monteiro (2009).

existence of equilibrium in the classical CAPM where agents are assumed to have homogeneous expectation about the return distribution. Won et al. (2008) provide a general equilibrium model for the recent developments of the CAPM which analyze the effects of heterogeneous expectation.[3] Allouch and Le Van (2008) and Martins-da-Rocha and Monteiro (2009) examine the equilibrium existence issue with possibly satiated preferences in the case where satiation occurs inside as well as outside the set of attainable and individually rational allocations. Since the literature is limited to the case of complete and transitive preferences, it is not relevant for studying the effect of satiation on equilibrium under non-ordered preferences. This paper further extends Won et al. (2008) to the case of asset pricing models with non-ordered and possibly satiated preferences.

The literature makes interesting attempts to address the existence issue with satiable preferences by using a weaker notion of equilibrium. For example, Mas-Colell (1992) introduces a notion of 'equilibrium with slack' which allows agents to keep some positive income unused. Kajii (1996) assumes that agents are endowed with a positive amount of fiat money which does not affect agents' welfare directly. These results, however, do not apply to asset pricing models because they fail to characterize conditions under which 'weak equilibrium' notion coincides with Walrasian equilibrium.

This paper is organized as follows. In the next section, we introduce an exchange economy where preferences need not be either ordered or nonsatiated, and illustrate that no equilibrium may exist in the presence of satiation. In Section 3, we provide the main result of the paper. In particular, a new condition is introduced which ensures the existence of equilibrium with possibly satiated preferences. Some concluding remarks are collected in Section 4.

[3] For example, see Jarrow (1980).

II. MODEL

2.1. Exchange Economies

We consider an exchange $\mathcal{E} = (X_i, \succ_i, e_i)_{i \in I}$ where $I = \{1, 2, \dots, m\}$ is the set of consumers, X_i the consumption set in $\mathbb{R}^\ell$ for consumer i over which the preferences $\succ_i$ are defined, and e_i the initial endowment of consumer i. The consumption set X_i need not be bounded from below. The preferences $\succ_i$ are irreflexive and may not be ordered. For a point $x_i \in X_i$, the preferred set $P_i(x_i)$ are defined as $P_i(x_i) = \{x_i' \in X_i : x_i' \succ_i x_i\}$.

A profile $x = (x_i, \dots, x_m)$ of individual consumption bundles is called an allocation of the economy $\mathcal{E}$. The set $X = \prod_{i \in I} X_i$ contains all the allocations of $\mathcal{E}$. Let $X_0 = \sum_{i \in I} X_i$ denote the aggregate consumption set, whose generic element is denoted by x_0. In particular, we set $e_0 = \sum_{i \in I} e_i$. For each $p \in \mathbb{R}^\ell \setminus \{0\}$, the open budget set of agent i is defined as $\mathcal{B}_i(p) = \{x_i \in X_i : p \cdot x_i < p \cdot e_i\}$. For a set S in $\mathbb{R}^\ell$, let $c\ell S$ (*intS*, *coS*, respectively) denote the *closure* (*interior, convex hull*, respectively) of *S*.

DEFINITION 2.1.: A *competitive equilibrium* for $\mathcal{E}$ is a pair $(p, x) \in (\mathbb{R}^\ell \setminus \{0\}) \times X$ such that

(i) $x_i \in c\ell\mathcal{B}_i(p)$ for all $i \in I$,

(ii) $P_i(x_i) \cap c\ell\mathcal{B}_i(p) = \emptyset$ for all $i \in I$,

(iii) $\sum_{i \in I}(x_i - e_i) = 0$.

An allocation $x \in X$ is said to be *attainable* when it satisfies the third condition of competitive equilibrium and to be *individually rational* when $e_i \notin P_i(x_i)$ for every $i \in I$. The *rationally attainable set A* of the economy $\mathcal{E}$ is the set of individually rational and attainable allocations:

$$A = \{x \in X : e_i \notin P_i(x_i), \forall i \in I \text{ and } \sum_{i \in I}(x_i - e_i) = 0\}.$$

To establish the existence of competitive equilibrium of the economy

$\mathcal{E}$, we need the following assumptions.

ASSUMPTIONS : For every $i \in I$,

A1. X_i is a closed convex set in $\mathbb{R}^\ell$.

A2. $P_i(x_i)$ is convex for every $x_i \in X_i$.

A3. P_i is lower hemicontinuous on X_i.

A4. $P_i(x_i)$ is open in X_i for every $x_i \in X_i$.

A5. $e_i \in int\, X_i$.

Assumption A1 is standard. By Assumption A2, preferences are convex. Notice that Assumption A3 is a weaker condition than the continuity of preferences. Assumption A4 implies that if $x_i' \in X_i$ is sufficiently close to an element $y_i \in X_i$ which is preferred to x_i, then $x_i' \in P_i(x_i)$. Assumption A5 indicates a strong survival condition for consumer i in goods markets. It is worth noting that the preferences need not to be either transitive or complete, and are possibly satiated.

We provide an example which motivates the study of non-ordered preferences in general equilibrium with asset markets. In particular, in this example, non-ordered preferences are a variation of a mean-variance utility function based on the idea of Ellickson (1994).

EXAMPLE 2.1.: We consider an exchange economy where there are two consumers with unbounded consumption sets $X_1 = X_2 = \mathbb{R}^2$ and endowments $e_1 = (1, 3/2)$ and $e_2 = (1, 1/2)$. They have preferences represented by utility functions:

$$u_1(a,b) = 3(a+b) - (a^2 + b^2),$$
$$u_2(a,b;\overline{a}) = [\lambda(\overline{a})a + (1/2)(3 - \lambda(\overline{a}))b] - \frac{1}{2}(a^2 + b^2),$$

where $\lambda(\overline{a}) = 4\overline{a}/3$ and $\overline{a}$ represents the reference consumption of good 1. The function λ reflects that, as consumer 2 increases consumption of good 1, he has stronger preferences for good 1.[4] In

[4] Consumer 2's preferences emulate the example of Ellickson (pp. 316-320, 1994).

equilibrium, it must hold that $\overline{a} = a$ (see Ellickson (1994)). It is worth noting that, when λ is a constant function, consumer 2's preferences are reduced to a mean-variance utility function. Consumer 2's preferences are not transitive because his indifference curves can cross each other, e.g., $(4/3,5/6) \in P_2(1,1)$, $(1,1) \in P_2(3/4,1)$, but $(4/3,5/6) \notin P_2(3/4,1)$. Consequently, this example is inconsistent with the standard CAPM.

It is easy to check that consumer1 is satiated at $s_1 = (3/2,3/2)$ while consumer 2 is satiated at $s_2 = (\lambda(\overline{a}), (3-\lambda(\overline{a}))/2)$. One can show that the economy satisfies A.1 through A.5 and has a competitive equilibrium (p^*, x^*) with $p^* = (1/2,1/2)$ and $x^* = (x_1^*, x_2^*) = ((5/4,5/4),(3/4,3/4))$. □

2.2. Sustainable Satiation

To handle the difficulty with satiation, for each $x \in A$, we define two index sets: $I(x) = \{i \in I : P_i(x_i) \neq \emptyset\}$ and $I^s(x) = I \setminus I(x)$. That is, $I^s(x)$ denotes the set of agents who are satiated at the allocation $x \in A$ and $I(x)$ the set of agents who are not satiated at x. For each $i \in I$, let $S_i = \{x_i \in X_i : P_i(x_i) = \emptyset\}$ denote the set of his satiation consumption bundles. The following lemma shows that each S_i is closed.

LEMMA 2.1.: The set S_i is closed for each $i \in I$.

PROOF : It is clear that

$$X_i \setminus S_i = \{x_i \in X_i : P_i(x_i) \neq \emptyset\} = \{x_i \in X_i : P_i(x_i) \cap X_i \neq \emptyset\} .$$

Since P_i is lower hemicontinuous, $X_i \setminus S_i$ is open and therefore S_i is closed. ■

As exemplified below, the presence of satiation consumption bundles may cause the non-existence of equilibrium. To ensure the existence of equilibrium with satiable preferences, we need to impose certain restrictions on the primitives of the economy. For a closed convex set $Z \subset \mathbb{R}^\ell$, let $N_Z(z)$ denote the Clarke normal cone of Z at $z \in Z$, i.e., $N_Z(z) = \{v \in \mathbb{R}^\ell : v \cdot (z' - z) \leq 0, \forall z' \in Z\}$.

DEFINITION 2.2.: The exchange economy $\mathcal{E}$ is said to admit *sustainable satiation* (SS) if each $x \in A$ with $I^s(x) \neq \emptyset$ has the property that $v \in \bigcap_{i \in I(x)}[-N_{c\ell P_i(x_i)}(x_i)]$ implies $v \cdot (x_j - e_j) \geq 0, \forall j \in I^s(x)$.

The SS condition implies that, for each $x \in A$ with $I^s(x) \neq \emptyset$, if there exists $p \in \mathbb{R}^\ell$ such that $p \cdot x_i' \geq p \cdot x_i$ whenever $x_i' \in c\ell P_i(x_i)$ for all $i \in I(x)$, it holds that $x_j \notin \mathcal{B}_j(p)$ for all $j \in I^s(x)$. In other words, if $p \in \mathbb{R}^\ell$ supports x_i for every $i \in I(x)$, the price p keeps the satiation consumption x_j at least as valuable as e_j for every $j \in I^s(x)$. The following example shows that the SS condition may not be dispensed with for the existence of equilibrium.

EXAMPLE 2.2.: We consider an exchange economy where there are two consumers with $X_1 = X_2 = \mathbb{R}^2$ and with $e_1 = (1/2, 1)$ and $e_2 = (3/2, 1)$. They have preferences represented by a utility function

$$u_1(a,b) = 3(a+b) - (a^2 + b^2),$$
$$u_2(a,b) = (a+b) - \frac{1}{2}(a^2 + b^2).$$

It is easy to check that consumer 1 is satiated at $s_1 = (3/2, 3/2)$ while consumer 2 is satiated at $s_2 = (1,1)$. An equilibrium price would be a price supporting some efficient allocation. In this economy, there is a unique normalized price $p = (1/2, 1/2)$ that supports the efficient allocations. The economy has no equilibrium because e_2 is costlier at p than any consumption bundle of agent 2 at the efficient allocations. In particular, the economy does not satisfy the SS condition.

Notice that when the previous endowments are replaced by $(e_1, e_2) = ((1, 3/2), (1, 1/2))$, the economy satisfies the SS condition and has a competitive equilibrium (p^*, x^*) with $p^* = (1/2, 1/2)$ and $x^* = (x_1^*, x_2^*) = ((5/4, 5/4), (3/4, 3/4))$. □

III. EXISTENCE OF EQUILIBRIUM

The notions of arbitrage prevail in the literature which investigates the existence of equilibrium in economies with unbounded-from-below choice sets. The arbitrage-based literature basically relies upon the assumption that preferences are transitive and not satiated. Thus, the notions of arbitrage are not directly relevant to equilibrium analysis with non-transitive and possibly satiated preferences.

Nonetheless, the arbitrage-based literature (Dana et al. (1999) and Allouch (2002), among others) provides a stepping stone for the new condition to be presented below. As mentioned in Allouch (2002), the *no strong unbounded arbitrage* (NSUA) condition of Dana et al. (1999) is necessary and sufficient for the utility set to be compact when preferences are complete, transitive, and non-satiated. Allouch (2002) introduces the *compactness with partial preorder* (CPP) condition which generalizes the NSUA condition to the case with incomplete and transitive preferences. The CPP condition can be further extended to the case where preferences may be non-transitive and satiable. To do this, as in Allouch (2002), we introduce the *augmented preference correspondence* $\hat{P}_i : X_i \to 2^{X_i}$ for each $i \in I$ by

$$\hat{P}_i(x_i) = co(P_i(x_i) \cup \{x_i\}) \setminus \{x_i\}.$$

It is obvious that $P_i(x_i) \subset \hat{P}_i(x_i)$, $\forall x_i \in X_i$, $\hat{P}_i$ is convex-valued, and under Assumption A3, $\hat{P}_i$ is lower hemicontinuous. Moreover, $P_i(x_i) \neq \emptyset$ if and only if $\hat{P}_i(x_i) \neq \emptyset$.

DEFINITION 3.1.: The exchange economy $\mathcal{E}$ satisfies *weak bounded arbitrage with satiation* (*WBAS*) if there exists an increasing sequence of compact rectangles $\{K_n\}$ in $\mathbb{R}^\ell$ with $\mathbb{R}^\ell \subset \bigcup_{n=1}^{\infty} K_n$ such that every sequence $\{x^n\}$ in A allows a subsequence $\{x^{n_k}\}$ and a sequence $\{y^{n_k}\}$ in X which satisfy

(W1) $y^{n_k} \to y \in A$,

(W2) For each n_k and each $i \in I(x^{n_k})$, $[\hat{P}_i(y_i^{n_k}) \cap K_{n_k}] \subset \hat{P}_i(x_i^{n_k})$, and

(W3) $\sum_{i \in I^s(y^{n_k})} (y_i^{n_k} - e_i) \in \sum_{i \in I(x^{n_k})} con[\hat{P}_i(x_i^{n_k}) \cap K_{n_k} - \{e_i\}]$ for every

n_k, whenever $\sum_{i \in I} \| x_i^{n_k} \| \to \infty$.[5]

The conditions (W1) and (W2) are a generalization of the NSUA condition of Dana et al. (1999) and the CPP condition of Allouch (2002). The condition (W3) is newly introduced to avoid difficulties arising from making arbitrage arguments in the case with both unbounded consumption sets and satiable preferences.

Note that when A is bounded, (W1)-(W3) of the WBAS condition trivially hold true.[6] Thus, the WBAS condition imposes restrictions on the economy when A is unbounded. Moreover, (W3) will hold vacuously when preferences are globally insatiable for all $i \in I$. Like the CPP condition, the WBAS condition requires that each sequence in A have a subsequence P_i-dominated by a sequence of allocations convergent to an attainable allocation. This condition will be used in handling the complications with unbounded allocations in A. Clearly, the CPP condition coincides with the WBAS condition when preferences are insatiable and transitive.

Allouch and Le Van (2008) and Martins-da-Rocha and Monteiro (2009) are also concerned about the effect of satiation on equilibrium. They assume that if an agent is satiated at allocations in A, then he must have satiation outside A as well. A prominent difference between the WBAS conditions and their assumptions lies in the fact that the former allows agents to have satiation only inside A. As mentioned in Won et al. (2008), this is a definite advantage of the WBAS condition in dealing with the capital asset pricing model without riskless assets. Further remarks are in order.

REMARK 3.1: To go further in comparing the WBAS condition to the assumptions of Allouch and Le Van (2008) and Martins-da-Rocha and Monteiro (2009), we consider the case that P_i finds satiation points both

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[5] For a set S in a vector space, $con(S)$ denotes the cone generated by the set S, i.e., $con(S) = \bigcup_{\lambda \geq 0} \lambda S$.

[6] This fact can be confirmed as follows. Since A is bounded, any sequence $\{x^n\}$ in A is bounded. The sequence has a convergent subsequence $\{x^{n_k}\}$. Then we have only to set $y^{n_k} = x^{n_k}$ for all n_k.

inside and outside *A*. In this case, P_i can be transformed into preferences which has satiation only outside *A*. To this, for each $i \in I$, let A_i denote the projection of *A* onto X_i. For each $i \in I$, we define the correspondence $Q_i : X_i \to 2^{X_i}$ such that for each $i \in I$ with $S_i \subset A_i$, $Q_i = P_i$, and for each $i \in I$ with $S_i \setminus A_i \neq \emptyset$, we pick a point $s_i \in S_i \setminus A_i$ and define

$$Q_i(x_i) = \begin{cases} P_i(x_i), & \text{if } x_i \in X_i \setminus S_i, \\ \{s_i\}, & \text{if } x_i \in S_i \setminus \{s_i\}, \\ \emptyset, & \text{if } x_i = s_i. \end{cases}$$

The correspondence Q_i is the same as P_i except that it has satiation only at the point s_i outside A_i. We claim that Q_i is lower hemicontinuous. By Assumption A3, Q_i is trivially lower hemicontinuous for each $i \in I$ with $S_i \subset A_i$. To deal with the other case, we choose an open set *V* in X_i. Then it holds that

$$\begin{aligned} &\{y_i \in X_i : Q_i(y_i) \cap V \neq \emptyset\} \\ &= \{y_i \in X_i \setminus S_i : Q_i(y_i) \cap V \neq \emptyset\} \cup \{y_i \in S_i : Q_i(y_i) \cap V \neq \emptyset\} \\ &= \{y_i \in X_i \setminus S_i : P_i(y_i) \cap V \neq \emptyset\} \cup \{y_i \in S_i \setminus \{s_i\} : \{s_i\} \cap V \neq \emptyset\}. \end{aligned}$$

This yields

$$\begin{aligned} &\{y_i \in X_i : Q_i(y_i) \cap V \neq \emptyset\} \\ &= \begin{cases} \{y_i \in X_i : P_i(y_i) \cap V \neq \emptyset\} \setminus S_i, & \text{if } s_i \notin V. \\ (\{y_i \in X_i : P_i(y_i) \cap V \neq \emptyset\} \setminus S_i) \cup (S_i \setminus \{s_i\}), & \text{if } s_i \in V. \end{cases} \end{aligned}$$

The result implies that $\{y_i \in X_i : Q_i(y_i) \cap V \neq \emptyset\}$ is open in X_i and therefore, Q_i is lower hemicontinuous in X_i for each $i \in I$ with $S_i \setminus A_i \neq \emptyset$. Thus, Q_i is lower hemicontinuous for all $i \in I$.

To discuss the results of Allouch and Le Van (2008) and Martins-da-Rocha and Monteiro (2009) from the viewpoint of the current paper, let $\mathcal{E}_Q$ denote the economy $\mathcal{E}$ where each P_i is replaced by Q_i. Clearly, an equilibrium of the economy $\mathcal{E}_Q$ is an equilibrium of the economy $\mathcal{E}$.

Suppose that *A* is bounded. Then as remarked above, (W1) and (W2) of the WBAS condition hold trivially for $\mathcal{E}_Q$. Since satiation occurs only outside *A*, the SS condition and (W3) of the WBAS condition vacuously hold for the economy $\mathcal{E}_Q$. Thus, if *A* is bounded, $\mathcal{E}_Q$ satisfies the SS and WBAS conditions in a trivial way. In this case, $\mathcal{E}_Q$ also satisfies Assumptions A1-A5. By the main theorem below, $\mathcal{E}_Q$, and thus $\mathcal{E}$, have an equilibrium. These observations imply that the current paper encompasses the equilibrium existence theorem of Allouch and Le Van (2008) as a special case. On the other hand, *A* may be unbounded in Martins-da-Rocha and Monteiro (2009). In this case, (W1) and (W2) of the WBAS condition do not imply the SCU condition of Martins-da-Rocha and Monteiro (2009), and vice versa. □

Now we present the main consequence of the paper.

THEOREM 3.1.: Suppose that the exchange economy $\mathcal{E}$ satisfies Assumptions A1-A5. Then under the SS and WBAS conditions, $\mathcal{E}$ has a competitive equilibrium.

PROOF : Let $\{K_n\}$ be an increasing sequence of compact rectangles which satisfy the WBAS condition. Without loss of generality, we can take a sufficiently large K_1 such that $\{e_1,\dots,e_m\} \subset int K_1$. We set $X_i^n = X_i \cap K_n$ and $X^n = \prod_{i \in I} X_i^n$, and define a correspondence $\hat{P}_i^n : X_i^n \to 2^{X_i^n}$ by $\hat{P}_i^n(x_i) = \hat{P}_i(x_i) \cap K_n$ for all $x_i \in X_i$. Notice that the correspondence $\hat{P}_i^n$ is lower hemicontinuous with convex values in X_i^n. We introduce the economy $\mathcal{E}^n = (X_i^n, \hat{P}_i^n, e_i)_{i \in I}$. Let Δ denote the set $\{p \in \mathbb{R}^\ell : \| p \| \le 1\}$ where $\| \cdot \|$ indicates the Euclidean norm. For every consumer $i \in I$, the correspondence $\mathcal{B}_i^n : \Delta \to 2^{X_i^n}$ is defined by

$$\mathcal{B}_i^n(p) = \{x_i \in X_i^n : p \cdot x_i < p \cdot e_i + 1 - \| p \|\}.$$

Notice that the correspondence $\mathcal{B}_i^n : \Delta \to 2^{X_i^n}$ has open graph with convex values on Δ. Moreover, it is nonempty-valued by Assumption A5, so that the correspondence $c\ell\mathcal{B}_i^n : \Delta \to 2^{X_i^n}$, which is defined by $c\ell\mathcal{B}_i^n(p) = c\ell[\mathcal{B}_i^n(p)]$, is upper hemicontinuous with nonempty, compact and convex values on Δ. Without loss of generality, it will be assumed from now on that $I(x) \neq \emptyset$ for all $x \in A$.[7]

CLAIM 1 : For each n, there exists a pair $(p^n, x^n) \in \Delta \times X^n$ such that $x^n \in A$ and, for every $i \in I$,

(a) $x_i^n \in c\ell \mathcal{B}_i^n(p^n)$,

(b) $\hat{P}_i^n(x_i^n) \cap \mathcal{B}_i^n(p^n) = \emptyset$.

PROOF : For each $x \in X$, we set $z := \sum_{i \in I} z_i$ where $z_i = x_i - e_i$. For each $(p, x) \in \Delta \times X^n$, let us define correspondences $\varphi_i^n : \Delta \times X^n \to 2^{\Delta \times X^n}$ and $A_i^n : \Delta \times X^n \to 2^{\Delta \times X^n}$ such that for each $(p, x) \in \Delta \times X^n$,

$$\varphi_i^n(p, x) = \hat{P}_i^n(x_i), \forall i \in I,$$
$$\varphi_0^n(p, x) = \{q \in \Delta : q \cdot z > p \cdot z\},$$
$$A_i^n(p, x) = \mathcal{B}_i^n(p), \forall i \in I,$$
$$A_0^n(p, x) = \Delta := X_0^n.$$

Then $\Gamma^n = \{(X_i^n, A_i^n, \varphi_i^n) : i \in I_0 := I \cup \{0\}\}$ is an abstract economy satisfying the conditions of Theorem A.1 in Won and Yannelis (2008). Thus, there exists a pair $(p^n, x^n) \in \Delta \times X^n$ for each n such that

(i) $x_i^n \in c\ell \mathcal{B}_i^n(p^n)$, $\forall i \in I$,

(ii) $\hat{P}_i^n(x_i^n) \cap \mathcal{B}_i^n(p^n) = \emptyset$, $\forall i \in I$,

(iii) $p^n \cdot z^n \geq p \cdot z^n, \forall p \in \Delta$, where $z^n := \sum_{i \in I} z_i^n$ with $z_i^n = x_i^n - e_i$.

What remains is to show that $x^n \in A$ for each n. To do this, we first prove that $e_i \notin P_i(x_i^n)$ for every n and every $i \in I$. Suppose otherwise, i.e., $e_i \in P_i(x_i^n)$ for some n and some $i \in I$. Then, obviously $e_i \notin \hat{P}_i^n(x_i^n)$. By ($ii$) above, we have $p^n \cdot e_i \geq p^n \cdot e_i + (1 - \| p^n \|)$, which implies $\| p^n \| = 1$. Since $e_i \in int\, X_i$, there exists $\hat{x}_i^n \in X_i^n$ such that $p^n \cdot \hat{x}_i^n < p^n \cdot e_i$ and so $\hat{x}_i^n \in \mathcal{B}_i^n(p^n)$. On the other hand, by A4, P_i is open-valued. Thus, for $\alpha \in (0,1]$ sufficiently close to 1, it holds that

[7] Suppose that $I(x) = \emptyset$ for some $x \in A$. Since $x \in A$, the set $\{x_i - e_i\}$ of vectors is linearly dependent. In this case, there exists a nonzero $p \in \mathbb{R}^\ell$. such that $p \cdot (x_i - e_i) = 0$. Recalling that x_i is a satiation consumption for each $i \in I$, we see that (p, x) is an equilibrium.

$\hat{x}_i^n(\alpha) := \alpha e_i + (1-\alpha)\hat{x}_i^n \in P_i(x_i^n) \cap K_n$ and thus $\hat{x}_i^n(\alpha) \in \hat{P}_i^n(x_i^n)$. Noting that $\hat{x}_i^n(\alpha) \in \mathcal{B}_i^n(p^n)$, we obtain $\hat{x}_i^n(\alpha) \in \hat{P}_i^n(x_i^n) \cap \mathcal{B}_i^n(p^n)$, which is a contradiction to (*ii*) above. This establishes that *x* is individually rational.

Next we need to show that $z^n = 0$ for every *n*. Suppose otherwise. Then it is inferred from (*iii*) that $p^n \cdot z^n > 0$ and $\| p^n \| = 1$. But (*i*) implies $p^n \cdot z^n \le 0$, which leads to a contradiction. Hence, we conclude that $x^n \in A$ for each *n*. □

By the WBAS condition, there exist a subsequence $\{x^{n_k}\}$ of $\{x^n\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in A$ which satisfy the conditions (W1)-(W3). Without loss of generality, we can assume that $p^{n_k} \to p^*$. Clearly, $\{\sum_{i\in I} \| x_i^{n_k} \|\}$ is either bounded or unbounded. We will begin with the case that $\sum_{i\in I} \| x_i^{n_k} \| \to \infty$ to exploit the condition (W3). The other case will be analyzed in the end of the proof.

CLAIM 2 : It holds that $p^* \cdot y_i = p^* \cdot e_i, \forall i \in I(y)$.

PROOF : We first show that $I(y) \subset I(y^{n_k}) \subset I(x^{n_k})$ for sufficiently large n_k, To show that $I(y) \subset I(y^{n_k})$, take any $i \in I(y)$, so that $P_i(y_i) \neq \emptyset$. By Lemma 2.1, the set S_i is closed. This implies that, for sufficiently large n_k, $y_i^{n_k} \notin S_i$ and therefore, $P_i(y_i^{n_k}) \neq \emptyset$, i.e., $i \in I(y^{n_k})$. Accordingly we have $I(y) \subset I(y^{n_k})$. Now to show that $I(y^{n_k}) \subset I(x^{n_k})$, take any $i \in I(y^{n_k})$, so that $P_i(y_i^{n_k}) \neq \emptyset$, which implies $\hat{P}_i(y_i^{n_k}) \neq \emptyset$. Since $y_i^{n_k} \to y_i$ and $\hat{P}_i$ is locally non-satiated at $y_i^{n_k}$, we see that $\hat{P}_i(y_i^{n_k}) \cap K_{n_k} \neq \emptyset$ for sufficiently large n_k. For such n_k, by (W2) of the WBAS condition, we have $\hat{P}_i(x_i^{n_k}) \neq \emptyset$, implying $P_i(x_i^{n_k}) \neq \emptyset$, i.e., $i \in I(x^{n_k})$. Thus we obtain $I(y^{n_k}) \subset I(x^{n_k})$. Hence, it follows that $I(y) \subset I(y^{n_k}) \subset I(x^{n_k})$ for sufficiently large n_k.

For each n_k and each $i \in I(y^{n_k})$, let $\hat{y}_i^{n_k}$ be a point in $P_i(y_i^{n_k})$. For each $\alpha \in (0,1]$, we have $\hat{y}_i^{n_k}(\alpha) := \alpha \hat{y}_i^{n_k} + (1-\alpha) y_i^{n_k} \in \hat{P}_i(y_i^{n_k})$. Observe that $y_i^{n_k} \to y_i$ implies $\hat{y}_i^{n_k}(\alpha) \in K_{n_k}$ for all α sufficiently close to 0 and sufficiently large n_k. Therefore $\hat{y}_i^{n_k}(\alpha) \in \hat{P}_i(y_i^{n_k}) \cap K_{n_k}$, which, in view of (W2) of the WBAS condition, implies that $\hat{y}_i^{n_k}(\alpha) \in \hat{P}_i(x_i^{n_k})$, and so $\hat{y}_i^{n_k}(\alpha) \in \hat{P}_i^{n_k}(x_i^{n_k})$. From (b) of Claim 1, it follows that $p^{n_k} \cdot \hat{y}_i^{n_k}(\alpha) \ge p^{n_k} \cdot e_i + (1 - \| p^{n_k} \|)$. Letting $\alpha \to 0$ gives $p^{n_k} \cdot y_i^{n_k} \ge p^{n_k} \cdot e_i + (1 - \| p^{n_k} \|)$. Since $I(y) \subset I(y^{n_k})$, this relation also holds for every $i \in I(y)$, i.e., $p^{n_k} \cdot y_i^{n_k} \ge p^{n_k} \cdot e_i + (1 - \| p^{n_k} \|)$, $\forall i \in I(y)$ for sufficiently large n_k.

In the limit, we get

$$p^{*} \cdot (y_i - e_i) \geq (1 - \| p^{*} \|), \quad \forall i \in I(y). \tag{1}$$

Moreover, since $p^{n_k} \cdot (y_i^{n_k} - e_i) \geq (1 - \| p^{n_k} \|) \geq 0, \quad \forall i \in I(y_i^{n_k})$ and $I(y) \subset I(y^{n_k})$ for sufficiently large n_k, it follows that $\sum_{i \in I(y^{n_k})} p^{n_k} \cdot (y_i^{n_k} - e_i) \geq \sum_{i \in I(y)} p^{n_k} \cdot (y_i^{n_k} - e_i)$. As $n_k \to \infty$, then we have

$$\lim_{n_k \to \infty} \sum_{i \in I(y^{n_k})} p^{n_k} \cdot (y_i^{n_k} - e_i) \geq \sum_{i \in I(y)} p^{*} \cdot (y_i - e_i). \tag{2}$$

On the other hand, since $y^{n_k} \to y$ and $y \in A$, there exists a sequence $\{\varepsilon^{n_k}\}$ converging to 0 in $\mathbb{R}^{\ell}$ such that, for each n_k, $\sum_{i \in I(y^{n_k})} (e_i - y_i^{n_k}) = \sum_{i \in I^s(y^{n_k})} (y_i^{n_k} - e_i) + \varepsilon^{n_k}$. By (W3) of the WBAS condition, for each $i \in I(x^{n_k})$, there exist $\lambda_i^{n_k} \geq 0$ and $\hat{x}_i^{n_k} \in \hat{P}_i(x_i^{n_k}) \cap K_{n_k}$ for each n_k such that $\sum_{i \in I^s(y^{n_k})} (y_i^{n_k} - e_i) = \sum_{i \in I(y^{n_k})} \lambda_i^{n_k} (\hat{x}_i^{n_k} - e_i)$. This gives the relation $\sum_{i \in I(y^{n_k})} (e_i - y_i^{n_k}) = \sum_{i \in I(x^{n_k})} \lambda_i^{n_k} (\hat{x}_i^{n_k} - e_i) + \varepsilon^{n_k}$.

However, for each $i \in I(x^{n_k})$, since $\hat{x}_i^{n_k} \in \hat{P}_i^{n_k}(x_i^{n_k})$, by (b) of Claim 1, we see that $p^{n_k} \cdot \hat{x}_i^{n_k} \geq p^{n_k} \cdot e_i + (1 - \| p^{n_k} \|) \geq p^{n_k} \cdot e_i$, and therefore $\sum_{i \in I(x^{n_k})} \lambda_i^{n_k} p^{n_k} \cdot (\hat{x}_i^{n_k} - e_i) \geq 0$, which implies $\sum_{i \in I(y^{n_k})} p^{n_k} \cdot (e_i - y_i^{n_k}) \geq p^{n_k} \cdot \varepsilon^{n_k}$. Passing to the limit, we have $\lim_{n_k \to \infty} \sum_{i \in I(y^{n_k})} p^{n_k} \cdot (e_i - y_i^{n_k}) \geq 0$, i.e.,

$$\lim_{n_k \to \infty} \sum_{i \in I(y^{n_k})} p^{n_k} \cdot (y_i^{n_k} - e_i) \leq 0. \tag{3}$$

From (1), (2), and (3), it is deduced that $\| p^{*} \| = 1$ and $\sum_{i \in I(y)} p^{*} \cdot (y_i - e_i) = 0$. These results along with (1) imply that $p^{*} \cdot y_i = p^{*} \cdot e_i, \forall i \in I(y)$.

□

CLAIM 3 : It holds that $P_i(y_i) \cap \mathcal{B}_i(p^{*}) = \emptyset, \forall i \in I(y)$.

PROOF : Suppose to the contrary that, for some $i \in I(y)$, there is $\hat{x}_i \in P_i(y_i) \cap \mathcal{B}_i(p^{*})$. Observe that, since P_i is lower hemicontinuous and $\mathcal{B}_i$ has open graph, $P_i \cap \mathcal{B}_i$ is lower hemicontinuous. Since $I(y) \subset I(y^{n_k})$ for sufficiently large n_k, we can pick a sequence $\{\hat{x}_i^{n_k}\}$ in X_i converging to $\hat{x}_i$ such that $\hat{x}_i^{n_k} \in P_i(y_i^{n_k}) \cap \mathcal{B}_i(p^{n_k})$ for sufficiently large

n_k. For such n_k, we set $\hat{x}_i^{n_k}(\alpha) := \alpha\hat{x}_i^{n_k} + (1-\alpha)y_i^{n_k}$ with $\alpha \in (0,1]$, which belongs to $\hat{P}_i(y_i^{n_k}) \cap \mathcal{B}_i(p^{n_k})$ for all α sufficiently close to 1. Observe that, since $y_i^{n_k} \to y_i$ and $\hat{x}_i^{n_k} \to \hat{x}_i$, for each $\alpha \in (0,1]$, $\hat{x}_i^{n_k}(\alpha) \in K_{n_k}$ for sufficiently large n_k. Thus, for such α, $\hat{x}_i^{n_k}(\alpha) \in \hat{P}_i(y_i^{n_k}) \cap K_{n_k}$ and therefore, by (W2) of the WBAS condition, $\hat{x}_i^{n_k}(\alpha) \in \hat{P}_i(x_i^{n_k})$. Consequently, it follows that $\hat{x}_i^{n_k}(\alpha) \in \hat{P}_i^{n_k}(x_i^{n_k}) \cap \mathcal{B}_i^{n_k}(p^{n_k})$ for α sufficiently close to 1 and sufficiently large n_k. This is a contradiction to (b) of Claim 1, which proves the current claim. □

CLAIM 4 : It holds that $P_i(y_i) \cap c\ell\mathcal{B}_i(p^*) = \emptyset, \forall i \in I$.

PROOF : This is obvious for all $i \in I^s(y)$. For each $i \in I(y)$, since $P_i(y_i)$ and $\mathcal{B}_i(p^*)$ are open and $\mathcal{B}_i(p^*) \neq \emptyset$ by Assumption A5, it is also immediate from Claim 3 that $P_i(y_i) \cap c\ell\mathcal{B}_i(p^*) = \emptyset$. Hence, we have $P_i(y_i) \cap c\ell\mathcal{B}_i(p^*) = \emptyset, \forall i \in I$.

CLAIM 5 : It holds that $y_i \in c\ell\mathcal{B}_i(p^*)$ for every $i \in I$.

PROOF : By Claim 2, we have $y_i \in c\ell\mathcal{B}_i(p^*), \forall i \in I(y)$. What remains is to show that $y_i \in c\ell\mathcal{B}_i(p^*), \forall i \in I^s(y)$. By Claim 2 and Claim 4, $y_i' \in P_i(y_i)$ implies that $p^* \cdot y_i' > p^* \cdot e_i = p^* \cdot y_i$ for every $i \in I(y)$. Thus, $p^* \in \bigcap_{i \in I(y)}[-N_{c\ell P_i(y_i)}(y_i)]$ and so, by the SS condition, we obtain $p^* \cdot y_i \geq p^* \cdot e_i, \forall i \in I^s(y)$. Since $y \in A$, the claim follows from Claim 2. □

Consequently, we see that $y_i \in c\ell\mathcal{B}_i(p^*), P_i(y_i) \cap c\ell\mathcal{B}_i(p^*) = \emptyset$, and $y \in A$ for every $i \in I$. Hence, we conclude that (p^*, y) is an equilibrium of $\mathcal{E}$.

Now we turn to the case that $\{\sum_{i \in I} \| x_i^{n_k} \|\}$ is bounded. Without loss of generality, we can assume that $\{x_i^{n_k}\}$ converges to a point x_i^* for each $i \in I$. It follows from Claim 1 that $x^* \in A$. By replacing $\{y^{n_k}\}$ with $\{x^{n_k}\}$ in the arguments of Claim 2 which lead to the relation (1), we obtain $p^* \cdot (x_i^* - e_i) \geq 1 - \| p^* \|, \forall i \in I(x^*)$. On the other hand, (a) of Claim 1 implies that $p^* \cdot (x_i^* - e_i) \leq 1 - \| p^* \|, \forall i \in I$. Thus, we have $p^* \cdot x_i^* = p^* \cdot e_i + 1 - \| p^* \|$ for all $i \in I(x^*)$.

For each $p \in \Delta$, let $\hat{\mathcal{B}}_i(p)$ denote the set $\{x_i \in X_i : p \cdot x_i < p \cdot e_i +$

$1-\|p\|\}$. By replacing $\mathcal{B}_i$ by $\hat{\mathcal{B}}_i$ in the arguments of Claims 3 and 4, we see that $P_i(x_i^*)\cap c\ell\hat{\mathcal{B}}_i(p^*)=\emptyset$, $\forall i\in I$. As a consequence, for each $i\in I(x_i^*)$, $p^*\cdot x_i > p^*\cdot e_i + 1-\|p\| = p^*\cdot x_i^*$, $\forall x_i\in P_i(x_i^*)$. By the same arguments of Claim 5, we have $p^*\cdot x_i^* \geq p^*\cdot e_i$ for every $i\in I^s(x^*)$. Combining it with the fact that $x^*\in A$ and $p^*\cdot x_i^* = p^*\cdot e_i + 1-\|p^*\|$, $\forall i\in I(x^*)$, we obtain $\|p^*\|=1$ and so $P_i(x_i^*)\cap c\ell\mathcal{B}_i(p^*)=\emptyset$, $\forall i\in I$. Moreover, $p^*\cdot x_i^* = p^*\cdot e_i$, $\forall i\in I(x^*)$, with which $p^*\cdot x_i^*\geq p^*\cdot e_i$, $\forall i\in I^s(x^*)$ implies $p^*\cdot x_i = p^*\cdot e_i$, $\forall i\in I$. Thus $x_i^*\in c\ell\mathcal{B}_i(p^*)$ for every $i\in I$. Hence, we conclude that (p^*, x^*) is an equilibrium of $\mathcal{E}$.

■

IV. CONCLUSION

We have investigated the equilibrium existence problem with the economy where consumptions need not be bounded from below, preferences are not necessarily either complete or transitive, and satiation occurs. In particular, the paper includes as a special case the classical existence results with non-ordered preferences, the arbitrage-based equilibrium theory, and the recent developments of the CAPM. An important category of its applications is the recent developments of the capital asset pricing models without riskless assets. One example is Won et al. (2008) which examine the effect of heterogeneous expectations on equilibrium prices in the CAPM. Further applications can be made in the case in which preferences are subject to Knightian uncertainty[8] and mean-preserving-spread risk aversion.[9]

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[8] For example, Dow and Werlang (1992), Rigotti and Shannon (2005), and Mukerji and Tallon (2001) address risk-sharing problems under Knightian uncertainty or ambiguity aversion.

[9] For discussion on mean-preserving-spread risk aversion, see Boyle and Ma (2006).

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