

EQUILIBRIUM IN FINANCIAL MARKETS WITH MARKET FRICTIONS

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Redundant assets have complicated implications to equilibrium, when available asset markets are subject to market frictions. We show the existence of equilibrium in two-period asset markets where portfolio choices are subject to portfolio constraints. The main consequences of the paper are differentiated from the literature in two respects. First, asset markets are allowed to have a large multiplicity of alternative portfolios in equilibrium. A portfolio decomposition technique is developed to resolve the large multiplicity problem. Second, we provide a survival condition with asset markets for the existence of equilibrium. Remarkably, this cannot be dispensed within the constrained asset markets, even when the endowment of goods is in the interior of the consumption set for each agent.

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I. INTRODUCTION

Redundant assets, such as call and put options, stock index futures, and forward contracts, are introduced through costly financial innovation.¹

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¹ An asset is redundant if its return vector is linearly dependent on the return vectors of other assets.

They are useless in the case where available asset markets are free from market frictions, because they do not contribute to risk sharing. That is the reason why they are ignored in the literature of general equilibrium with incomplete markets, such as Cass(1984), Werner(1985), Geanakoplos and Polemarchakis(1987), Gottardi and Hens(1996) among others. Why then do they deserve costly production in real markets? Empirical studies show that they are frequently mispriced.² If the only source of mispricing is market friction, then it is evident that redundant assets contribute to risk sharing in a nontrivial manner.

Redundant assets have complicated implications to equilibrium and asset pricing when available asset markets are subject to market frictions. In particular, the role of redundant assets is not yet fully understood in the constrained incomplete markets where they contribute to risk sharing and create the large set of alternative portfolios with the same returns. The purpose of this paper is to establish the existence of equilibrium in two-period asset markets which are subject to portfolio constraints. The main consequences of this paper are differentiated from the literature in two respects. First, a portfolio decomposition technique is developed to deal with the large multitude of alternative portfolios, which raises the technical barrier to fixed point arguments. This innovation makes it possible to treat the unbounded set of alternative equilibrium portfolios in the same way as in unconstrained asset markets. Second, we provide a survival condition with asset markets for the existence of equilibrium. This condition is unnecessary in the unconstrained markets. Remarkably, this cannot be dispensed within the constrained asset markets, even when the endowment of goods is in the interior of the consumption set for each agent.

The literature of general equilibrium with incomplete markets does not apply to the existence of equilibrium with constrained asset markets in two respects. First, redundant assets may contribute to risk sharing, and therefore, the law of one price fails under portfolio constraints in general. Thus, the approach of Werner(1985), Geanakoplos and Polemarchakis (1987), among others does not apply to constrained asset markets. Second,

² Mispricing refers to the violation of the law of one price. For example, see Jarrow and O'Hara(1989), and Pontiff(1996).

equilibrium may not exist when the endowment of goods is in the interior of the consumption set for each agent. Gottardi and Hens(1996) address exhaustively the survival problem with incomplete markets.³ The approach of Gottardi and Hens(1996), however, is not valid here, because they start with the existence theorem of the unconstrained incomplete markets where the endowments of goods satisfy the interiority condition.⁴

The following is an example of how Cass, Siconolfi, and Villanacci (2001) notice the difficulty with redundant assets in constrained markets.

“In this context, Assumption 1 is not at all innocuous.⁵ When their portfolio holdings are constrained, households may very well benefit from the opportunities afforded by the availability of additional bonds whose yields are not linearly independent.”

The real difficulty with redundant assets arises when there exists a large set of alternative portfolios that have real effect on equilibrium. In unconstrained markets, the unbounded multiplicity problem with alternative portfolios disappears as soon as the redundant assets are removed from the asset structure. The dilemma with constrained markets is that they cannot be removed, because they contribute to risk sharing in general. However it can be circumvented by examining the role of redundant assets from a little different perspective. Let V denote the subspace of portfolios spanned by the rows of the return matrix and V^\perp the set of zero-income portfolios.⁶ Then, the whole space of portfolios has the direct sum of V and V^\perp . By the law of one price, the value of zero-income portfolios is zero. Therefore, the set of no arbitrage prices lies in

³ The consequence of Gottardi and Hens(1996) subsumes the existence theorems of Werner(1985) and Geanakoplos and Polemarchakis(1987) as a special case.

⁴ They construct a sequence of economies by perturbing the original endowments of goods in a way that the perturbations belong to the interior of the consumption set. The limit of a sequence of equilibria for the perturbed economies becomes a quasi-equilibrium of the economy. It becomes equilibrium of the economy under the minimum wealth and resource relatedness conditions. As illustrated later, however, equilibrium may not exist under portfolio constraints although the endowments of goods are in the interior of the consumption set for all agents. Thus, the approach of Gottardi and Hens(1996) is not useful for constrained asset markets.

⁵ Assumption 1 of Cass, Siconolfi and Villanacci(2001) is a condition which assumes away redundant assets.

⁶ Zero-income portfolios are a portfolio which yields zero income in all the states of nature. Thus, the kernel of the return matrix is the set of zero-income portfolios.

V . Now, we decompose portfolios into two parts, one in V and the other in V^\perp . The set V coincides with the set of incomes spanned by an asset structure with the redundant assets removed. Thus, the projection of portfolios onto V is virtually equivalent to the removal of the redundant assets from the asset structure in unconstrained markets.

This intuition is developed into a sophisticated technique to address the large multiplicity problem in constrained markets. We begin with identifying the subspace, N of V^\perp , which contains zero-income portfolios with zero value in equilibrium. The set N causes the large multiplicity problem in equilibrium when $N \neq \{0\}$. Then artificial portfolio constraints are built from the original portfolio constraints by projecting the set of feasible portfolios onto the orthogonal complement of N so that the large multiplicity of zero-value portfolios is removed in equilibrium. Thus, the economy with the artificial portfolio constraints has no multiplicity problem in equilibrium. As shown later, the original economy and the artificial economy share the same set of equilibrium prices. Therefore, we have only to show the existence of equilibrium in the artificial economy.

There exist relatively a few studies on the existence of equilibrium under portfolio constraints. To our knowledge, Siconolfi(1986) is the first person who investigates the existence issue under general convex constraints in an extensive way. His work provides many results which motivate future research, like the indeterminacy problem with portfolio constraints, which is addressed in Cass, Siconolfi and Villanacci(2001). Won and Hahn(2000) take a distinct approach to the existence of equilibrium in the framework of Siconolfi(1986).⁷ Siconolfi(1986) and Won and Hahn(2000), however, impose restrictions on the risk-sharing role of redundant assets to prevent the large multiplicity problem with portfolios. Formally speaking, the restrictive assumption of Siconolfi has the form $C_i \cap V^\perp = \{0\}$ for each agent i , where C_i is the set of portfolios which are feasible unlimitedly in a positive direction for agent i under the portfolio constraints.⁸ This condition limits the risk-sharing role

⁷ The minimum income condition of Won and Hahn(2000) is different from that of Siconolfi(1986), and one does not imply the other. For details, see Won and Hahn(2000).

⁸ As shown later, C_i is the maximum cone in the portfolio constraint set.

of redundant assets without any economic rationale. To take an easy example, non-negative wealth constraints are not covered in Siconolfi (1986), Won and Hahn(2000), and Cass, Siconolfi and Villanacci(2001). Balasko, Cass, and Siconolfi(1990), Benveniste and Ketterer(1992), and Polemarchakis and Siconolfi(1997) assume that portfolio constraints are represented by linear homogeneous equations. The linear restrictions, however, do not cover short-selling or wealth constraints.

The rest of this paper is organized as follows. The constrained asset markets under study are described in Section 2. In Section 3, based on the results of portfolio decomposition, we construct an artificial economy which has the same equilibrium as the original economy, except that it is free from the large multiplicity problem. We also discuss an extension of the law of one price to constrained asset markets. Section 4 is devoted to proving the existence of equilibrium. Concluding remarks are made in the last section.

II. THE ECONOMY

We consider an economy which persists over two periods. Assets are traded in the first period (denoted by 0), and consumptions arise in the second period (denoted by 1).⁹ Assets pay the monetary returns to the holder in the second period.¹⁰ Asset payoffs are contingent upon the event $s \in S = \{1, \dots, \mathbf{S}\}$, which is revealed in the second period. In each state $s \in S$, there is a market for \mathbf{L} commodities.

Let $I = \{1, 2, \dots, \mathbf{I}\}$ denote the set of agents, $J = \{1, 2, \dots, \mathbf{J}\}$ the set of financial assets, and $L = \{1, 2, \dots, \mathbf{L}\}$ the set of consumption goods. Each agent $i \in I$ has the consumption set $X_i := \mathbb{R}_+^{\mathbf{L}}$, an initial endowment of goods $e_i \in X_i$, and the preferences represented by a utility function $u_i : X_i \rightarrow \mathbb{R}$. For a collection of points $\{y(1), \dots, y(\mathbf{S})\}$ in $\mathbb{R}^{\mathbf{L}}$, we set $y = (y(1), \dots, y(\mathbf{S}))$. Utility functions are assumed to satisfy the following properties.

ASSUMPTION 2.1: The following hold true.

⁹ As discussed later, the model covers the case where consumption arises in both period.

¹⁰ Alternatively we can assume that assets pay units of the numeraire good because nominal assets can be converted into real assets and vice versa. For details, see Magill and Shafer(1991).

- (i) Each u_i is continuous, strictly increasing, and quasiconcave.¹¹
(ii) $e_i(s) > 0$ for each i and $s \in S$, and $\sum_{i \in I} e_i \gg 0$.¹²

The first condition of Assumption 2.1 is quite standard. The second condition states that each agent has the positive endowment of at least one good in each state, and the total endowment of every good is positive.

Each asset $j \in J$ pays r_{sj} at state s . The vector of asset returns in state s is given by a J -dimensional row vector $r(s) = (r_{sj})_{j \in J}$, and the return of asset j by a S -dimensional column vector $r_j = (r_{sj})_{s \in S}$. The asset payoffs are described by an $\mathbf{S} \times \mathbf{J}$ matrix $R = [(r(s))_{s \in S}]$. Here, either $\mathbf{S} \geq \mathbf{J}$ or $\mathbf{S} < \mathbf{J}$ holds. This means that it does not matter whether financial markets are potentially complete or not. Market incompleteness is rather represented by the opportunity set Θ_i of portfolios in \mathbf{R}^J . We set $\Theta = \sum_{i \in I} \Theta_i$.

For vectors p and x_i in \mathbf{R}^L , we set,

$$p \square (x_i - e_i) = \begin{bmatrix} p(1) \cdot (x_i(1) - e_i(1)) \\ \vdots \\ p(\mathbf{S}) \cdot (x_i(\mathbf{S}) - e_i(\mathbf{S})) \end{bmatrix}, \quad R(q) = \begin{bmatrix} -q \\ R \end{bmatrix}.$$

The budget correspondence \mathcal{B}_i of agent i is defined by:

$$\mathcal{B}_i(p, q) := \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : \begin{bmatrix} 0 \\ p \square (x_i - e_i) \end{bmatrix} \leq R(q) \cdot \theta_i \right\}.$$

For a given pair $(p, q) \in \mathbf{R}_+^{\mathbf{LS}} \times \mathbf{R}^J$, each agent $i \in I$ solves the optimization problem:

$$\max_{(x_i, \theta_i) \in \mathcal{B}_i(p, q)} u_i(x_i),$$

¹¹ The function u_i is strictly increasing if for any x, x' in X_i with $x - x' \in \mathbf{R}_+^{\mathbf{SL}}$ and $x \neq x'$, $u_i(x) > u_i(x')$.

¹² Let v and v' be vectors in an Euclidean space. Then $v \geq v'$ implies that $v - v' \in \mathbf{R}_+^{\mathbf{SL}}$; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that $v - v' \in \mathbf{R}_{++}^{\mathbf{SL}}$.

The demand correspondence ξ_i for agent i is defined by:

$$\xi_i(p, q) = \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : (x_i, \theta_i) \in \arg \max_{(x'_i, \theta'_i) \in B_i(p, q)} u_i(x'_i) \right\}.$$

Competitive equilibrium of the economy is defined as follows.

DEFINITION 2.2: A profile $(p, q, x, \theta) \in \mathbb{R}_+^{LS} \times \mathbb{R}^J \times (\prod_{i \in I} X_i) \times (\prod_{i \in I} \Theta_i)$ is a *competitive equilibrium* if (i) $(x_i, \theta_i) \in \xi_i(p, q)$ for every $i \in I$, (ii) $\sum_{i \in I} (x_i - e_i) = 0$, and (iii) $\sum_{i \in I} \theta_i = 0$.

Let V denote the subspace spanned by the row vectors of the return matrix R and V^\perp be its kernel (i.e., $V^\perp = \{\theta \in \mathbb{R}^J : R \cdot \theta = 0\}$). Redundant assets exist if and only if $V^\perp \neq \{0\}$. In particular, some assets are redundant if the rank of the return Matrix R is less than the minimum of J and S . Portfolios in V^\perp are called, *zero-income portfolios*, which generate zero income transfer in each state of the second period. In particular, portfolios in $\Theta_i \cap V^\perp$ are called, *constrained zero-income portfolios*, for agent i . A portfolio θ in \mathbb{R}^J has the direct sum $\hat{\theta} + \tilde{\theta}$, where $\hat{\theta} \in V$ and $\tilde{\theta} \in V^\perp$. The portfolio $\tilde{\theta}$ does not affect the size of income transfers but may matter to the feasibility of θ under the portfolio constraints.

Let A be a non-empty convex set in \mathbb{R}^m for some positive integer m . The recession cone of A is the set $Y(A) = \{v \in E : A + v \subset A\}$. An element $v \in Y(A)$ is a direction of recession of A . The lineality space $\mathcal{L}(A)$ of A is the set $\{v \in \mathbb{R}^m : \lambda v \in Y(A) \text{ for all } \lambda \in \mathbb{R}\}$. If A is closed, then the following hold.¹³

- i) $Y(A)$ is closed.
- ii) $Y(A) \subset A$ and $\mathcal{L}(A) \subset Y(A)$.
- iii) $v \in Y(A)$ if and only if $x + \lambda v \in A$ for some $x \in A$ and all $\lambda \geq 0$.

¹³ For more details on the analysis of convex sets, see Rockafellar(1970).

We set $C_i = Y(\Theta_i)$ and $L_i = \mathcal{L}(\Theta_i)$ for each $i \in I$. A portfolio in $C_i \cap V^\perp$ is a zero-income portfolio that is a direction of recession of Θ_i . Let N denote the lineality space of the cone $\sum_{i \in I} (C_i \cap V^\perp)$. If $N \neq \{0\}$, there exists a set of nonzero portfolios in $C_i \cap V^\perp$ that jointly span N . As shown later, if $N \neq \{0\}$, then there exists a large multiplicity of optimal portfolios in equilibrium. Let M denote the orthogonal complement of N in V^\perp , where $N = \mathcal{L}\left(\sum_{i \in I} (C_i \cap V^\perp)\right)$. For each $i \in I$, let $\hat{\Theta}_i$ denote the projections of Θ_i onto $V + M$. We make the following assumption for each $i \in I$.

ASSUMPTION 2.2: The set $\Theta_i \subset \mathbb{R}^J$ is a closed convex cone with vertex with $0 \in \Theta_i$, and $\hat{\Theta}_i$ is closed.¹⁴

Note that market frictions, such as short-selling constraints, bid-ask spreads, and proportional transaction costs, can be represented as a convex cone with vertex (See Luttmer(1996)). The linear constraints of Balasko, Cass, and Siconolfi(1990), Benveniste and Ketterer(1992), and Polemarchakis and Siconolfi(1997) are a special case of Assumption 2.2. In addition to the convexity of the portfolio constraints, Siconolfi (1986) and Won and Hahn(2000) impose the extra requirement that $C_i \cap V^\perp = \{0\}$ for each $i \in I$. Notice that if $C_i \cap V^\perp = \{0\}$ for each $i \in I$, then $N = \{0\}$. Thus, Siconolfi(1986) and Won and Hahn(2000) cannot cover the case where portfolio constraints lead to the large portfolio multiplicity problem (i.e., $N \neq \{0\}$).

Let A be a non-empty convex subset in \mathbb{R}^m for some positive integer m . We denote the closure of A by $cl(A)$, the interior of A by $int(A)$, and the boundary of A with respect to the relative topology by ∂A .

III. PORTFOLIO DECOMPOSITION AND THE EXTENDED LAW OF ONE PRICE

Zero-income portfolios contain information on how one asset is replicated by other assets. In frictionless markets, they have no value in

¹⁴ For a positive integer m , a set A in \mathbb{R}^m is a cone if $\lambda v \in A$ for all $v \in A$ and all $\lambda \geq 0$. It is a cone with vertex if $A - v$ is a cone for some $v \in \mathbb{R}^m$.

equilibrium, and therefore, are a realization of linear pricing. In the case where asset markets are subject to portfolio constraints, zero-income portfolios may not follow the linear pricing rule but still have important information on asset pricing in equilibrium. However, since redundant assets are involved in risk sharing and create the unbounded set of zero-income portfolios, it is impossible in general to remove them without perturbing equilibrium. A systematic way of decomposing portfolios is developed to treat the large multiplicity of alternative portfolios with the same value. The consequence of portfolio decomposition enables us to construct the artificial economy, which is free from the large multiplicity of zero-value portfolios in equilibrium and has the same set of equilibrium allocations of goods as the original economy. These results will play a crucial role in investigating the existence of equilibrium with the original economy.

If $N \neq \{0\}$, a large multiplicity of equilibrium prices arise. Specifically, N consists of zero-income portfolios which turn out to have no value in equilibrium, and therefore, cause the large multiplicity problem. In general, we can consider two ways of handling the problem with zero-income portfolios: the projection method and the exclusion method. The former involves projecting away the zero-income portfolios which cause the large multiplicity problem. The latter is to simply remove redundant assets from the asset structure and then price them by the law of one price. This is the way of handling redundant assets in the classical literature of incomplete markets such as Werner(1985) and Balasko and Cass(1989). In unconstrained asset markets, both ways of handling zero-income portfolios are the same in that they do not affect the opportunity set of income transfers. The following example, however, shows that the exclusion method is not appropriate in the presence of portfolio constraints because it changes drastically the opportunity set of income transfers.

EXAMPLE 3.1: We consider two asset structures represented by the payoff matrix:

$$R_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The asset structure 1 is a sub-structure of the asset structure 2, which does not contain the redundant assets 3 and 4. Both asset structures have \mathbb{R}^2 as the income spanning space in a frictionless market. The removal of redundant assets from the asset structure 2 does not affect the opportunity set of income transfers. To apply the projection method, we decompose \mathbb{R}^4 into the subspace V spanned by the rows of R_2 and its orthogonal complement V^\perp in \mathbb{R}^4 . A simple linear algebra shows that portfolios in V generates the set of income transfers \mathbb{R}^2 .

The two ways of treating zero-income portfolios have different consequences under short selling restrictions. Suppose that no short sales are allowed. Then, the set of feasible portfolios is equal to \mathbb{R}_+^2 under the asset structure 1 and to \mathbb{R}_+^4 under the asset structure 2. The opportunity set of income transfers is $\mathbb{R}_+^2 \cap \{(x, y) \in \mathbb{R}^2 : y \leq x\}$ under the asset structure 1 and \mathbb{R}_+^2 under the asset structure 2.

Now each $\theta \in \mathbb{R}_+^4$ is uniquely decomposed as $\theta = \hat{\theta} + \tilde{\theta}$, where $\hat{\theta} \in V$ and $\tilde{\theta} \in V^\perp$. The point $\hat{\theta}$ is the projection of θ onto V . Let $\hat{\Theta}$ denote the projection of \mathbb{R}_+^4 onto V . Then $\hat{\Theta}$ does not contain any elements of V^\perp except the origin. Since $R_2 \cdot \theta = R_2 \cdot \hat{\theta}$ for all $\theta \in \mathbb{R}_+^4$, the opportunity set of income transfers for $\hat{\Theta}$ is equal to \mathbb{R}_+^2 . Thus, the projection method does not distort the opportunity set of income transfers.

This is not the case, however, with the exclusion method. If assets 3 and 4 are removed from the asset structure 2, then it is reduced to the asset structure 1, and therefore, has $\mathbb{R}_+^2 \cap \{(x, y) \in \mathbb{R}^2 : y \leq x\}$ as the set of income transfers. Thus, the removal of redundant assets can lead to a drastic change in the set of risk-sharing opportunities under the short-selling restrictions.

The following proposition presents a consequence of portfolio decomposition.

PROPOSITION 3.1: The following results hold true under Assumption 2.2.

- (i) $\Theta = N + \sum_{i \in I} \hat{\Theta}_i$,
- (ii) $\left[\sum_{i \in I} (\Upsilon(\hat{\Theta}_i) \cap M) \right] \cap \left[-\sum_{i \in I} (\Upsilon(\hat{\Theta}_i) \cap M) \right] = \{0\}$,
- (iii) $N = \mathcal{L}\left(\sum_{i \in I} (\Theta_i \cap V^\perp)\right)$.

PROOF : See the appendix.

The results of Proposition 3.1 will play a crucial role in characterizing equilibrium prices and proving the existence of equilibrium. We make the following assumption.

ASSUMPTION 3.1: For each $i \in I$, there exists $w_i \in C_i$ which satisfies $R \cdot w_i > 0$.

Assumption 3.1 states that each C_i contains a non-trivial portfolio which yields non-negative contingent incomes.

We will examine the law of one price from a little distinct perspective to have an idea about how to extend it to constrained markets. The law of one price states that if two portfolios θ_1 and θ_2 in \mathbb{R}^J have the same return in each state of the second period, then they have the same price. Formally, $q \in \mathbb{R}^J$ satisfies the law of one price if $R \cdot \theta_1 = R \cdot \theta_2$ implies $q \cdot \theta_1 = q \cdot \theta_2$. Put differently, $q \in \mathbb{R}^J$ satisfies the law of one price if $R \cdot \theta = 0$ implies $q \cdot \theta = 0$ (i.e., if $q \cdot \theta = 0$ for all $\theta \in V^\perp$). We see that $q \in V$, if and only if it satisfies the law of one price. The law of one price in frictionless markets is equivalent to the condition that zero-income portfolios have zero value. In unconstrained markets, we have $\Theta_i = \mathbb{R}^J$ for all $i \in I$. Then $C_i = \mathbb{R}^J$, and therefore, $C_i \cap V^\perp = V^\perp$ for all $i \in I$. Thus, by analogy from the unconstrained markets, we would be tempted to generalize the law of one price to the case of frictional markets as follows; a price $q \in \mathbb{R}^J$ satisfies ‘the *à la* law of one price’ with market frictions if $\theta_i \in C_i \cap V^\perp$ implies $q \cdot \theta_i = 0$ for all $i \in I$. However, such *naïveté* is not acceptable, because zero-income portfolios are not free in equilibrium, in general. We extend the ‘classical’ law of one price to the constrained asset markets.

DEFINITION 3.1: An asset price q satisfies the *extended law of one price with respect to* N , if $q \cdot v = 0$ for all $v \in N$. If $N = V^\perp$, then the asset price q satisfies the *classical law of one price*.

Let \mathcal{E} denote the original economy and $\hat{\mathcal{E}}$ the economy which is the

same as \mathcal{E} , except that each Θ_i is replaced by $\hat{\Theta}_i$. We show that equilibrium prices satisfy the extended law of one price, and both the economies \mathcal{E} and $\hat{\mathcal{E}}$ have the same set of equilibria except for the nominal difference in asset allocations. Let Q^* be the set of equilibrium asset prices of the economy \mathcal{E} .

THEOREM 3.1: The following hold.

- (i) Equilibrium prices satisfy the extended law of one price with respect to N , i.e., $Q^* \subset V + M$.
- (ii) The pair $(p, q, x, \hat{\theta})$ is an equilibrium of $\hat{\mathcal{E}}$, if and only if there exists $\eta_i \in N$ for each $i \in I$, such that $(p, q, x, \hat{\theta} + \eta)$ is an equilibrium of \mathcal{E} .

PROOF : See the appendix.

No arbitrage conditions in the literature provide a useful framework for asset pricing and equilibrium analysis. First, we consider a typical form of arbitrage for unconstrained asset markets.

DEFINITION 3.2: Suppose that $\Theta_i \in \mathbb{R}^J$ for all $i \in I$. Then an asset price $q \in \mathbb{R}^J$ admits *no arbitrage* if there is no $v \in \mathbb{R}^J$ which satisfies $R(q) \cdot v > 0$.

The no arbitrage condition of Definition 3.2 provides a suitable conceptual framework for studying asset pricing, portfolio choice problem, and equilibrium in frictionless markets. Specifically, Werner(1985) among others shows the existence of equilibrium with incomplete markets by taking advantage of Definition 3.2.

When asset markets face portfolio constraints, however, the no arbitrage conditions for frictionless markets are no longer useful, because the extended law of one price fails. We introduce a notion of arbitrage which is appropriate in characterizing equilibrium prices in constrained asset markets.

DEFINITION 3.3: A price vector $q \in \mathbb{R}^J$ admits *no constrained*

arbitrage for agent i , in the economy \mathcal{E} , if there is no $v_i \in C_i$, such that $R(q) \cdot v_i > 0$. A price vector $q \in \mathbb{R}^J$ admits no *constrained arbitrage* for the economy \mathcal{E} if it admits no constrained arbitrage for every agent $i \in I$.

The above definition is used in Jouini and Kallal(1995, 1999), Luttmer (1996), Chen(1995), Pham and Touzi(1999), Cvitanic and Karatzas (1993), and Broadie, Cvitanic and Soner(1998) among others. Clearly, the no constrained arbitrage condition holds in equilibrium.

Let Q_i denote the set of prices which admit no constrained arbitrage prices for agent i . We set $Q = \bigcap_{i \in I} Q_i$. The set Q denotes the set of prices which admits no constrained arbitrage for the economy \mathcal{E} . Since the no constrained arbitrage condition is fulfilled in equilibrium, we have $Q^* \subset Q$. Let Q^N denote the set of prices which admit no arbitrage for unconstrained market. Noting that $\Theta_i \subset \mathbb{R}^J$ for all $i \in I$, we have $Q^N \subset Q$. But the converse is not true in general.¹⁵

The following result shows that no constrained arbitrage prices satisfy the extended law of one price with respect to N .

PROPOSITION 3.2: Prices in Q satisfy the extended law of one price with respect to N .

PROOF : See the appendix.

One can verify that under Assumption 3.1, $q \in \mathbb{R}^J$ admits no constrained arbitrage for agent i , if and only if, $q \cdot v_i > 0$ for all $v_i \in C_i$ which satisfy $R \cdot v_i > 0$. If q satisfies the no constrained arbitrage condition, it is straightforward to see that the 'only if' part holds true. Suppose that there is $v_i \in C_i$ which satisfy $R(q) \cdot v_i > 0$. Then there are two possibilities, (i) $q \cdot v_i \leq 0$ and $R \cdot v_i > 0$ or (ii) $q \cdot v_i < 0$ and $R \cdot v_i = 0$. The case (i) leads to an immediate contradiction. Consider the case (ii). By Assumption 3.1, we can choose $w_i \in C_i$ which satisfies $R \cdot w_i > 0$. Then there exists a small number $\alpha > 0$ such that $\alpha w_i + (1 - \alpha)v_i \in C_i$, $q(\alpha w_i + (1 - \alpha)v_i) < 0$ and $R \cdot (\alpha w_i + (1 - \alpha)v_i) > 0$, which

¹⁵ If $\Theta_i = C_i$ and Θ_i is a strict subset of $\{v \in \mathbb{R}^J : R \cdot v > 0\}$ for all $i \in I$, then $Q \subsetneq Q^N$.

is impossible. The following is an alternative expression of Definition 3.3.

DEFINITION 3.3' : A price vector $q \in \mathbb{R}^J$ admits no constrained arbitrage for agent i , in the economy, \mathcal{E} , if $q \cdot v_i > 0$ for all $v_i \in C_i$ which satisfy $R \cdot v_i > 0$. A price vector $q \in \mathbb{R}^J$ admits no constrained arbitrage for the economy \mathcal{E} if it admits no constrained arbitrage for every agent $i \in I$.

From now on, Definition 3.3' will be adopted in characterizing no constrained arbitrage prices. For each $i \in I$, we set $\Gamma_i = \{v_i \in Y(\hat{\Theta}_i) : R \cdot v_i > 0\}$.¹⁶ Now we set $\Gamma = \sum_{i \in I} (\Gamma_i \cup \{0\})$ and $\bar{\Gamma} = \sum_{i \in I} cl(\Gamma_i)$. The sets Q and Γ are characterized as following.

LEMMA 3.2: The sets Q and Γ have the following property.

- (i) $Q = \{q \in V + M : q \cdot v > 0 \text{ for all nonzero } v \in \Gamma\}$.
- (ii) The set $\bar{\Gamma}$ is a closed convex pointed cone.¹⁷

PROOF : See the appendix.

We define the sets of normalized prices.

$$\begin{aligned}\bar{\Delta} &= \{(p, q) \in \mathbb{R}_+^{SL} \times cl(Q) : \|q\| = 1, \sum_{l \in L} p_l(s) = 1 \text{ for all } s \in S\}, \\ \Delta &= \{(p, q) \in \mathbb{R}_{++}^{SL} \times Q : \|q\| = 1, \sum_{l \in L} p_l(s) = 1 \text{ for all } s \in S\}\end{aligned}$$

Where $\|\cdot\|$ denotes the Euclidean norm.¹⁸

¹⁶ By Assumption 3.1, $\Gamma_i \neq \emptyset$.

¹⁷ A cone C is pointed if $C \cap (-C) = \{0\}$.

¹⁸ The set of normalized asset prices in Δ and $\bar{\Delta}$ are not convex. Assumption 2.2 and 3.1 are so general that they may not exclude the case that $cl(Q)$ is a half-space. If $cl(Q)$ is a half-space, then a convex normalization of $cl(Q)$ and Q is not available. As shown later in the proof of the existence of equilibrium, however, the non-convex normalization involves technical difficulties with fixed-point arguments.

IV. THE EXISTENCE OF EQUILIBRIUM

If agents are endowed with a positive amount of some commodities in each state of the second period, they always survive in unconstrained asset markets.¹⁹ This is not the case, however, with constrained asset markets. As illustrated below, equilibrium may fail to exist in constrained asset markets, when each agent has the endowment of commodities in the interior of the consumption set. We provide a survival condition with the constrained asset markets which is indispensable for the existence of equilibrium. An important corollary is that the survival condition is unnecessary for the existence of equilibrium in markets where agents are endowed with a positive amount of some commodities in the beginning period.²⁰

Another difficulty arises from the failure of the Cass trick. The method of Cass(1984) does not apply to the case where no agent is allowed to behave in equilibrium as if in complete markets. It should be noted that such an ideal agent is only an instrument to verify the boundary behavior of excess demand functions, as in complete markets. We demonstrate that if asset markets are subject to portfolio constraints, then goods and asset markets are jointly responsible for the boundary behavior of the excess demand correspondence. In particular, *survival conditions with portfolios* are needed to guarantee not only the upper hemicontinuity of demand correspondences but also the desired boundary behavior of excess demand under portfolio constraints. We assume that \mathcal{E} satisfies the following survival condition with asset markets.

ASSUMPTION 4.1: For each $q \in Q$, $\min_i q \cdot \Theta_i < 0$ for all $i \in I$.

This condition states that agents are allowed to transfer income to any state of the second period through asset markets when the no arbitrage condition holds. It is worth noting that Assumption 4 is concerned about prices in Q alone but not about prices in the boundary of Q . It is shown

¹⁹ Gottardi and Hens(1996) treat exhaustively the survival problem with unconstrained asset markets.

²⁰ This result shows that the survival conditions introduced in Siconolfi(1986), and Cass, Siconolfi and Villanacci(2001) are unnecessary for the existence of equilibrium.

below that Assumption 4.1 holds true in an economy where each agent is allowed to consume and have the positive endowments of goods in the first period. Moreover, Assumption 4.1 cannot be dispensed with as shown below.

EXAMPLE 4.1: We demonstrate that Assumption 4.1 is virtually fulfilled in an economy where agents are allowed to consume and have a positive endowment of goods in the beginning period. Let \mathcal{E}' be the economy which is the same as \mathcal{E} , except that consumption is allowed in the first period. More precisely, the consumption space is augmented by adding \mathbb{R}^L to \mathbb{R}^{SL} . It is assumed that agent i has the consumption set $\mathbb{R}_+^L \times X_i$ and the endowment of goods $(e_i(0), e_i)$ where $e_i(0) \in \mathbb{R}_+^L$ and $e_i(0) > 0$ for each $i \in I$. Let v_i denote the utility function in $\mathbb{R}_+^L \times X_i$ for agent i . Then for a given price pair $(p, q) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}^J$, agent i is supposed to choose $(\bar{y}_i, \bar{\theta}_i)$ which maximizes $v_i(y_i)$ in the budget set:

$$\begin{aligned} p(0) \cdot (y_i(0) - e_i(0)) + q \cdot \theta_i &\leq 0, \\ p(s) \cdot (y_i(s) - e_i(s)) &\leq r(s) \cdot \theta_i, \forall s \in S, \\ \theta_i &\in \Theta_i, \quad y \in Y_i. \end{aligned}$$

To transform the economy \mathcal{E}' into the economy where no consumption arises in the beginning period, we add state 0 to the second period and introduce asset 0 which pays one unit of money in state 0 and nothing in the other states of the second period. Let $q_0 > 0$ and $\eta_i(0)$ denote its price and the amount of asset 0 held by agent i , respectively. For each $y_i(0)$ and $p(0)$, we set $\eta_i(0) = p(0) \cdot (y_i(0) - e_i(0))$ and let (q_0, q) denote asset prices. Then the budget set for agent i is transformed as:

$$\begin{aligned} (q_0, q) \cdot (\eta_i(0), \theta_i) &\leq 0, \\ p(0) \cdot (y_i(0) - e_i(0)) &\leq \eta_i(0) = (1, 0_J) \cdot (\eta_i(0), \theta_i), \\ p(s) \cdot (y_i(s) - e_i(s)) &\leq r(s) \cdot \theta_i = (0, r(s)) \cdot (\eta_i(0), \theta_i), \forall s \in S, \\ (\eta_i(0), \theta_i) &\in \mathbb{R} \times \Theta_i, \quad y \in Y_i, \end{aligned}$$

where 0_J is the zero in \mathbb{R}^J . The second inequality can be considered

as the budget set which faces agent i in the state 0. No consumption arises in the initial period, and moreover, agent i has the portfolio constraint set $\mathbb{R} \times \Theta_i$ in the transformed economy. Since $(-1, 0_j) \in \mathbb{R} \times \Theta_i$ and $(q_0, q) \cdot (-1, 0_j) = -q_0 < 0$, the transformed economy satisfies Assumption 4.1.

EXAMPLE 4.2: To illustrate that Assumption 4.1 cannot be dispensed with, we consider a two state two-agent economy where two Arrow-Debreu securities are traded and only one good is consumed in each state. Since the good is also used as a numeraire, the payoff matrix is given by the 2×2 identity matrix. Both agents have the same endowment of goods and distinct preferences.

$$\begin{aligned} u_1(x) &= 2\sqrt{x_1} + \sqrt{x_2}, & e_1 &= (1, 1), \\ u_2(x) &= \sqrt{x_1} + 2\sqrt{x_2}, & e_2 &= (1, 1), \end{aligned}$$

We assume that each agent faces the same portfolio constraint as following.

$$\Theta_i = \{(a, b) \in \mathbb{R}^2 : a + b \geq 0\}$$

Clearly, $C_i = \Theta_i$ for all $i = 1, 2$ and $V^\perp = N = \{0\}$. Thus, it is easy to check that the economy satisfies Assumption 2.1, 2.2 and 3.1.

Since each asset is an Arrow-Debreu security, we have $\Gamma_i = \mathbb{R}_+^2 \setminus \{0\}$, and therefore, $Q_i = Q = \mathbb{R}_{++}^2$ for each $i = 1, 2$. We set $Q^0 = \{q = (q_1, q_2) \in Q : q_1 = q_2 > 0\}$. Then it is easy to check that $q \in Q^0$, if and only if, $\min q \cdot \Theta_i = 0$ for each $i = 1, 2$. Since $Q^0 \subset Q$, Assumption 4.1 is violated.

Now we claim that there exists no equilibrium of the economy. Since there exists only one good in each state of the second period, the utility maximization problem for agent 1 is reduced to the following relations.

$$\begin{aligned} \text{Agent 1:} \quad & \max_{(a,b) \in \Theta_1} 2\sqrt{a+1} + \sqrt{b+1} & s.t. \quad & q_1 a + q_2 b \leq 0, \\ \text{Agent 2:} \quad & \max_{(a,b) \in \Theta_2} \sqrt{a+1} + 2\sqrt{b+1} & s.t. \quad & q_1 a + q_2 b \leq 0. \end{aligned}$$

Let $\mathcal{A}_i(q)$ be the budget set for agent i at the price q . We set $v_1(a, b) = 2\sqrt{a+1} + \sqrt{b+1}$ and $v_2(a, b) = \sqrt{a+1} + 2\sqrt{b+1}$ for each $(a, b) \geq (-1, -1)$, and $\mathcal{A}_i(q) = \{(a, b) \in \mathbb{R}^2 : q_1 a + q_2 b \leq 0, (a, b) \in \Theta_i\}$ for each $i = 1, 2$. The function v_i can be considered a utility function defined over portfolios which is derived from u_i . The set $\mathcal{A}_i(q)$ is the budget set for agent i at the price q .

Suppose that there exists an equilibrium $\{(q_1, q_2), (a_1, b_1), (a_2, b_2)\}$. The no arbitrage condition implies that $q_1 > 0$ and $q_2 > 0$. Since $a_1 + b_1 \geq 0$ and $a_2 + b_2 \geq 0$, the market clearing condition implies that $a_1 + b_1 = 0$ and $a_2 + b_2 = 0$. Then the first order conditions for utility maximization at equilibrium choices are reduced to the relations:

$$\begin{aligned} \lambda(q_1 - q_2) - \frac{1}{\sqrt{1+a_1}} + \frac{1}{2\sqrt{2-a_1}} &= 0, \quad \lambda > 0, \\ \mu(q_1 - q_2) - \frac{1}{2\sqrt{1-a_1}} + \frac{1}{\sqrt{2+a_1}} &= 0, \quad \mu > 0, \\ (q_1 - q_2)a_1 &= 0, \end{aligned}$$

where the last equation is derived from the budget constraint for each agent, and λ and μ are the Lagrangian multipliers for agent 1 and 2, respectively.

The last equation of the previous relations implies that $q_1 = q_2$ or $a_1 = 0$. If $q_1 = q_2$, then the first and second equations are not compatible. Suppose that $a_1 = 0$. Then $\lambda(q_1 - q_2) = 1 - \frac{1}{2\sqrt{2}} > 0$ and $\mu(q_1 - q_2) = \frac{1}{2} - \frac{1}{\sqrt{2}} < 0$, which is impossible. Therefore, we conclude that the economy has no equilibrium.

Let K be a closed rectangle in $\mathbb{R}^{\text{SL}} \times \mathbb{R}^{\text{J}}$ with center at the origin. We set $P = (\mathbb{R}_+^{\text{L}} \setminus \{0\})^{\text{S}}$. The set P denotes the set of nonnegative prices which excludes the zero price in each contingency of the second period. For each $(p, q) \in P \times \text{cl}(Q)$, we define the budget set $\mathcal{B}_i(p, q; K) = \mathcal{B}_i(p, q) \cap K$ for every $i \in I$, which is a compact truncation of the budget set $\mathcal{B}_i(p, q)$. We are ready to define demand correspondences with respect to the truncated budget set:

$$\xi_i(p, q; K) = \left\{ (x_i, \theta_i) \in X_i \times \Theta_i : (x_i, \theta_i) \in \arg \max_{(x, \theta) \in \mathcal{B}_i(p, q; K)} u_i(x) \right\}.$$

As implicitly shown in Example 4.2, the correspondence $\xi_i(p, q; K)$ need not be upper hemicontinuous on the set of asset prices which do not allow income transfer from the beginning period to the next period. To circumvent this problem, we need to take some steps. For a point $(p, q) \in P \times cl(Q)$, which satisfies either $(p, q) \notin R_{++}^{SL} \times Q$ or $\min q \cdot \Theta_i = 0$, we define the set:

$$\varphi_i(p, q; K) = \left\{ (x_i, \theta_i) \in X_i \times \Theta_i \left| \begin{array}{l} \exists \{(p^n, q^n)\} \text{ in } R_{++}^{SL} \times Q \text{ such that } (p^n, q^n) \rightarrow (p, q) \\ \min q^n \cdot \Theta_i < 0 \text{ for each } n, \text{ and} \\ (x_i^n, \theta_i^n) \rightarrow (x_i, \theta_i) \text{ for some } (x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K) \end{array} \right. \right\}$$

We claim that $\varphi_i(p, q; K)$ is closed. Let $\{(x_i^n, \theta_i^n)\}$ be a sequence in $\varphi_i(p, q; K)$, which converges to some point (x_i, θ_i) in $X_i \times \Theta_i$. Then there exists $(p^{n,m}, q^{n,m}) \rightarrow (p, q)$ such that $(p^{n,m}, q^{n,m}) \in R_{++}^{SL} \times Q$ for each m , and $(x_i^{n,m}, \theta_i^{n,m}) \rightarrow (x_i^n, \theta_i^n)$ which satisfies $(x_i^{n,m}, \theta_i^{n,m}) \in \xi_i(p^{n,m}, q^{n,m}; K)$ for each m . Recalling that $(x_i^n, \theta_i^n) \rightarrow (x_i, \theta_i)$, by the diagonal sequence theorem there exists a sequence $n_1 < n_2 < \dots < n_k < \dots$ such that $(x_i^{n_k}, \theta_i^{n_k}) \rightarrow (x_i, \theta_i)$. Thus, we have $(x_i, \theta_i) \in \varphi_i(p, q; K)$.²¹

Let $\hat{\varphi}_i(p, q; K)$ denote the convex hull of $\varphi_i(p, q; K)$. Since $\varphi_i(p, q; K)$ is compact, so is $\hat{\varphi}_i(p, q; K)$. Now we introduce an artificial correspondence $\xi'_i(p, q; K)$ on $P \times cl(Q)$ as follows.

$$\xi'_i(p, q; K) = \begin{cases} \xi_i(p, q; K), & \text{if } (p, q) \in R_{++}^{SL} \times Q \text{ and } \min q \cdot \Theta_i < 0, \\ \hat{\varphi}_i(p, q; K), & \text{otherwise.} \end{cases}$$

²¹ The reader is referred to Kantorovich and Akilov(1982) for the diagonal sequence theorem.

Let $\hat{\xi}_i(p, q; K)$ be the convex hull of $\xi'_i(p, q; K)$. Then $\hat{\xi}_i(p, q; K) = \xi_i(p, q; K)$ if $\min q \cdot \Theta_i < 0$, and $\hat{\xi}_i(p, q; K) = \hat{\phi}_i(p, q; K)$ if $\min q \cdot \Theta_i = 0$.

LEMMA 4.1: Each $\hat{\xi}_i(p, q; K)$ is nonempty-valued, compact-valued, convex-valued, and upper hemicontinuous at each $(p, q) \in P \times cl(Q)$.

PROOF : See the appendix.

The Cass trick is not valid in general under portfolio constraints. Nevertheless we can show that there exists at least one agent that has 'large' purchasing power around the boundary of $\bar{\Delta}$. This leads to the explosion of aggregate demand on the relevant boundary of $\bar{\Delta}$ in frictional markets. Let $\{K_n\}$ be an increasing sequence of rectangles in $\mathbb{R}^{SL} \times \mathbb{R}^J$ which satisfies $\cup_n K_n = \mathbb{R}^{SL} \times \mathbb{R}^J$.

PROPOSITION 4.1: Let $\{(p^n, q^n)\}$ be a sequence of prices in $\bar{\Delta}$ convergent to a point $(p, q) \in \bar{\Delta} \setminus \Delta$. Then for a sequence of allocations $\{(x^n, \theta^n)\}$ such that $(x_i^n, \theta_i^n) \in \hat{\xi}_i(p^n, q^n; K_n)$ for all n and all $i \in I$, we have $\sum_{i \in I} \|x_i^n\| \rightarrow \infty$.

PROOF : See the appendix.

Proposition 4.1 states that if the sequence of optimal choices $\{(x_i^n, \theta_i^n)\}$ is bounded for each i , then $\{(p^n, q^n)\}$ in $\bar{\Delta}$ converges to the point $(p, q) \in \Delta$ (i.e., $p \gg 0$ and $q \in Q$). This result is used to show that equilibrium prices of the economy $\hat{\mathcal{E}}$ are in Δ . Before we state the main theorem, it is necessary to provide the following preliminary existence theorem for the artificial economy $\hat{\mathcal{E}}$.

THEOREM 4.1: Under Assumptions 2.1, 2.2, 3.1 and 4.1, there exists equilibria for the economy $\hat{\mathcal{E}}$.

PROOF : See the appendix.

We are ready to provide the main existence theorem for the original economy \mathcal{E} . The following is immediate from Theorem 3.1 and Theorem 4.1.

THEOREM 4.2: Under Assumption 2.1, 2.2, 3.1 and 4.1, there exists equilibria for the economy \mathcal{E} .

V. CONCLUDING REMARKS

We have shown the existence of equilibrium in the asset markets where portfolio constraints are expressed as a convex cone, and we have demonstrated that equilibrium prices satisfy the extended law of one price in constrained asset markets. To do this, we rely on the results of portfolio decomposition, described in Proposition 3.1, and introduce survival conditions with portfolios. It is worth noting that any extra restrictions, such as $C_i \cap V^\perp = \{0\}$ for all $i \in I$, are not imposed on available portfolios. The large multiplicity problem with portfolio is allowed to occur in equilibrium. Thus, when the portfolio constraints are described as a convex cone with a vertex, the consequence of Theorem 4.1 subsumes as a special case the existence theorems of Siconolfi(1986), Benveniste and Ketterer(1990), Won and Hahn(2000), Cass, Siconolfi and Villanacci (2001) among others.

This paper is restricted to the case where the market value of the endowments of goods is always positive in each state of the second period, and the set of constrained portfolios is a convex cone with a vertex. It is a conceivable topic to extend the survival condition of Assumption 4.1 by weakening the positivity condition on the endowments of goods. Another challenging issue is to provide the existence of equilibrium with constrained markets in a multi-period economy.²²

²² Equilibrium may not exist even in unconstrained multi-period markets because the prices of long-lived assets are involved in spanning state-contingent incomes. See Magill and Quinzii(1996).

APPENDIX

The proof of Proposition 3.1 relies on the following technical lemma.

LEMMA A: For each $i \in I$, let A_i be a closed convex set in \mathbb{R}^J , $\Upsilon(A_i)$ its recession cone and L the lineality space of $\sum_{i \in I} \Upsilon(A_i)$. Then we have:

$$L = \sum_{i \in I} (\Upsilon(A_i) \cap L).$$

PROOF: It is clear that $\sum_{i \in I} (\Upsilon(A_i) \cap L) \subset L$. We have only to show the converse. Let v be a point in L . Pick $v_i \in \Upsilon(A_i)$ such that $v = \sum_{i \in I} v_i$. We introduce a set:

$$H = \{x \in \mathbb{R}^J : x = \alpha_0(-v) + \sum_{i \in I} \alpha_i v_i \text{ for some } \alpha = (\alpha_0, \dots, \alpha_I) \in \mathbb{R}_+^{I+1}\}.$$

We claim that H is a subspace of \mathbb{R}^J . Let x and y be points in H . Then there exist α and β in \mathbb{R}_+^{I+1} such that:

$$x = \alpha_0(-v) + \sum_{i \in I} \alpha_i v_i, \quad y = \beta_0(-v) + \sum_{i \in I} \beta_i v_i.$$

For any real numbers a and b , we set:

$$c = \max\{|a\alpha_i + b\beta_i|; i = 0, \dots, I\}.$$

Since $-v + \sum_{i \in I} v_i = 0$, we see that $c + a\alpha_i + b\beta_i \geq 0$ for all $i = 0, \dots, I$ and

$$ax + by = (c + a\alpha_0 + b\beta_0)(-v) + \sum_{i \in I} (c + a\alpha_i + b\beta_i)v_i$$

It implies $ax + by \in H$ for any real numbers a and b . Therefore H is a

subspace of E .

Since $-v \in L$ and $v_i \in Y(A_i)$ for all $i \in I$, the set H is in a convex cone $\sum_{i \in I} Y(A_i) + L$. On the other hand, the cone $\sum_{i \in I} Y(A_i) + L$ has the lineality space L . These results imply $H \subset L$. In particular, we see each $v_i \in L$ and therefore, $v_i \in Y(A_i) \cap L$ and $v \in \sum_{i \in I} (Y(A_i) \cap L)$. ■

PROOF OF PROPOSITION 3.1: (i) First show that $\sum_{i \in I} \Theta_i \subset N + \sum_{i \in I} \hat{\Theta}_i$. Let θ_i be a point in Θ_i for each $i \in I$. We have the decomposition $\theta_i = \hat{\theta}_i + \eta_i$ where $\hat{\theta}_i \in V + M$ and $\eta_i \in N$. Then we see:

$$\sum_{i \in I} \theta_i = \sum_{i \in I} \eta_i + \sum_{i \in I} \hat{\theta}_i \in N + \sum_{i \in I} \hat{\Theta}_i.$$

Conversely, let θ be a point in $N + \sum_{i \in I} \hat{\Theta}_i$. Then there exist $\eta \in N$ and $\hat{\theta}_i \in \hat{\Theta}_i$ for each $i \in I$ such that $\theta = \eta + \sum_{i \in I} \hat{\theta}_i$. We choose $\eta_i \in N$ such that $\hat{\theta}_i + \eta_i \in \Theta_i$. On the other hand, by Lemma A we have:

$$N = \sum_{i \in I} ((C_i \cap V^\perp) \cap N) = \sum_{i \in I} (C_i \cap N).$$

Thus there exists $\eta'_i \in C_i \cap N$ such that $\eta - \sum_{i \in I} \eta_i = \sum_{i \in I} \eta'_i$. Since $\eta'_i \in C_i$, $\hat{\theta}_i + \eta_i + \eta'_i$ must be in Θ_i . Thus we see that:

$$\theta = \eta + \sum_{i \in I} \hat{\theta}_i = \eta - \sum_{i \in I} \eta_i + \sum_{i \in I} (\hat{\theta}_i + \eta_i) = \sum_{i \in I} (\hat{\theta}_i + \eta_i + \eta'_i) \in \sum_{i \in I} \Theta_i.$$

It implies that $N + \sum_{i \in I} \hat{\Theta}_i \subset \sum_{i \in I} \Theta_i$. □

(ii) For each $i \in I$, let ψ_i denote the vertex of Θ_i . Since C_i is the recession cone of Θ_i , we have $\Theta_i = C_i + \psi_i$. The point ψ_i has the decomposition $\psi_i = \hat{\psi}_i + \tilde{\psi}_i$ where $\hat{\psi}_i \in V + M$ and $\tilde{\psi}_i \in N$. Let \hat{C}_i denote the projection of C_i onto $V + M$ for each $i \in I$. Since $\Theta_i = C_i + \psi_i$, we have $\hat{C}_i = Y(\hat{\Theta}_i)$ and $\hat{C}_i = \hat{\Theta}_i - \hat{\psi}_i$ for each $i \in I$. Suppose there exists a nonzero vector $\phi \in \mathbb{R}^J$ such that.

$$\phi \in \left(\sum_{i \in I} (\hat{C}_i \cap M) \right) \cap \left(- \sum_{i \in I} (\hat{C}_i \cap M) \right)$$

For each $i \in I$, we pick ϕ_i and ϕ'_i in $\hat{C}_i \cap M$ such that $\phi = \sum_{i \in I} \phi_i$ and $-\phi = \sum_{i \in I} \phi'_i$. Then there exists η_i and η'_i in N for all $i \in I$ such that $\phi_i + \eta_i \in C_i$ and $\phi'_i + \eta'_i \in C_i$. By Lemma A we can choose $\tilde{\eta}_i \in C_i \cap N$ such that $-\sum_{i \in I} \tilde{\eta}_i = \sum_{i \in I} (\eta_i + \eta'_i)$. It follows that for all $i \in I$,

$$\phi_i + \eta_i \in C_i \cap V^\perp, \quad \phi'_i + \eta'_i + \tilde{\eta}_i \in C_i \cap V^\perp,$$

and

$$\sum_{i \in I} (\phi_i + \phi'_i + \eta_i + \eta'_i + \tilde{\eta}_i) = \sum_{i \in I} (\phi_i + \phi'_i) = \phi - \phi = 0.$$

The last relation gives the relation:

$$\sum_{i \in I} (\phi'_i + \eta'_i + \tilde{\eta}_i) = - \sum_{i \in I} (\phi_i + \eta_i) \in \sum_{i \in I} (C_i \cap V^\perp).$$

Thus we see that $\sum_{i \in I} (\phi_i + \eta_i) \in \sum_{i \in I} (C_i \cap V^\perp)$ and $-\sum_{i \in I} (\phi_i + \eta_i) \in \sum_{i \in I} (C_i \cap V^\perp)$. Noting that N is the lineality space of $\sum_{i \in I} (C_i \cap V^\perp)$, we have $\sum_{i \in I} (\phi_i + \eta_i) \in N$. Since $\phi_i \in M$ for all $i \in I$, the above relation implies that $\sum_{i \in I} \phi_i \in M \cap N$ and therefore, $\sum_{i \in I} \phi_i = 0$. That is, $\phi = 0$, which is impossible.

(iii) Since $C_i \cap V^\perp \subset \Theta_i \cap V^\perp$ for each i , by definition, $N = \mathcal{L}\left(\sum_{i \in I} (C_i \cap V^\perp)\right) \subset \mathcal{L}\left(\sum_{i \in I} (\Theta_i \cap V^\perp)\right)$. We apply (i) and (ii) of this proposition to $\Theta_i \cap V^\perp$'s. Then there exist a subspace N' and Ψ_i in M for each i such that:

$$\begin{aligned} \sum_{i \in I} (\Theta_i \cap V^\perp) &= N' + \sum_{i \in I} \Psi_i \\ \left[\sum_{i \in I} (\Upsilon(\Psi_i) \cap M) \right] \cap \left[- \sum_{i \in I} (\Upsilon(\Psi_i) \cap M) \right] &= \{0\} \end{aligned}$$

These results imply that N' is the maximal subspace in

$\sum_{i \in I} (\Theta_i \cap V^\perp)$ or $N' = \mathcal{L}\left(\sum_{i \in I} (\Theta_i \cap V^\perp)\right)$. On the other hand, $N' + \sum_{i \in I} \Psi_i = \sum_{i \in I} (\Theta_i \cap V^\perp) \subset \sum_{i \in I} \Theta_i = N + \sum_{i \in I} \hat{\Theta}_i$. By (ii) of this proposition, we have $N' \subset N$ and therefore, $\mathcal{L}\left(\sum_{i \in I} (\Theta_i \cap V^\perp)\right) \subset N$. ■

PROOF OF THEOREM 3.1: (i) Let q be a price in Q^* . Suppose that for some $v \in N$, $q \cdot v \neq 0$. By Lemma A, we can choose $v_i \in C_i \cap N$ such that $v = \sum_{i \in I} v_i$. If $q \cdot v < 0$, there exists $i \in I$ such that $q \cdot v_i < 0$. By Assumption 3.1, we can choose $w_i \in C_i$ with $R \cdot w_i > 0$. Clearly, there exists $\lambda > 0$ such that $\lambda v_i + w_i \in C_i$ and $q \cdot (\lambda v_i + w_i) < 0$. It follows that:

$$\begin{aligned} \theta_i + \lambda v_i + w_i &\in \Theta_i, \\ q \cdot (\theta_i + \lambda v_i + w_i) &< q \cdot \theta_i \leq 0, \\ R \cdot (\theta_i + \lambda v_i + w_i) &= R \cdot (\theta_i + w_i) > R \cdot \theta_i \end{aligned}$$

This implies that there exists $x'_i \in X_i$ such that $u_i(x'_i) > u(x_i)$ and $(x'_i, \theta_i + \lambda v_i + w_i) \in \mathcal{B}_i(p, q)$. It contradicts the optimality of (x_i, θ_i) in $\mathcal{B}_i(p, q)$. If $q \cdot v > 0$, then $q \cdot (-v) < 0$. Since $-v \in N$, we can proceed with $-v$ in the same way as before to reach the same conclusion. Thus, $q \cdot v = 0$ for all $v \in N$ and therefore, $q \in V + M$.

(ii) Suppose that (p, q, x, θ) is an equilibrium for \mathcal{E} . For each $i \in I$, we have the decomposition $\theta_i = \hat{\theta}_i + \eta_i$ where $\hat{\theta}_i \in V + M$ and $\eta_i \in N$. Since $q \in V + M$ and $\eta_i \in N$ for each $i \in I$, we see that $R(q) \cdot (\hat{\theta}_i + \eta_i) = R(q) \cdot \hat{\theta}_i$. Therefore the pair $(p, q, x, \hat{\theta})$ is an equilibrium for $\hat{\mathcal{E}}$.

Conversely, suppose that $(p, q, x, \hat{\theta})$ is an equilibrium for $\hat{\mathcal{E}}$. We can choose $z_i \in N$ for each $i \in I$ such that $\hat{\theta}_i + z_i \in \Theta_i$. By Lemma A, there exists $z'_i \in C_i \cap N$, such that:

$$-\sum_{i \in I} z_i = \sum_{i \in I} z'_i.$$

We set $\theta_i = \hat{\theta}_i + z_i + z'_i$ for each $i \in I$. It follows that $\theta_i \in \Theta_i$ for each $i \in I$ and $\sum_{i \in I} \theta_i = \sum_{i \in I} \hat{\theta}_i = 0$. In summary we see that for each $i \in I$,

$$\theta_i \in \Theta_i, \quad R(q) \cdot \theta_i = R(q) \cdot \hat{\theta}_i, \text{ and } \sum_{i \in I} \theta_i = 0$$

Therefore the pair (p, q, x, θ) is an equilibrium for \mathcal{E} .

PROOF OF PROPOSITION 3.2: Let $v \in N$. By definition, there exists $v_i \in C_i \cap V^\perp$ such that $v = \sum_{i \in I} v_i$. This implies that $R \cdot v_i = 0$ for all $i \in I$. Let q be a price in Q . By Definition 3.3, we must have $-q \cdot v_i \leq 0$ or $q \cdot v_i \geq 0$ for all $i \in I$, and therefore, $q \cdot v \geq 0$. On the other hand, $-v$ is in N . By applying the same arguments to $-v$, we obtain $q \cdot (-v) \geq 0$. Therefore, we can conclude that $q \cdot v = 0$ or q satisfies the extended law of one price. ■

PROOF OF LEMMA 3.2: (i) Let $q \in \{q' \in V + M : q' \cdot v > 0 \text{ for all nonzero } v \in \Gamma\}$. Since $\Gamma_i \subset \Gamma$ for each $i \in I$, we see that $q \cdot v_i > 0$ for all $v_i \in \Gamma$. Since Θ_i is a cone with vertex, $\Upsilon(\hat{\Theta}_i)$ equals \hat{C}_i , the projection of C_i onto $V + M$. Thus, we have $q \in Q_i$ for all $i \in I$ and therefore, $q \in Q$. To prove the converse, let $q \in Q$. Then $q \in Q_i$ for all $i \in I$. For a nonzero point $v \in \Gamma$, there exists $v_i \in \Gamma_i \cup \{0\}$ for each $i \in I$ such that $v = \sum_{i \in I} v_i$. Then we have $q \cdot v_i > 0$ for all $i \in I$ with $v_i \neq 0$ and therefore, $q \cdot v > 0$. Thus we have $q \in \{q' \in V + M : q' \cdot v > 0 \text{ for all nonzero } v \in \Gamma\}$.

(ii) Clearly, $\bar{\Gamma}$ is convex. For closedness, it suffices to show that $\{cl(\Gamma_i) : i \in I\}$ are positively semi-independent, since $cl(\Gamma_i)$ is itself a closed convex cone. Pick $\theta_i \in cl(\Gamma_i)$ for every $i \in I$ such that $\sum_{i \in I} \theta_i = 0$. Then $R \cdot (\sum_{i \in I} \theta_i) = 0$. Since $\theta_i \in cl(\Gamma_i)$ implies $R \cdot \theta_i \geq 0$, we see that $R \cdot \theta_i = 0$ for every $i \in I$. It follows that $\theta_i \in \Upsilon(\hat{\Theta}_i) \cap M$ for every $i \in I$. By Proposition 3.1 we see that each θ_i is equal to 0. Hence $\{cl(\Gamma_i) : i \in I\}$ are positively semi-independent. To show that $\bar{\Gamma}$ is pointed, pick $\bar{\theta} \in \bar{\Gamma} \cap (-\bar{\Gamma})$. Then there exist $\theta_i \in cl(\Gamma_i)$ and $\theta'_i \in cl(\Gamma_i)$ such that $\bar{\theta} = \sum_{i \in I} \theta_i$ and $-\bar{\theta} = \sum_{i \in I} \theta'_i$, respectively. Since $R \cdot \theta_i \geq 0$ and $R \cdot \theta'_i \geq 0$, we have $R \cdot \bar{\theta} \geq 0$ and $R \cdot (-\bar{\theta}) \geq 0$, which implies $R \cdot \bar{\theta} = 0$. It follows that $\theta_i \in \Upsilon(\hat{\Theta}_i) \cap M$ for all $i \in I$ and therefore, $\bar{\theta} \in \sum_{i \in I} (\Upsilon(\hat{\Theta}_i) \cap M)$. By Proposition 3.1, we have $\bar{\theta} = 0$. ■

PROOF OF LEMMA 4.1: Since K is compact and u_i is continuous, $\hat{\xi}_i(p, q; K)$ is nonempty and compact. If ξ'_i is upper hemicontinuous, so is the convex hull $\hat{\xi}_i$.²³ Thus we have only to show the upper hemicontinuity of ξ'_i . Choose a sequence $\{(p^n, q^n, x_i^n, \theta_i^n)\}$ which converges to (p, q, x_i, θ_i) and satisfies $(x_i^n, \theta_i^n) \in \xi'_i(p^n, q^n; K)$ for all n .

Suppose that $(p, q) \in P \times cl(Q) \setminus R_{++}^{SL} \times Q$ or $\min q \cdot \theta_i = 0$. Then there are two possibilities:

- (a) $(p^n, q^n) \in R_{++}^{SL} \times Q$ and $\min q^n \cdot \theta_i < 0$ for infinitely many n or
- (b) $(p^n, q^n) \in P \times cl(Q) \setminus R_{++}^{SL} \times Q$ or $\min q^n \cdot \theta_i = 0$ for infinitely many n .

If (a) holds, then $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K)$ for infinitely many n . By definition, $(x_i, \theta_i) \in \varphi_i(p, q; K)$ and therefore, $(x_i, \theta_i) \in \xi'_i(p, q; K)$. If (b) holds, then $(x_i^n, \theta_i^n) \in \varphi_i(p^n, q^n; K)$ for infinitely many n . Without loss of generality, we may assume that $(x_i^n, \theta_i^n) \in \varphi_i(p^n, q^n; K)$ for all n . Then there exists $\{(p^{n,m}, q^{n,m})\}$ in $R_{++}^{SL} \times Q$ for each n such that $\min q^{n,m} \cdot \theta_i < 0$ for all m , $(p^{n,m}, q^{n,m}) \rightarrow (p^n, q^n)$ and $(x_i^{n,m}, \theta_i^{n,m}) \rightarrow (x_i^n, \theta_i^n)$ for some $(x_i^{n,m}, \theta_i^{n,m}) \in \xi_i(p^{n,m}, q^{n,m}; K)$. Recalling that $(x_i^n, \theta_i^n) \rightarrow (x_i, \theta_i)$, by the diagonal sequence theorem there exists a sequence $n_1 < n_2 < \dots < n_k < \dots$ such that $(x_i^{n_k}, \theta_i^{n_k}) \rightarrow (x_i, \theta_i)$. Thus we have $(x_i, \theta_i) \in \varphi_i(p, q; K)$ and therefore, $(x_i, \theta_i) \in \xi'_i(p, q; K)$.

We turn to the case that $(p, q) \in R_{++}^{SL} \times Q$ and $\min q \cdot \theta_i < 0$. Suppose that $(x_i, \theta_i) \in \mathcal{B}_i(p, q; K) \setminus \xi_i(p, q; K)$. Then there is $(\bar{x}_i, \bar{\theta}_i) \in \mathcal{B}_i(p, q; K)$ such that $u_i(\bar{x}_i) > u_i(x_i)$. It follows from the continuity of preferences that for some $\alpha \in (0, 1)$, $u_i(\alpha \bar{x}_i) > u_i(x_i)$. Since $e_i > 0$ and $p(s) \gg 0$ for all $s \in S$, we see that $q \cdot (\alpha \bar{\theta}_i) \leq 0$ and $p \square (\alpha \bar{x}_i - e_i) \ll R \cdot (\alpha \bar{\theta}_i)$.

Since $\min q \cdot \theta_i < 0$, we can choose $b_i \in \Theta_i$ with $q \cdot b_i < 0$. Then there exists a positive number $\beta < 1$ which satisfies $q \cdot (\beta \alpha \bar{\theta}_i + (1 - \beta)b_i) < 0$, $p \square (\beta \alpha \bar{x}_i - e_i) \ll R \cdot (\beta \alpha \bar{\theta}_i + (1 - \beta)b_i)$ and $u_i(\beta \alpha \bar{x}_i) > u_i(x_i)$. Then for sufficiently large n , we have

$$u_i(\beta \alpha \bar{x}_i) > u_i(x_i^n) \text{ and } (\beta \alpha \bar{x}_i, \beta \alpha \bar{\theta}_i + (1 - \beta)b_i) \in \mathcal{B}_i(p^n, q^n; K).$$

This contradicts the optimality of (x_i^n, θ_i^n) in $\mathcal{B}_i(p^n, q^n; K)$. ■

²³ For details on this point, see Hildenbrand(1974).

PROOF OF PROPOSITION 4.1: Since $\hat{\xi}_i(p^n, q^n; K_n)$ is the convex hull of $\xi'_i(p^n, q^n; K_n)$ for each n , we have only to verify the current proposition for ξ'_i . Thus without loss of generality we may assume that $(x_i^n, \theta_i^n) \in \hat{\xi}_i(p^n, q^n; K_n)$ for all n . To the contrary, suppose that $\{\sum_{i \in I} \|x_i^n\|\}$ is bounded. Then, $\{x_i^n\}$ is bounded for every $i \in I$. Decompose $\theta_i^n = \hat{\theta}_i^n + \tilde{\theta}_i^n$ where $\hat{\theta}_i^n \in V$ and $\tilde{\theta}_i^n \in V^\perp$. Then we see that for every n ,

$$p^n \square (x_i^n - e_i) = R \cdot \theta_i^n = R \cdot \hat{\theta}_i^n$$

Suppose that $\{\hat{\theta}_i^n\}$ is unbounded. Since $\{\hat{\theta}_i^n / \|\hat{\theta}_i^n\|\}$ is bounded, it has a subsequence converging to a nonzero vector $v_i \in V$. Recalling that $\{x_i^n\}$ and $\{p^n\}$ are bounded, we have $R \cdot v_i = 0$.

This implies $v_i = 0$, which is impossible. Since the sequence $\{\hat{\theta}_i^n\}$ is bounded, without loss of generality we can take $\{(x_i^n, \hat{\theta}_i^n)\}$ as a convergent subsequence. Let $(x_i, \hat{\theta}_i)$ denote its limit point in $X_i \times V$.

On the other hand, we see that for each $i \in I$ and each n ,

$$q^n \cdot \hat{\theta}_i^n + q^n \cdot \tilde{\theta}_i^n = 0.$$

Since $\{q^n \cdot \hat{\theta}_i^n\}$ is bounded, so is $\{q^n \cdot \tilde{\theta}_i^n\}$. Without loss of generality, we assume that $q^n \cdot \tilde{\theta}_i^n$ converges to a number c_i for each $i \in I$. Then $q \cdot \hat{\theta}_i + c_i = 0$ for each $i \in I$.

Let $L(s) = \{l \in L : p_l(s) = 0\}$ for each $s \in S$. Define $\delta(s) \in \mathbb{R}^L$ such that $\delta_l(s) > 0$ for every $l \in L(s)$ and $\delta_l(s) = 0$ for every $l \in L \setminus L(s)$. Set $\delta = (\delta(1), \dots, \delta(s))$. Then we see that $p \square \delta = 0$ and $p \square (x_i + \delta) \leq p \square e_i + R \cdot \hat{\theta}_i$. Since $p(s) > 0$ and $e_i(s) \gg 0$ for each $s \in S$, we have $p \square (\alpha x_i + \delta) \ll p \square e_i + R \cdot (\alpha \hat{\theta}_i)$ for all $\alpha \in (0, 1)$. It is also worth noting that if $p(s) \in \partial \mathbb{R}_+^L$ for some $s \in S$, then $\delta > 0$.

We can consider two possibilities for $\{(p^n, q^n)\}$; either (a) $(p^n, q^n) \in \mathbb{R}_{++}^{\text{SL}} \times Q$ and $\min q^n \cdot \Theta_i < 0$ for infinitely many n or (b) $(p^n, q^n) \in P \times cl(Q) \setminus \mathbb{R}_{++}^{\text{SL}} \times Q$ or $\min q^n \cdot \Theta_i = 0$ for infinitely many n . Suppose that (a) holds. Then $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K^n)$ for infinitely many n . Suppose that (b) holds. Then $(x_i^n, \theta_i^n) \in \varphi_i(p^n, q^n; K^n)$ for infinitely many n . By definition, there exists a sequence $\{(p^{n,m}, q^{n,m})\}$ in $\mathbb{R}_{++}^{\text{SL}} \times Q$

such that $\min q^{n,m} \cdot \Theta_i < 0$, $(p^{n,m}, q^{n,m}) \rightarrow (p^n, q^n)$ and $(x_i^{n,m}, \theta_i^{n,m}) \rightarrow (x_i^n, \theta_i^n)$ with $(x_i^{n,m}, \theta_i^{n,m}) \in \xi_i(p^{n,m}, q^{n,m}; K^n)$ for each m . Since $x_i^n \rightarrow x_i$ and $(p^n, q^n) \rightarrow (p, q)$, by the diagonal sequence theorem there exists a sequence $n_1 < n_2 < \dots < n_k < \dots$ such that $(x_i^{n_k}, \theta_i^{n_k}) \rightarrow (p, q)$ and $x_i^{n_k} \rightarrow x_i$. In particular, $(x_i^{n_k}, \theta_i^{n_k})$ is in $\xi_i(p^{n_k}, q^{n_k}; K^n)$.

Thus without loss of generality, we may assume that $(p^n, q^n) \in \mathbb{R}_{++}^{\text{SL}} \times Q$, $\min q^n \cdot \Theta_i < 0$, and $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K^n)$ for all n and all $i \in I$. Since $(p, q) \in \bar{\Delta} \setminus \Delta$, either 1) $p^n(s) \rightarrow p(s) \in \partial \mathbb{R}_+^L \setminus \{0\}$ for some $s = S$ or 2) $q^n \rightarrow q \in cl(Q) \setminus Q$ and $p^n \rightarrow p \gg 0$.

(CASE1) Suppose that $p^n(s) \rightarrow p(s) \in \partial \mathbb{R}_+^L \setminus \{0\}$ for some $s = S$. Then we can set δ in a way that $\delta(s) > 0$ and $\delta(s') = 0$ if $s' \neq s$. Since $\sum_{i \in I} e_i \gg 0$, there exists $i \in I$ such that $p(s) \cdot e_i(s) > 0$. Then we have $p(s) \cdot [\alpha(x_i(s) + \delta(s))] < p(s) \cdot e_i(s) + r(s) \cdot (\alpha \hat{\theta}_i)$ and $u_i[\alpha(x_i + \delta)] > u_i(x_i)$ for some $\alpha \in (0, 1)$. It follows that $u_i[\alpha(x_i^n + \delta)] > u_i(x_i^n)$ and $(\alpha(x_i^n + \delta), \alpha \theta_i^n) \in \mathcal{B}_i(p^n, q^n; K_n)$ for sufficiently large n , which contradicts the fact that $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K_n)$.

(CASE 2) Suppose that $q^n \rightarrow q \in cl(Q) \setminus Q$ and $p^n \rightarrow p \gg 0$. First of all, we claim that there exists $\gamma \in \Gamma \setminus \{0\}$ such that $q \cdot \gamma = 0$. Since $\Gamma \subset \bar{\Gamma}$ and $q \in cl(Q)$, $q \cdot \gamma' \geq 0$ for all $\gamma' \in \Gamma$. Now suppose that $q \cdot \gamma' > 0$ for all nonzero $\gamma' \in \Gamma$. Then by (i) of Lemma 3.2, we must have $q \in Q$, which is impossible. For each $i \in I$, we pick $\gamma_i \in \Gamma_i \cup (0)$ such that $\gamma = \sum_{i \in I} \gamma_i$. Since $q \cdot \gamma_i \geq 0$, $q \cdot \gamma = 0$ implies that $q \cdot \gamma_i = 0$ for each $i \in I$. There exists $i \in I$ such that $\gamma_i \neq 0$ and therefore, $R \cdot \gamma_i > 0$.

Since $\min q^n \cdot \Theta_i < 0$, there exists $\zeta_i^n \in \Theta_i$ for each n which satisfies $q^n \cdot \zeta_i^n < 0$. On the other hand, $R \cdot \gamma_i > 0$. Thus, there is a state $s^* \in S$ such that $r(s^*) \cdot \gamma_i > 0$. Thus we can pick a consumption $\tau \in \mathbb{R}_{++}^L$ such that $p(s^*) \cdot \tau < r(s^*) \cdot \gamma_i$. Choose $w \in \mathbb{R}_+^{\text{SL}}$ such that $w(s) = \tau$ if $s = s^*$ and $w(s) = 0$ if $s \in S \setminus \{s^*\}$. Clearly, $u_i(x_i + w) > u_i(x_i)$. Then there exists $\bar{\alpha} \in (0, 1)$ which satisfies $u_i(\alpha x_i + w) > u_i(x_i)$ for all $\alpha \in [\bar{\alpha}, 1]$. Recalling that $q^n \cdot \theta_i^n \leq 0$ and $q^n \cdot \zeta_i^n < 0$, we have $q^n \cdot (\alpha \theta_i^n + (1 - \alpha) \zeta_i^n) < 0$ for sufficiently large n . Since $p \gg 0$, $q \cdot \gamma_i = 0$ and $R \cdot \gamma_i > 0$, we can choose $\alpha \in (\bar{\alpha}, 1)$ and $\{\gamma_i^n\}$ in Θ_i such that $\gamma_i^n \rightarrow \gamma_i$, and for sufficiently large n ,

$$0 < -q^n \cdot [\alpha \theta_i^n + (1 - \alpha) \zeta_i^n + \gamma_i^n],$$

$$p^n \square (\alpha x_i + w - e_i) < R \cdot [\alpha \theta_i^n + (1 - \alpha) \zeta_i^n + \gamma_i^n].$$

This implies that for sufficiently large n , $u_i(\alpha x_i + w) > u_i(x_i^n)$ and $(\alpha x_i^n + w, \alpha \theta_i^n + (1 - \alpha) \zeta_i^n + \gamma_i^n) \in \mathcal{B}_i(p^n, q^n; K_n)$. This contradicts the fact that $(x_i^n, \theta_i^n) \in \xi_i(p^n, q^n; K_n)$. ■

Proof of Theorem 4.1

First we take some preliminary steps to prove Theorem 4.1. To apply fixed point theorems, we need to find a compact convex set of nonzero prices that generates $cl(Q)$. If $\bar{\Gamma}$ has the empty interior in $V + M$, then $cl(Q)$ is not a pointed cone, which makes it impossible to find such a price simplex. We set:

$$Q^\circ = \{q \in V + M : q \cdot v > 0 \text{ for all } v \in \bar{\Gamma} \setminus \{0\}\}.$$

By (ii) of Lemma 4.2, Q° is an open set in $V + M$ which is not a subspace. Lemma 8 of Debreu(1962) enables us to find a nondecreasing sequence of pointed convex cones in $Q^\circ \cup \{0\}$ whose union contains Q° .

LEMMA B: There exists a nondecreasing sequence $\{Q^n\}$ of pointed closed convex cones which satisfies in $Q^n \subset Q^\circ \cup \{0\}$ for all n and $Q^\circ \cup \{0\} = \bigcup_n Q^n$.

PROOF: By Lemma 8 of Debreu(1962), there exists a sequence $\{Q^n\}$ of closed, convex cones in $Q^\circ \cup \{0\}$ such that $n > m$ implies $Q^m \subset Q^n$ and $\bigcup_n Q^n$ contains the relative interior of $Q^\circ \cup \{0\}$. Since Q° is open in $V + M$, we see that $Q^\circ \subset \bigcup_n Q^n \subset Q^\circ \cup \{0\}$ and therefore, $\bigcup_n Q^n = Q^\circ \cup \{0\}$. We claim that each Q^n is pointed. Suppose that Q^m is not pointed for some m . Then $\mathcal{L}(Q^m) \neq \{0\}$. We also see that $\mathcal{L}(Q^m) \subset Q^{n'}$ for all $n' \geq m$. It implies $\mathcal{L}(Q^m) \subset \bigcup_n Q^n$ and therefore, $\mathcal{L}(Q^m) \subset Q^\circ \cup \{0\}$. Since Q° is not a subspace, neither is $cl(Q)$. These facts imply that $\mathcal{L}(Q^m)$ is in the relative boundary of $Q^\circ \cup \{0\}$. Since $\mathcal{L}(Q^m) \neq \{0\}$, it contradicts the openness of Q° in $V + M$. ■

Recalling that Q° is open in $V + M$, we may assume that each Q^n has the nonempty interior in $V + M$. For each n , we define the set:

$$\Gamma^n = \{\gamma \in V + M : q \cdot \gamma \geq 0, \forall q \in Q^n\}.$$

Each Γ^n is a pointed closed convex cone. Lemma B leads to the following result.

LEMMA C: The sequence $\{\Gamma^n\}$ is nonincreasing such that $\bar{\Gamma} \subset \Gamma^n$ for all n and $\bigcap_n \Gamma^n = \bar{\Gamma}$.

PROOF : Since $\{Q^n\}$ is nondecreasing, $\{\Gamma^n\}$ is nonincreasing. By Lemma B, $\bar{\Gamma}$ is in Γ^n for each n and therefore, $\bar{\Gamma} \subset \bigcap_n \Gamma^n$. Suppose that there exists $\gamma \in (\bigcap_n \Gamma^n) \setminus \bar{\Gamma}$. Then $\gamma \in \bar{\Gamma}$ for all n and $\gamma \notin \bar{\Gamma}$. We recall that Q° is open in $V + M$ and $\bar{\Gamma} = \{\gamma \in V + M : q \cdot \gamma \geq 0, \forall q \in Q^\circ\}$. Thus $\gamma \notin \bar{\Gamma}$ implies that $q \cdot \gamma < 0$ for some $q \in Q^\circ$. By Lemma B, there exists m which satisfies $q \in Q^m$. Since $\gamma \in \Gamma^m$, we have $q \cdot \gamma \geq 0$, which is contradictory. We conclude that $\bigcap_n \Gamma^n = \bar{\Gamma}$. ■

Since Q^n is pointed, Γ^n has the nonempty interior in $V + M$. Pick a portfolio θ_i^0 in the relative interior of $cl(\Gamma_i)$ for each $i \in I$. Then a portfolio $\theta^0 = \sum_{i \in I} \theta_i^0$ is in the interior of $\bar{\Gamma}$. Clearly $q \cdot \theta^0 > 0$ for all q in $cl(Q) \setminus \{0\}$. By Lemma C, θ^0 is in the relative interior of Γ^n in $V + M$ for sufficiently large n . Then we see that $q \cdot \theta^0 > 0$ for all nonzero $q \in Q^n$. We define the sets of normalized prices.

$$\begin{aligned} \Delta^n &= \{(p, q) \in \mathbb{R}_+^{\text{SL}} \times Q^n : \|q\| = 1, \sum_{l \in L} p_l(s) = 1 \text{ for all } s \in S\}, \\ \tilde{\Delta}^n &= \{(p, q) \in \mathbb{R}_+^{\text{SL}} \times Q^n : q \cdot \theta^0 = 1, \sum_{l \in L} p_l(s) = 1 \text{ for all } s \in S\}. \end{aligned}$$

Clearly, Δ^n is closed and $\Delta^n \subset \bar{\Delta}$ for all n .

PROOF OF THEOREM 4.1: (STEP 1) First, we apply the fixed point theorem to the truncated demand correspondences of the economy $\hat{\varepsilon}$. Let K be a rectangle in $\mathbb{R}^{\text{SL}} \times \mathbb{R}^J$ sufficiently large such that it contains $(\sum_{i \in I} e_i, 0)$. Let Δ' denote a nonempty, convex and compact set in

$P \times cl(Q)$. For **notational convenience**, we will keep the same notation \mathcal{B}_i , ξ_i and $\hat{\xi}_i$ for the budget, demand and artificial demand correspondences in the economy $\hat{\mathcal{E}}$ as in the economy \mathcal{E} .

Let $m = (p, q, z, w) \in \Delta' \times K$ and define the correspondence $\Psi(\cdot; K) = \mu^K \times \varsigma^K : \Delta' \times K \rightarrow \Delta' \times K$ by $\mu(m; K) = \{(p, q) \in \Delta' : q \cdot w \geq q' \cdot w, p \square z \geq p' \square z, \forall (p', q') \in \Delta'\}$, $\varsigma(m; K) = \sum_{i \in I} \hat{\xi}_i(p, q; K) - (\sum_{i \in I} e_i, 0)$. By Lemma 4.1, we can show that $\Psi(\cdot; K)$ is nonempty-valued, compact-valued, convex-valued, and upper hemicontinuous. By Kakutani's fixed point theorem, there is a fixed point $\bar{m} = (\bar{p}, \bar{q}, \bar{z}, \bar{w}) \in \Psi(\bar{m}; K)$.

(STEP 2) Let $\{K_n\}$ be an increasing sequence of rectangles in $\mathbb{R}^{SL} \times \mathbb{R}^J$ such that $\bigcup_n K_n = \mathbb{R}^{SL} \times \mathbb{R}^J$. By applying the result of STEP 1 to the case where Δ' and K are replaced by $\tilde{\Delta}^n$ and K_n , there exists a fixed point (p^n, q^n, z^n, w^n) for $\Psi(\cdot; K_n)$. Since $(z^n, w^n) \in \varsigma(p^n, q^n, z^n, w^n; K_n)$, there is an allocation (x^n, θ^n) such that $(x_i^n, \theta_i^n) \in \hat{\xi}_i(p^n, q^n; K_n)$ for each i , $\sum_{i \in I} (x_i^n - e_i) = z^n$ and $\sum_{i \in I} \theta_i^n = w^n$. Recalling that $q^n \cdot w^n \leq 0$, we see that $q \cdot w^n \leq 0$ for each $q \in Q^n$ which satisfies $q \cdot \theta^0 = 1$. In particular, we obtain $q \cdot w^n \leq 0$ for all $q \in Q^n$. The latter implies that $-w^n \in \Gamma^n$.

(STEP3) We show that $\{x_i^n\}$ and $\{\theta_i^n\}$ are bounded for all $i \in I$. Suppose that $\sum_{i \in I} \|x_i^n\| + \sum_{i \in I} \|\theta_i^n\| + \|w^n\| \rightarrow \infty$, where $\|\cdot\|$ denotes the Euclidean norm. We set $a^n = 1 / (\sum_{i \in I} \|x_i^n\| + \sum_{i \in I} \|\theta_i^n\| + \|w^n\|)$. Then $a^n \rightarrow 0$. Since $\{a^n w^n\}$, and $\{a^n x_i^n\}$ and $\{a^n \theta_i^n\}$ are bounded for each $i \in I$, they have a subsequence convergent to a point \dot{w}, \dot{x}_i , and $\dot{\theta}_i$, respectively. In particular, $\dot{x}_i \geq 0$ and $\dot{\theta}_i \in Y(\hat{\Theta}_i)$ for each $i \in I$. On the other hand, $\{p^n\}$ is bounded so that it has a subsequence convergent to a point p^* in \mathbb{R}^{LS} .

We claim that $-\dot{w} \in \bar{\Gamma}$. Suppose not. Then by Lemma C, there exists \bar{n} such that $-\dot{w} \notin \Gamma^{\bar{n}}$ and $\Gamma^n \subset \Gamma^{\bar{n}}$ for all $n \geq \bar{n}$. Since $\Gamma^{\bar{n}}$ is closed, there exists an open neighborhood B of $-\dot{w}$ in $V+M$ such that $\Gamma^{\bar{n}} \cap B = \emptyset$. Thus we have $\Gamma^n \cap B = \emptyset$ for all $n \geq \bar{n}$. Since $-a^n w^n \in \Gamma^n$, it implies that $-a^n w^n \notin B$ for all $n \geq \bar{n}$. It contradicts the fact that $-\dot{w}$ is an accumulation point $\{-a^n w^n\}$.

Recalling that $p \square \sum_{i \in I} (x_i^n - e_i) \leq R \cdot w^n$ for all $p \in \mathbb{R}_+^{SL} \setminus \{0\}$, we see

that:

$$p \square \sum_{i \in I} (a^n x_i^n - a^n e_i) \leq R \cdot (a^n w^n).$$

Passing to the limit, we have $p \square \sum_{i \in I} \dot{x}_i \leq R \cdot \dot{w}$. Since $-\dot{w} \in \bar{\Gamma}$, we have $R \cdot \dot{w} \leq 0$. This implies that $p \square \sum_{i \in I} \dot{x}_i \leq 0$ for all $p \in \mathbb{R}_+^{\text{SL}} \setminus \{0\}$, and therefore, $\sum_{i \in I} \dot{x}_i \leq 0$. Since $\dot{x}_i \geq 0$ for all $i \in I$, we see that $\dot{x}_i = 0$ for all $i \in I$, and $R \cdot \dot{w} = 0$.

Now by multiplying by a^n the relation $p^n \square (x_i^n - e_i) = R \cdot \theta_i^n$, we have $p^n \square (a^n x_i^n - a^n e_i) = R \cdot (a^n \theta_i^n)$. Passing to the limit we see that $0 = p \square \dot{x}_i = R \cdot \dot{\theta}_i$ or $\dot{\theta}_i \in M$ for each $i \in I$. Recalling that $\dot{\theta}_i \in Y(\hat{\Theta}_i)$, we must have $\dot{\theta}_i \in Y(\hat{\Theta}_i) \cap M$ for each i . Again by multiplying the relation $\sum_{i \in I} \theta_i^n = w^n$ by a^n and passing to the limit, we obtain $\sum_{i \in I} \dot{\theta}_i = \dot{w}$. Since $-\dot{w}_i \in \bar{\Gamma}$, there exists $\dot{w}_i \in cl(\Gamma_i)$ for each i such that $\sum_{i \in I} \dot{w}_i = -\dot{w}$. Recalling that $R \cdot \dot{w} = 0$ and $R \cdot \dot{w}_i \geq 0$, we obtain $R \cdot \dot{w}_i = 0$ for all $i \in I$. Thus we have $\dot{w}_i \in Y(\hat{\Theta}_i) \cap M$ for all $i \in I$.

It follows that $\sum_{i \in I} (\dot{\theta}_i + \dot{w}_i) = 0$ and $\dot{\theta}_i + \dot{w}_i \in Y(\hat{\Theta}_i) \cap M$ for all $i \in I$. By (ii) of Proposition 3.1, we must have $\dot{\theta}_i = \dot{w}_i = 0$ for all $i \in I$. In particular, $\dot{w}_i = 0$ for all $i \in I$ implies $\dot{w} = 0$. Thus we see that $\dot{w} = 0$, and $\dot{x}_i = 0$ and $\dot{\theta}_i = 0$ for each $i \in I$. But this result leads to the following contradiction

$$1 = a^n \frac{1}{a^n} = \sum_{i \in I} \|a^n x_i^n\| + \sum_{i \in I} \|a^n \theta_i^n\| + \|a^n w^n\| \rightarrow \sum_{i \in I} \|\dot{x}_i\| + \sum_{i \in I} \|\dot{\theta}_i\| + \|\dot{w}\| = 0$$

Therefore, we conclude that $\{x_i^n\}$ and $\{\theta_i^n\}$ are bounded for all $i \in I$.

(STEP 4) Now we set:

$$\dot{q}^n = \frac{q^n}{\|q^n\|}.$$

Clearly, $\{\dot{q}^n\}$ is bounded and $(p^n, \dot{q}^n) \in \Delta^n \subset \bar{\Delta}$ for all n . Without loss of generality, we can assume that $(p^n, \dot{q}^n, z^n, x^n, \theta^n) \rightarrow ((p^*, q^*), z^*, x^*, \theta^*) \in \Delta \times \mathbb{R}_+^{\text{SL}} \times \mathbb{R}_+^{\text{SLI}} \times (\prod_{i \in I} \Theta_i)$.

Since each $\{\theta_i^n\}$ is bounded, so is $\{w^n\}$. Thus it has a subsequence convergent to a point w^* . By the same argument used in Step 3, we can verify that $-w^* \in \bar{\Gamma}$. Thus $R \cdot w^* \leq 0$ and therefore, $z^* \leq 0$. Since $q^n \cdot w^n = 0$ for all n , we have $q^* \cdot w^* = 0$.

(STEP 5) For a sufficiently large rectangle K in $\mathbb{R}^{\text{SL}} \times \mathbb{R}^{\text{J}}$, each (x_i^*, θ_i^*) is in the interior of K . By Lemma 4.1, we see $(x_i^*, \theta_i^*) \in \hat{\xi}_i(p^*, q^*; K)$. Since $(p^n, q^n) \in \bar{\Delta}$ for all n , Proposition 4.1 allows us to have $(p^*, q^*) \in \Delta$ and therefore, $(x_i^*, \theta_i^*) \in \xi_i(p^*, q^*; K)$. Since K is not binding at (x_i^*, θ_i^*) , it is in $\xi_i(p^*, q^*)$.

Now we check the attainability condition for allocations. We choose $w_i \in cl(\Gamma_i)$ for each i such that $-w^* = \sum_{i \in I} w_i$. Recall that $R \cdot w_i \geq 0$ and $q^* \cdot w_i \geq 0$ for each $i \in I$. Since $q^* \cdot (-w^*) = 0$ and $R \cdot w^* \leq 0$, we have $R(q^*) \cdot w_i = 0$ for all $i \in I$. It follows that $(x_i^*, \theta_i^* + w_i) \in \xi_i(p^*, q^*)$ and $\sum_{i \in I} (\theta_i^* + w_i) = 0$.

On the other hand, the Walras' law implies that $p^* \square z^* = 0$. Since $z^* \leq 0$ and $p^n \gg 0$, we see that $z^* = 0$ or $\sum_{i \in I} (x_i^* - e_i) = 0$. Therefore a vector $(p^*, q^*, x^*, \theta^* + w) \in \Delta \times \mathbb{R}_+^{\text{SLI}} \times (\prod_{i \in I} \Theta_i)$ is an equilibrium of the economy $\hat{\mathcal{E}}$. ■

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