Fractional Group Identification

Wonki Jo Cho* Chang Woo Park†

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Abstract

We study group identification problems, the objective of which is to classify agents into (racial) groups based on opinions. Our point of departure from the literature is to allow memberships to be fractional, to qualify the extent of belonging. Examining implications of independence of irrelevant opinions, we provide characterizations of three nested families of decision rules. First, the independence axiom, together with a full-range condition, characterizes the weighted-average rules, namely those rules obtained by taking a weighted average of the one-vote rules by Miller (2008). Additionally imposing weak agentwise change leads to a subfamily of agentwise weighted-average rules, for which an agent's membership is determined only by the opinions that directly concern him. In the presence of symmetry, a rule should be a convex combination of five rules, which includes the liberal rule (self-identification) and the almost-column rule. Convex combinations of the latter two constitute a further subfamily of symmetrized agentwise weighted-average rules that value an agent's self-opinion and others' opinions differently. The main results of Cho and Ju (2017) follow as a corollary to ours.

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*Department of Economics, Korea University, Anam-ro 145, Seongbuk-gu, Seoul, South Korea; chowonki@korea.ac.kr

†School of Economics, Sogang University, Baekbeom-ro 35, Mapo-gu, Seoul, South Korea, 04107; chwpark@sogang.ac.kr

1 Introduction

Ethno-racial identity is a multi-faceted issue and it is not rare to find individuals seeing themselves as belonging to several groups, whether it be due to racial or cultural reasons. About 38.5 percent of Brazilians acknowledge their mixed heritage.[1] The 2010 U.S. Census shows that over 9 million people self-reported multiple racial backgrounds and they are projected to be the fastest growing demographic group in the United States.[2] In the United Kingdom, at least 3.5 percent of the babies born in 2005 are multi-racial. Motivated by these accounts, we study the problem of classifying individuals into (racial) groups, where their membership is permitted to be fractional to reflect varying degrees of belonging.

The group identification literature, initiated by a seminal work of Kasher and Rubinstein (1997), takes a social choice approach to the latter problem. A typical group identification problem consists of individual opinions on who belong to which group and the objective is to aggregate them into a single social decision. One may propose some properties that a reasonable (decision) rule should satisfy and characterize a rule or a family of rules by means of such properties. Starting from the binary case where the question is essentially to partition the agents into members and non-members (Kasher and Rubinstein, 1997; Samet and Schmeidler, 2003; Sung and Dimitrov, 2005; Dimitrov et al., 2007; Houy, 2007; and Çengelci and Sanver, 2010), the literature has developed to allow the group under study to vary (Miller, 2008; Cho and Ju, 2015) and identify three or more groups simultaneously (Cho and Ju, 2017).

A limitation with the existing studies is that they only deal with deterministic membership. If an agent belongs to a group, his identity is fully determined by that group. Yet an individual's identity may not be so clear-cut; it may spread across several groups, each constituting a differing portion of identity. To take account of this possibility, we study fractional membership in a multinary identification model. For instance, $1/2$ of an agent's identity may come from group a, $1/3$ from group b, and $1/6$ from group c. Though simple, our departure leads to a new, natural family of rules

[1] *Demographic Census 2000.* Brazilian Institute of Geography and Statistics.

[2] *The two or more races population: 2010.* United States Census Bureau. Published September 2012.

that can accommodate various degrees of self-determination and social consent. Our characterizations justify the rules from an axiomatic perspective.

Specifically, we extend the multinary model of Cho and Ju (2017), where there are three or more groups to be identified. Each agent has an opinion, stating to which one of the groups each agent (including himself) belongs. An (identification) problem is simply a profile of such opinions and is represented by an $n \times n$ matrix, where entry (i, j) is agent i's opinion about agent j (n is the number of agents). A (social) decision specifies the agents' fractional membership to all groups; that is, to each agent corresponds a profile of non-negative fractions adding up to one and representing the extent to which the agent belongs to the groups. Our main axiom for rules is independence of irrelevant opinions, which requires that identification of a given group should rely only on the opinions about that group. The axiom is implicit in the binary model of Kasher and Rubinstein (1997) and Samet and Schmeidler (2003) and the variable group model of Miller (2008). Cho and Ju (2017) explicitly formulate the axiom in a context where multiple groups are identified at the same time. The axiom has a natural analog in our fractional model: if the opinions on group a, say, remain the same, so should the fractional decisions on group a (while the fractional decisions on the other groups may vary).

With independence of irrelevant opinions imposed, a new family of rules emerge that value individual opinions differently to obtain a social decision. A typical rule in this family is associated with a profile of weights, each corresponding to an agent. Given a problem, the rule determines agent i's membership by taking a weighted average of all entries in the problem, using i's weights the rule is equipped with. We call these rules the weighted-average rules. We show that the weighted-average rules are the only rules satisfying independence of irrelevant opinions and a full-range condition called deterministic full range (Theorem 1). Deterministic full range requires that for each agent i and each group a, the range of a rule should include a decision where i is a full (i.e., with fraction 1) member of group a; and the axiom is weaker than unanimity. The one-vote rules (Miller, 2008), which determine each agent's identity using one fixed entry of a problem, are special (degenerate) cases of the weighted-average rules. Thus, our characterization of the weighted-average rules generalizes that of the one-vote rules by Cho and Ju (2007).

A drawback of the weighted-average rules is that agent i's membership may be determined by agent j's opinion about agent h. When we pursue identification on an individual basis, this is not desirable and we may wish to rule out such anomalies. The agentwise weighted-average rules are a subfamily of the weighted-average rules such that for each agent i, the weights determining his membership are positive only for those entries of a problem that directly concern him, namely those in column i. We characterize the agentwise weighted-average rules by independence of irrelevant opinions, deterministic full range, and weak agentwise change (Theorem 2). The last of the three properties requires that there exist at least one case where opinions about i are decisive enough to influence his membership in some deterministic fashion. Weak agentwise change is implied by some axioms studied in the literature, such as agentwise unanimity by Kasher and Rubinstein (1997) ("consensus" as the authors call it) and a different type of an independence axiom by Samet ans Schmeidler (2003).

Finally, we also consider symmetry, the requirement that all agents be treated symmetrically, with names having no impact on membership. Once symmetry is imposed in addition to independence of irrelevant opinions and deterministic full range, a rule should be a convex combination of five rules (Theorem 3). While some rules among the five are rather peculiar, two are interesting and can serve as a building block for constructing a rule that suits our purpose at hand. One is the liberal rule, which puts each agent in the group of his own choice. Also known as self-identification, the liberal rule is the most common way of collecting information on ethnicity and race (for example, most censuses use the liberal rule). The other rule is what we call the almost-column rule, according to which agent i's membership is the simple average of the entries in column i except entry (i, i), that is, all opinions about i except his self-opinion. Taking a convex combination of the liberal and almost-column rules, we obtain the symmetrized agentwise weighted-average rules, a further subfamily of the agentwise weighted-average rules. Each symmetrized agentwise weighted-average rule has the advantage of determining an agent's identity valuing his self-opinion and the others' opinions about him differently while satisfying symmetry. Clearly, under the assumption of deterministic decisions, only the liberal rule satisfies all the axioms and the main characterization of Cho and Ju (2017) follows as a corollary.

Our results indicate that the small departure of permitting decisions to be fractional unveils a new family of rules that afford the flexibility of basing individual identity on several opinions. This strength cannot be achieved by deterministic rules because of their very deterministic nature: two or more entries of a problem cannot be decisive at the same time. The weighted-average rules, on the other hand, can compromise on their decisiveness by assigning weights to them and the distribution of weights represents the emphasis we place on one's self-opinion and the others' opinions.

The connection between our results and Cho and Ju (2017) parallels that between Gibbard-Satterthwaite Impossibility Theorem (Gibbard, 1973; Satterthwaite, 1975) and Gibbard Random Dictatorship Theorem (Gibbard, 1977) in social choice theory. In a voting environment where agents submit ordinal preferences over alternatives, the first theorem says, each strategy-proof voting scheme is dictatorial (assuming that the scheme includes at least three outcomes). When a voting scheme is allowed to be probabilistic, the second theorem shows, strategy-proofness (together with ex post efficiency) implies random dictatorship: a fixed distribution over agents determines the dictator at random, who in turn chooses his most preferred alternative. In sum, the passage from the deterministic to the probabilistic setup adds randomizations over the voting schemes that are already available in the deterministic setup. The weighted-average rules generalize the one-vote rules in much the same way as random dictatorship does dictatorship. The distinction, of course, is that symmetry is compatible with independence of irrelevant opinions in the deterministic identification model whereas a similar axiom (anonymity) is not compatible with strategy-proofness in the deterministic voting model.

Related Literature

Group identification begins with an axiomatic analysis of Kasher and Rubinstein (1997). In a binary setup where agents are to be identified as members or non-members of a group under question, they characterize the liberal rule and derive an impossibility result. Samet and Schmeidler (2003) propose and characterize the consent rules that depending on the choice of parameters, can embed varying degrees of liberalism and social consent in decisions. Other papers studying the binary model include Sung and Dimitrov (2005), Dimitrov et al. (2007), Houy (2007), and Çengelci and Sanver (2010). On a more general domain where opinions can be "neutral", Ju (2010, 2013) explores decisiveness of an agent or a group of agents and characterizes self-dependency, a hallmark of liberalism, by a weaker set of axioms.

Most closely related to this paper is Cho and Ju (2017), who consider the multi-nary problem of identifying three or more groups simultaneously. Noting an implicit assumption in the binary model, they introduce independence of irrelevant opinions, an adaptation of Arrow's (1951) independence axiom in preference aggregation theory to the group identification setting. They show that the one-vote rules uniquely satisfy the axiom (together with non-degeneracy) and that only the liberal rule obtains once symmetry is additionally imposed. Clearly, the weighted-average rules in this paper are a fractional counterpart of the one-vote rules and our characterizations extend Cho and Ju (2017).

On the other hand, Miller (2008) considers a variable group model with the focus on consistency (or separability as Miller calls it), a relational property that binds decisions across groups; e.g., the decision on group "a and b" should be the conjunction of the decisions on group "a" and on group "b". The one-vote rules first appear in this context and Miller (2008) shows that they are the only rules satisfying consistency (together with non-degeneracy). While different axioms are imposed, Miller's (2008) findings are very similar to Cho and Ju (2017). To clarify the distinction, Cho and Ju (2015) introduce an extended setup that subsumes the two papers and derive stronger characterizations.

The rest of the paper proceeds as follows. In Section 2, we set up the model. In Section 3, we investigate the connection between independence of irrelevant opinions and a stronger, simplifying property called decomposability. Main characterizations are in Section 4.

2 The Model

A finite set of agents seek to determine their memberships to several groups, with membership being allowed to be fractional. Let $N \equiv \{1, \cdots, n\}$ be the set of agents and $G \equiv \{k_1, \cdots, k_m\}$ the set of groups ($n \geq 2$ and $m \geq 3$). Agents are denoted by i and j, and groups by k and ℓ. Each agent $i \in N$ has an **opinion** $\boldsymbol{P_i} \equiv (P_{ij}) \in G^N$, where for all $j \in N$, $P_{ij} = k \in G$ means that i believes j to be a member of group k. A (multinary) **problem** is an opinion profile $\boldsymbol{P} \equiv (P_i)_{i \in N}$. We often treat individual opinions as row vectors $(1 \times n)$ and problems as matrices $(n \times n)$, with P_{ij} being entry (i, j) of P. Let $\boldsymbol{\mathcal{P}} \equiv G^{N \times N}$ be the set of all problems. For each $P \in \mathcal{P}$ and each $i \in N$, let $\boldsymbol{P^i} \equiv (P_{ji})_{j \in N}$ be column i of P, which is the opinions about i; and let $\boldsymbol{P^{-i}} \equiv (P^j)_{j \in N \setminus \{i\}}$ be all the columns of P but column i.

Let $\boldsymbol{\Delta(G)}$ be the set of distributions over G. A (social) **decision** is a profile $x \equiv (x_i)_{i \in N}$, where for each $i \in N$, $x_i \equiv (x_{ik})_{k \in G} \in \Delta(G)$ is agent i's fractional membership to all groups. Let $\boldsymbol{X} \equiv (\Delta(G))^N$ be the set of all decisions. Given decision $x \in X$, for each $i \in N$ and each $k \in G$, we interpret x_{ik} as an index measuring the extent to which agent i belongs to group k (in particular, we do not interpret x_{ik} as the probability of agent i belonging to group k). An agent may affirm affiliation to multiple racial groups and our formulation with fractions attempts to capture the composition of his identity.

A (social decision) **rule** is a mapping $f : \mathcal{P} \to X$, associating with each problem a decision. Decision $x \in X$ is **deterministic** if for each $i \in N$, x_i is a degenerate distribution (i.e., for some $k \in G$, $x_{ik} = 1$). A rule is **deterministic** if it always produces a deterministic decision. Decisions and rules are **fractional** if they are not deterministic. In the deterministic group identification models (Samet and Schmeidler, 2003; Miller, 2008; Cho and Ju, 2017), both binary and multinary, the following rules have been studied. The **liberal rule**, denoted $\boldsymbol{L}$, classifies each agent into the group of his own choice; i.e., for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $P_{ii} = k$ implies $L_{ik}(P) = 1$. A rule f is a **one-vote rule** if each agent's identity is determined by a single fixed entry for all problems; i.e., for each $i \in N$, there is $(j, h) \in N^2$ such that for each $P \in \mathcal{P}$ and each $k \in G$, $P_{jh} = k$ implies $f_{ik}(P) = 1$—we call (j, h) the **decisive entry** for i. The decisive entry for i is not required to be in column i, in which case, the membership decision for i is based on an opinion about some other agent. An **agentwise one-vote rule** is a one-vote rule such that for each $i \in N$, the decisive entry for agent i is in column i.

We can easily extend the one-vote rules to the fractional setup by introducing "fractional decisiveness". A rule f is a **weighted-average rule** if for each $i \in N$, there exists $\alpha_i \in \Delta(N^2)$ such that for each $P \in \mathcal{P}$ and each $k \in G$, $f_{ik}(P) = \sum_{(j,h) \in N^2 : P_{jh}=k} \alpha_i(j,h)$ (where $\alpha_i(j,h)$ is the weight assigned to entry (j,h) by α_i). We call $(\alpha_i)_{i \in N}$ the **weights associated with $\boldsymbol{f}$**. By definition, a weighted-average rule is a convex combination of the one-vote rules. As is the case with the one-vote rules, a weighted-average rule is allowed to put positive weights on the opinions not about agent i (i.e., the entries outside column i) when determining i's membership, which is not desirable when we seek a decision on an individual basis. A rule f is an **agentwise weighted-average rule** if it is an weighted-average rule with the associated weights $(\alpha_i)_{i \in N}$ such that for each $i \in N$, α_i puts positive weights only on the opinions about i (i.e., $\sum_{j \in N} \alpha_i(j,i) = 1$). A rule f is a **symmetrized agentwise weighted-average rule** if there is $s, t \in [0,1]$ with $s + t = 1$ such that for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = s\mathbf{1}_{\{P_{ii}=k\}} + \frac{t}{n-1} |\{j \in N \backslash \{i\} : P_{ji} = k\}|$, where $\mathbf{1}_{\{P_{ii}=k\}}$ is an indicator function. Under a symmetrized agentwise weighted-average rule, each i's membership is a weighted average of his self-opinion and the others' opinions about him; the others' opinions about him are all valued equally (but not necessarily as equally as i's self-opinion); and the weights (s,t) applied when taking a weighted average is the same across all agents.

Below we use the following notation. For each $k \in G$, $\boldsymbol{k_{n \times n}}$ is a problem in $\mathcal{P}$ consisting of k's only; a row vector $\boldsymbol{k_{1 \times n}}$ and a column vector $\boldsymbol{k_{n \times 1}}$ are similarly defined. For all $x_i, y_i \in \Delta(G)$, let $\boldsymbol{||x_i - y_i||} = \max_{k \in G} |x_{ik} - y_{ik}|$. Note that $||x_i - y_i|| = 1$ if and only if for some $k \in G$, one of x_{ik} and y_{ik} is 1 and the other is 0. In particular, if there is $k \in G$ with $x_{ik}, y_{ik} > 0$, then $||x_i - y_i|| < 1$.

Axioms

How should the membership to a group be affected when the opinions about the other groups change? Does the identity of a Hispanic depend on whether other agents view him as a White or as an Asian? In the context of ethnic classification, each ethnicity is treated as an independent entity. Therefore, it is natural to argue that when determining a group, changes in the opinions about the other groups should be dismissed as irrelevant. In the deterministic identification model, Cho and Ju (2017) introduce the latter inter-group independence and call it independence of irrelevant opinions, noting its resemblance to Arrow's (1951) independence axiom in preference aggregation theory. In our fractional setting, the independence axiom requires that an agent's *fractional* membership to a group should remain the same regardless of changes in the opinions about the other groups. A formal expression of this idea is as follows.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{P}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for each $i \in N$, $f_{ik}(P) = f_{ik}(P')$.

Concerning fairness, we require that all agents be treated symmetrically. This idea is expressed by a permutation over the agents, which represents a change in their names. Let $\pi : N \to N$ be a permutation. For each $P \in \mathcal{P}$, let $\boldsymbol{P_\pi} \equiv (P_{\pi(i),\pi(j)})_{i,j \in N}$ and $\boldsymbol{f_\pi(P)} \equiv \big(f_{\pi(i)}(P)\big)_{i \in N}$ (for each $x \in X$, x_π is defined similarly). The following property, due to Samet and Schmeidler (2003), says that name changes should not affect the membership decision for the agents.

Symmetry. For each $P \in \mathcal{P}$ and each permutation $\pi : N \to N$, $f(P_\pi) = f_\pi(P)$.

In the deterministic case, independence of irrelevant opinions has strong implications when combined with the following axiom, which requires that no agent's membership should be fixed.

Non-degeneracy. For each $i \in N$, there are $P, P' \in \mathcal{P}$ such that $f_i(P) \neq f_i(P')$.

Non-degeneracy, however, is too weak in the fractional setup to pin down a family of rule we are interested in. Thus, we consider a slightly stronger requirement: when restricted to deterministic decisions, a rule should have a full range for each agent.

Deterministic Full Range. For each $i \in N$ and each $k \in G$, there is $P \in \mathcal{P}$ such that $f_{ik}(P) = 1$.

Deterministic full range says nothing about what decision a rule should assign to particular problems. Yet with independence of irrelevant opinions imposed, the axiom indeed prescribes reasonable decisions to those problems that consist of a single group. The following property is stronger than deterministic full range.

Unanimity. For each $i \in N$ and each $k \in G$, $f_{ik}(k_{n\times n}) = 1$.

While independence of irrelevant opinions turns out to pin down quite a small class of rules, it has no bite on how opinions about agent i should be valued when agent j's membership is determined. In fact, none of the axioms introduced so far, including independence of irrelevant opinions, compels a rule to distinguish columns from rows, which is why a rule satisfying independence of irrelevant opinions may put a positive weight on the information that does not directly concern agent j. Now we introduce two axioms that allow us to remedy such problems. In a binary setup, Kasher and Rubinstein (1997) consider an agent-by-agent version of the above unanimity property, requiring that if all agents have the same opinion about an agent, then the social decision should respect that opinion. This property can be adapted to the multinary setup as follows.

Agentwise Unanimity. For each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, if $P^i = k_{n\times 1}$, then $f_{ik}(P) = 1$.[3]

To weaken agentwise unanimity, consider problems $P, P' \in \mathcal{P}$. Suppose that for some agent i, P and P' differ only in the opinions about i (i.e., $P^i \neq P'^i$ and $P^{-i} = P'^{-i}$). If the membership decision for i is never affected as the problem changes from P to P', one could argue that opinions about i play no role in determining his identity, which is not desirable. Thus, we require that there exist a case where opinions about i are decisive enough to influence his membership in some deterministic fashion.

Weak Agentwise Change. For each $i \in N$, there are $P, P' \in \mathcal{P}$ such that $P^i \neq P'^i$, $P^{-i} = P'^{-i}$, and $||f_i(P) - f_i(P')|| = 1$.

Even when combined with unanimity, weak agentwise change is weaker than agentwise unanimity. Samet and Schmeidler (2003) propose a property saying that to determine an agent's membership, a rule should focus only on the opinions about him—they call the property "independence". Clearly, the independence axiom has direct bearing

[3] Kasher and Rubinstein (1997) call this property consensus.

on decisions across agents and it is stronger than weak agentwise change in the presence of unanimity.

3 Decomposability

A simple way of solving a multinary problem is to transform it into a list of "binary" problems (one for each group), obtain a "binary" decision for each binary problem, and then combine all binary decisions into a multinary decision. This approach, called *decomposition*, turns out to be very close to the requirement of independence of irrelevant opinions in the deterministic setup (Cho and Ju, 2017). In this section, we re-examine that relationship in the fractional case.

Given $P \in \mathcal{P}$, for each $k \in G$, the **binary problem concerning group $\boldsymbol{k}$ derived from $\boldsymbol{P}$**, denoted $\boldsymbol{B^{P,k}} \in \{0,1\}^{N\times N}$, is defined as for all $i,j \in N$, $B^{P,k}_{ij} = 1$ if $P_{ij} = k$ and $B^{P,k}_{ij} = 0$ otherwise. Let $\mathcal{B} \equiv \{0,1\}^{N\times N}$ be the set of all binary problems. Our definition of opinion requires that each agent be a member of one and only one group. Thus, each multinary problem P can alternatively be represented by m binary problems $(B^{P,k})_{k\in G}$. For each $B \in \mathcal{B}$, let $|\boldsymbol{B}| \equiv \sum_{i,j\in N} B_{ij}$ be the number of ones in problem B. A **binary decision** is a profile $b \equiv (b_i)_{i\in N} \in [0,1]^N$, where for each $i \in N$, b_i is agent i's fractional membership to the group under question. A list of m binary decisions $(b^k)_{k\in G}$ translates to a proper multinary decision if and only if for each $i \in N$, $\sum_{k\in G} b^k_i = 1$.

Since multinary problems can be expressed as a collection of binary problems, one may ask if a rule can also be expressed as a function operating on binary problems. Formally, an **approval function** is a mapping $\varphi : \mathcal{B} \to [0,1]^N$, associating with each binary problem $B \in \mathcal{B}$ a binary decision $\varphi(B) \in [0,1]^N$. A natural counterpart of the weighted-average rules for approval functions are those with fixed weights on the entries of a binary problem: an approval function φ is a **weighted-average approval function** if for each $i \in N$, there exists $\alpha_i \in \Delta(N^2)$ such that for each $B \in \mathcal{B}$, $\varphi_i(B) = \sum_{(j,h)\in N^2 : B_{jh}=1} \alpha_i(j,h)$.

The following property requires that a rule be expressed in the form of an approval function.

Decomposability. There is an approval function $\varphi : \mathcal{B} \to [0,1]^N$ such that for each

$P \in \mathcal{D}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$.

In this case, we say that $\boldsymbol{f}$ **is represented by** $\boldsymbol{\varphi}$. Clearly, a weighted-average rule is decomposable, represented by a weighted-average approval function. Further, decomposability is stronger than independence of irrelevant opinions. An approval function can serve to represent a decomposable rule only if it satisfies a number of properties. Exploring the latter properties is instructive since it simplifies our investigation of independence of irrelevant opinions and decomposability to that of properties of approval functions, which are more analytically tractable. To define such properties, an approval function φ is **m-unit-additive** if for all m binary problems $B^1, \ldots, B^m \in \mathcal{B}$, $\sum_{k \in G} B^k = 1_{n\times n}$ implies $\sum_{k\in G} \varphi(B^k) = 1_{1\times n}$. It is **unanimous** if $\varphi(0_{n\times n}) = 0_{1\times n}$ and $\varphi(1_{n\times n}) = 1_{1\times n}$. Given $B \in \mathcal{B}$, the **dual binary problem of** $\boldsymbol{B}$ is denoted $\overline{\boldsymbol{B}} \equiv 1_{n\times n} - B$ (i.e., for all $i, j \in N$, $\left(\overline{B}\right)_{ij} = 1 - B_{ij}$); similarly, given $b \in [0,1]^N$, the **dual binary decision of** $\boldsymbol{b}$ is denoted $\overline{\boldsymbol{b}} = 1_{1\times n} - b$. The **dual of** $\boldsymbol{\varphi}$, denoted by $\boldsymbol{\varphi^d}$, is the approval function such that for all $B \in \mathcal{B}$, $\varphi^d(B) = \overline{\varphi(\overline{B})}$. We say that φ is **self-dual** if $\varphi = \varphi^d$. Finally, φ is **monotonic** if for all $B, B' \in \mathcal{B}$ such that $B \leq B'$, $\varphi(B) \leq \varphi(B')$.

Now we show that the ability for an approval function to represent a decomposable rule is equivalent to m-unit-additivity.

Proposition 1. *An approval function represents a decomposable rule if and only if it is m-unit-additive.*

Proof. First, we prove the "only if" part. Suppose that an approval function φ represents a decomposable rule f. Let $B^1, \ldots, B^m \in \mathcal{B}$ be such that $\sum_{k\in G} B^k = 1_{n\times n}$. Then there exists $P \in \mathcal{P}$ such that for all $k \in G$, $B^k = B^{P,k}$. Since φ represents f, for each $i \in N$ and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$. Thus, by the definition of fractional decisions, $\sum_{k\in G} \varphi_i(B^k) = \sum_{k\in G} \varphi_i(B^{P,k}) = \sum_{k\in G} f_{ik}(P) = 1$ and φ is m-unit-additive.

Next, to prove the "if" part, let φ be an m-unit-additive approval function. Define a rule f using φ as follows: for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$. Then f is well-defined by m-unit-additivity of φ. Thus, f is decomposable and is represented by φ. $\square$

In the deterministic setup of Cho and Ju (2017), decomposability is equivalent to the combination of independence of irrelevant opinions and non-degeneracy. The latter equivalence, however, fails in our fractional model. The following example makes this point.

Example 1 (A rule satisfying independence of irrelevant opinions and non-degeneracy, but not decomposability)**.** We consider a variant of the one-vote rules. Fix $a \in G$ and for each $i \in N$, $(j_i, h_i) \in N^2$. A rule f^* is such that each agent $i \in N$ belongs to group a with at least fraction $\frac{1}{2}$ and the remaining fraction is determined solely by entry (j_i, h_i). That is, for each $P \in \mathcal{P}$ and each $i \in N$, (i) if $P_{j_i h_i} = a$, then $f^*_{ia}(P) = 1$; and (ii) if $P_{j_i h_i} = b \in G \backslash \{a\}$, then $f^*_{ia}(P) = \frac{1}{2} = f^*_{ib}(P)$. Clearly, f^* is independent of irrelevant opinions and non-degenerate. However, f^* is not decomposable. To see this, suppose, by contradiction, that f^* is decomposable and is represented by an approval function φ. Consider agent $1 \in N$. Since $f^*_1(a_{n\times n}) = 1$, decomposability implies that $\varphi_1(1_{n\times n}) = 1$. Now let $b \in G \backslash \{a\}$. Since $f^*_{1b}(b_{n\times n}) = \frac{1}{2}$, decomposability again implies $\varphi_1(1_{n\times n}) = \frac{1}{2}$, a contradiction. $\triangle$

For our purpose, it is enough to identify an axiom that when combined with independence of irrelevant opinions, implies decomposability. It turns out that deterministic full range suffices. To prove this, we first establish that under the assumption of independence of irrelevant opinions, deterministic full range is equivalent to unanimity.

Lemma 1. *In the presence of independence of irrelevant opinions, deterministic full range is equivalent to unanimity.*

Proof. We only show that under the assumption of independence of irrelevant opinions, deterministic full range implies unanimity (the converse is clear). Let f be a rule satisfying independence of irrelevant opinions and deterministic full range. For each $k \in G$, define an approval function φ^k as follows: for each $B \in \mathcal{B}$ and each $i \in N$, $\varphi^k_i(B) = f_{ik}(P)$, where $P \in \mathcal{P}$ is such that $B^{P,k} = B$. The m approval functions $\left(\varphi^k\right)_{k\in G}$ are well-defined because f satisfies independence of irrelevant opinions.

Step 1: *For each $i \in N$ and each $k \in G$, $\varphi^k_i(0_{n\times n}) = 0$.*

Let $i \in N$ and $k \in G$. By deterministic full range, there exists $P \in \mathcal{P}$ such that $f_{ik}(P) = 1$. Let $\ell, h \in G \setminus \{k\}$ be distinct. Let $P' \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P'_{jj'} = k$ if and only if $P_{jj'} = k$; and (ii) $P'_{jj'} = h$ if and only if $P_{jj'} \neq k$. By independence of irrelevant opinions, $f_{ik}(P') = f_{ik}(P) = 1$, so that $f_{i\ell}(P') = 0$. Thus, $\varphi_i^\ell(0_{n\times n}) = \varphi_i^\ell(B^{P',\ell}) = f_{i\ell}(P') = 0$. Since our choice of k is arbitrary, the claim follows.

Step 2: *f is unanimous.*

Suppose, by contradiction, that there exist $i \in N$ and $k \in G$ such that $f_{ik}(k_{n\times n}) < 1$. Then, there is $\ell \in G \setminus \{k\}$ such that $f_{i\ell}(k_{n\times n}) > 0$. That is, $\varphi_i^\ell(0_{n\times n}) = \varphi_i^\ell(B^{k_{n\times n},\ell}) = f_{i\ell}(k_{n\times n}) > 0$, contradicting Step 1. □

Now we show that decomposability follows from independence of irrelevant opinions and deterministic full range.

Proposition 2. *Independence of irrelevant opinions and deterministic full range together imply decomposability.*

Proof. Let f be a rule satisfying independence of irrelevant opinions and deterministic full range. Then f is represented by a profile of m approval functions $(\varphi^k)_{k\in G}$ (as in the proof of Lemma 1). By Lemma 1, f is unanimous. Now, we proceed in two steps.

Step 1: *For each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, if k is not one of the entries of P, $f_{ik}(P) = 0$.*

Let $P \in \mathcal{P}$, $i \in N$, and $k \in G$. Assume that k is not one of entries of P. Let $\ell \in G\setminus\{k\}$ and consider $P' \equiv \ell_{n\times n}$. By Lemma 1, $f_{i\ell}(P') = 1$, so that $f_{ik}(P') = 0$. Now applying independence of irrelevant opinions to P and P', $f_{ik}(P) = f_{ik}(P') = 0$.

Step 2: $\varphi^1 = \varphi^2 = \cdots = \varphi^m$.

Suppose, by contradiction, that there are $k, \ell \in G$ such that $\varphi^k \neq \varphi^\ell$. Then there are $B \in \mathcal{B}$ and $i \in N$ such that $\varphi_i^k(B) \neq \varphi_i^\ell(B)$. Let $h \in G \setminus \{k, \ell\}$. Let $P \in \mathcal{P}$ be such that for all $j, j' \in N$, (i) $P_{jj'} = h$ if and only if $B_{jj'} = 0$; and (ii) $P_{jj'} = k$ if and only if $B_{jj'} = 1$. Similarly, let $P' \in \mathcal{P}$ be such that (i) $P'_{jj'} = h$ if and only if $B_{jj'} = 0$; and $P'_{jj'} = \ell$ if and only if $B_{jj'} = 1$. By construction, $B^{P,k} = B^{P',\ell} = B$. Also, since $B^{P,h} = B^{P',h}$, independence of irrelevant opinions implies $f_{ih}(P) = f_{ih}(P')$. Since for each $a \in G \setminus \{k, h\}$, a is not one of the entries of P, Step 1 implies $f_{ia}(P) = 0$. Thus, $\sum_{k'\in G} f_{ik'}(P) = f_{ik}(P) + f_{ih}(P) = 1$. That is, $f_{ih}(P) = 1 - f_{ik}(P) = 1 - \varphi_i^k(B^{P,k}) = 1 - \varphi_i^k(B)$. Similarly, for each $a \in G \setminus \{\ell, h\}$, $f_{ia}(P') = 0$, so that $f_{ih}(P') = 1 - \varphi_i^\ell(B)$. However, $\varphi_i^k(B) \neq \varphi_i^\ell(B)$ implies $f_{ih}(P) \neq f_{ih}(P')$, a contradiction. □

In the fractional setup, decomposability alone fails to imply some basic properties of approval functions, which stands in contrast with what is known in the deterministic setup. For instance, an approval function representing a decomposable rule may violate unanimity, monotonicity, and self-duality. The following example illustrates this point.

Example 2. Let $N = \{1, 2\}$ and $G = \{a, b, c\}$. Let $t \in [0, \frac{1}{2}]$ and define an approval function φ as follows: for each $B \in \mathcal{B}$ and each $i \in N$, $\varphi_i(B) = t$ if $B = 0_{2\times 2}$; $\frac{t+1}{4}$ if $|B| = 1$; $\frac{1-t}{2}$ if $|B| = 2$; $\frac{3-5t}{4}$ if $|B| = 3$; and $1 - 2t$ if $B = 1_{2\times 2}$. It is simple to show that φ is m-unit-additive (for each $t \in [0, \frac{1}{2}]$). Define a rule f by means of φ as follows: for each $P \in \mathcal{P}$, each $i \in N$, and each $k \in G$, $f_{ik}(P) = \varphi_i(B^{P,k})$. By construction, f is represented by φ and therefore, it is decomposable. Nevertheless, φ is not unanimous (neither is f) unless $t = 0$. Also, φ is monotonic if and only if $t \in [0, \frac{1}{3}]$. Finally, φ is not self-dual unless $t = 0$. △

The approval function defined in Example 2 can work to define a decomposable rule while failing several reasonable properties. Our next result indicates that we may escape those failures by additionally imposing deterministic full range.

Proposition 3. *Let f be a rule satisfying decomposability and deterministic full range. Then an approval function φ that represents f is unanimous, self-dual, and monotonic.*

Proof. Suppose that f satisfies decomposability and deterministic full range and that f is represented by φ. By Lemma 1, f is unanimous. By Proposition 1, φ is m-unit-additive.

To show that φ is unanimous, let $i \in N$. For each $k \in G$, $1 = f_{ik}(k_{n\times n}) = \varphi_i(B^{k_{n\times n},k}) = \varphi_i(1_{n\times n})$. Let $B^1 = 1_{n\times n}$ and for all $k \in G \setminus \{1\}$, $B^k = 0_{n\times n}$. Then, $1 = \sum_{k\in G} \varphi_i(B^k) = \varphi_i(1_{n\times n}) + (m-1)\varphi_i(0_{n\times n}) = 1 + (m-1)\varphi_i(0_{n\times n})$. Thus, $\varphi_i(0_{n\times n}) = 0$.

To show that φ is self-dual, let $B^1 = B$, $B^2 = \overline{B}$ and for all $k \in G \setminus \{1, 2\}$, $B^k = 0_{n\times n}$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, m-unit-additivity and unanimity imply $1_{1\times n} = \sum_{k\in G} \varphi(B^k) = \varphi(B) + \varphi(\overline{B})$. Thus, $\varphi(B) = \overline{\varphi(\overline{B})}$.

To show that φ is monotonic, suppose, by contradiction, that there are $i \in N$ and $B, B' \in \mathcal{B}$ with $B \leq B'$ such that $\varphi_i(B) > \varphi_i(B')$. Let $B^1 = B$ and $B^2 = \overline{B'}$. Let $B^3, \ldots, B^m \in \mathcal{B}$ be such that $\sum_{k\in G} B^k = 1_{n\times n}$ (such $B^3, \ldots, B^m$ exist since $B^1 + B^2 \leq$

$B + \overline{B} = 1_{n\times n}$). By m-unit-additivity and self-duality,

$$\begin{aligned} 1 &= \sum_{k\in G} \varphi_i(B^k) \\ &= \varphi_i(B) + \varphi_i(\overline{B'}) + \sum_{k\in G\setminus\{1,2\}} \varphi_i(B^k) \\ &> \varphi_i(B') + \varphi_i(\overline{B'}) + \sum_{k\in G\setminus\{1,2\}} \varphi_i(B^k) \\ &= 1 + \sum_{k\in G\setminus\{1,2\}} \varphi_i(B^k), \end{aligned}$$

a contradiction. □

4 Main Results

With preliminary observations on independence of irrelevant opinions at hand, we are now ready to explore its consequences in detail. In the deterministic case, independence of irrelevant opinions and non-degeneracy characterize the one-vote rules (Cho and Ju, 2017). An exact counterpart of the latter characterization in the fractional setup—namely that the two axioms characterize the weighted-average rules—does not hold (see Example 1). However, the weighted-average rules are the only rules satisfying independence of irrelevant opinions and the stronger axiom of deterministic full range.

Theorem 1. *A rule satisfies independence of irrelevant opinions and deterministic full range if and only if it is a weighted-average rule.*

Proof. In the proof, we use the following notation. A binary problem $B \in \mathcal{B}$ is a **unit binary problem** if $|B| = 1$ (i.e., there is only one unity in B). For all $j, h \in N^2$, let $\boldsymbol{U^{jh}} \in \mathcal{B}$ be the unit binary problem such that $U^{jh}_{jh} = 1$.

We omit the simple proof of the "if" part. To prove the "only if" part, let f be a rule satisfying independence of irrelevant opinions and deterministic full range. By Proposition 2, f is decomposable and is represented by an approval function φ. Recall that decomposability of f is equivalent to m-unit-additivity of φ. Moreover, since f is unanimous and decomposable, φ is unanimous, self-dual, and monotonic. Now it suffices to show that φ is a weighted-average approval function.

Step 1: For each $i \in N$ and each $B \in \mathcal{B}\backslash\{0_{n\times n}\}$, $\varphi_i(B) = \sum_{(j,h)\in N^2:B_{jh}=1} \varphi_i(U^{jh})$.

We prove the claim by induction. Clearly, $(*)$ the claim is true for each $B \in \mathcal{B}$ with $|B| = 1$. Let $\ell \in \mathbb{N}$ be such that $\ell < n^2$. Suppose that $(**)$ for each $i \in N$ and each $B \in \mathcal{B}$ with $|B| \leq \ell$, $\varphi_i(B) = \sum_{(j,h)\in N^2:B_{jh}=1} \varphi_i(U^{jh})$. Let $i \in N$ and let $B \in \mathcal{B}$ be such that $|B| = \ell + 1$. Define m binary problems $\left(B^k\right)_{k\in G}$ as follows: (i) $|B^1| = 1$, $|B^2| = \ell$, and $B^1 + B^2 = B$; (ii) $B^3 = \overline{B}$; and (iii) for each $k \in G\backslash\{1, 2, 3\}$, $B^k = 0_{n\times n}$. Since $\sum_{k\in G} B^k = 1_{n\times n}$, m-unit-additivity implies $\sum_{k\in G} \varphi_i(B^k) = 1$. By unanimity, for each $k \in G\backslash\{1, 2, 3\}$, $\varphi_i(B^k) = 0$. Now self-duality and the induction hypothesis $(**)$ imply $\varphi_i(B) = \overline{\varphi_i(\overline{B})} = 1 - \varphi_i(\overline{B}) = \varphi_i(B^1) + \varphi_i(B^2) = \sum_{(j,h)\in N^2:B_{jh}=1} \varphi_i(U^{jh})$. Finally, the claim follows from $(*)$ and the induction argument.

Step 2: φ is a weighted-average approval function.

Let $i \in N$. For each $(j, h) \in N^2$, let $\alpha_i(j, h) \equiv \varphi_i(U^{jh})$. By unanimity and Step 1, $\sum_{(j,h)\in N^2} \alpha_i(j, h) = \sum_{(j,h)\in N^2} \varphi_i(U^{jh}) = \varphi_i(1_{n\times n}) = 1$. Thus, $\alpha_i \in \Delta(N^2)$. Now by construction and Step 1, φ is a weighted-average approval function, with the associated weights $(\alpha_i)_{i\in N}$. □

Remark 1. The axioms in the theorem are logically independent. The variants of the one-vote rules in Example 1 satisfy independence of irrelevant opinions but not deterministic full range. It is simple to construct rules satisfying the latter axiom but not the former. △

The characterization of the one-vote rules in the deterministic setup follows as a simple corollary if we restrict Theorem 1 to the class of deterministic rules. To prove, suppose that f is a deterministic rule satisfying independence of irrelevant opinions and non-degeneracy. Then f satisfies unanimity (and hence deterministic full range). By Theorem 1, f is a weighted-average rule. Now the deterministic nature of f implies that it is a one-vote rule.

Corollary 1 (Cho and Ju, 2017)**.** *A deterministic rule satisfies independence of irrelevant opinions and non-degeneracy if and only if it is a one-vote rule.*

Neither independence of irrelevant opinions nor deterministic full range constrains the extent to which agent i's membership relies on opinions about the other agents.

Imposing weak agentwise change, we can ensure that positive weights are given only to the opinions about i.

Theorem 2. *A rule satisfies independence of irrelevant opinions, deterministic full range, and weak agentwise change if and only if it is an agentwise weighted-average rule.*

Proof. The simple proof of the "if" part is omitted. To prove the "only if" part, let f be a rule satisfying the three axioms. By Theorem 1, f is a weighted-average rule, with, say, the associated weights $(\alpha_i)_{i\in N} \in (\Delta(N^2))^N$. Let $i \in N$. It suffices to show that for each $(j,h) \in N^2$ with $h \neq i$, $\alpha_i(j,h) = 0$. Suppose, by contradiction, that for some $(j,h) \in N^2$ with $h \neq i$, $\alpha_i(j,h) > 0$. Let $P, P' \in \mathcal{P}$ be such that $P^i \neq P'^i$ and $P^{-i} = P'^{-i}$. Letting $k \equiv P_{jh} = P'_{jh} \in G$, $f_{ik}(P) \geq \alpha_i(j,h) > 0$ and $f_{ik}(P') \geq \alpha_i(j,h) > 0$. Thus, $||f_i(P) - f_i(P')|| < 1$, violating weak agentwise change. □

Remark 2. To verify the independence of the axioms in the theorem, Theorem 1 provides rules satisfying all but weak agentwise change. The variant of the one-vote rules in Example 1 such that for each $i \in N$, the decisive entry (j_i, h_i) for i is in column i (i.e., $h_i = i$), satisfies all but deterministic full range. Finally, for a rule satisfying all but independence of irrelevant opinions, fix $a \in G$ and consider the following: for each $P \in \mathcal{P}$ and each $i \in N$, (i) if there is $k \in G$ with $P^i = k_{n\times 1}$, then $f_{ik}(P) = 1$; (ii) otherwise, $f_{ia}(P) = 1$. △

Since agentwise unanimity implies both unanimity and weak agentwise change, we obtain the following corollary.

Corollary 2. *A rule satisfies independence of irrelevant opinions and agentwise unanimity if and only if it is an agentwise weighted-average rule.*

Now we impose a fairness axiom, symmetry. In the deterministic case, the axiom singles out the liberal rule in the family of one-vote rules. With fractional memberships permitted, however, a few other rules emerge and any convex combination of those rules satisfies symmetry as well as other axioms. Below we introduce four members in the family of one-vote rules, some of which are rather peculiar.

The **almost-column rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{ji})_{j \in N\setminus\{i\}}$ (i.e., for each $k \in G$, $f_{ik}(P) = \frac{1}{n-1}|\{j \in N\setminus\{i\} : P_{ji} = k\}|$). The **almost-row rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{ij})_{j \in N\setminus\{i\}}$. The **almost-diagonal rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{jj})_{j \in N\setminus\{i\}}$. The **almost-off-diagonal rule** is the rule f such that for each $P \in \mathcal{P}$ and each $i \in N$, $f_i(P)$ is the simple average of $(P_{jh})_{j,h \in N\setminus\{i\}, j \neq h}$ (this rule is well-defined only when $n \geq 3$).

Theorem 3. *Assume that there are at least three agents ($n \geq 3$). A rule satisfies independence of irrelevant opinions, deterministic full range, and symmetry if and only if it is a convex combination of the liberal, almost-column, almost-row, almost-diagonal, and almost-off-diagonal rules.*

Proof. We only prove the "only if" part. Let f be a rule satisfying the three axioms. By Theorem 1, f is a weighted-average rule, with the associated weights $(\alpha_i)_{i \in N} \in (\Delta(N^2))^N$. Throughout the proof, let $a, b \in G$ be distinct groups.

Step 1: *For each $i \in N$ and all $j, h, j', h' \in N\setminus\{i\}$ with $j \neq h$ and $j' \neq h'$, $\alpha_i(j,h) = \alpha_i(j',h')$ (this step applies only when $n \geq 3$).*

Let $i \in N$. Let $j, h, j', h' \in N\setminus\{i\}$ with $j \neq h$ and $j' \neq h'$. We first show that when $h \neq h'$, $\alpha_i(j,h) = \alpha_i(j,h')$. Let $P \in \mathcal{P}$ be such that $P_{jh} = a$ and all the other entries of P are b. Then $f_{ia}(P) = \alpha_i(j,h)$. Let $\pi : N \to N$ be a transposition that swaps h and h' only. Then $f_{ia}(P_\pi) = \alpha_i(j,h')$. Using $\pi(i) = i$ and symmetry, $f_i(P) = f_{\pi(i)}(P) = f_i(P_\pi)$, so that $\alpha_i(j,h) = f_{ia}(P) = f_{ia}(P_\pi) = \alpha_i(j,h')$.

Next, we show that when $j \neq j'$, $\alpha_i(j,h') = \alpha_i(j',h')$. Let $P \in \mathcal{P}$ be such that $P_{jh'} = a$ and all the other entries of P are b. Then $f_{ia}(P) = \alpha_i(j,h')$. Let $\pi : N \to N$ be a transposition that swaps j and j' only. Then $f_{ia}(P_\pi) = \alpha_i(j',h')$. Using $\pi(i) = i$ and symmetry, $f_i(P) = f_{\pi(i)}(P) = f_i(P_\pi)$, so that $\alpha_i(j,h') = f_{ia}(P) = f_{ia}(P_\pi) = \alpha_i(j',h')$.

Step 2: *For each $i \in N$ and all $j, h \in N\setminus\{i\}$, (i) $\alpha_i(j,j) = \alpha_i(h,h)$; (ii) $\alpha_i(j,i) = \alpha_i(h,i)$; and (iii) $\alpha_i(i,j) = \alpha_i(i,h)$.*

This can be proved by an argument similar to that in Step 1.

Step 3: *Concluding.*

By symmetry, $\alpha_1(1,1) = \alpha_2(2,2) = \cdots = \alpha_n(n,n)$. Let $s \equiv \alpha_1(1,1)$. By Steps 1 and 2, for each $i \in N$, we may let $t_i \equiv \alpha_i(j,j)$ for some $j \in N \backslash \{i\}$; $u_i \equiv \alpha_i(j,i)$ for some $j \in N \backslash \{i\}$; $v_i \equiv \alpha_i(i,j)$ for some $j \in N \backslash \{i\}$; and $w_i \equiv \alpha_i(j,h)$ for some $j,h \in N \backslash \{i\}$ with $j \neq h$. Again by symmetry, (t_i, u_i, v_i, w_i) does not depend on i and we can write $t \equiv t_1 = \cdots = t_n$, $u \equiv u_1 = \cdots = u_n$, $v \equiv v_1 = \cdots = v_n$, and $w \equiv w_1 = \cdots = w_n$. Since $\alpha_i \in \Delta(N^2)$, $s + (n-1)t + (n-1)u + (n-1)v + (n^2 - 3n + 2)w = 1$. Now it is clear that f is obtained by taking a convex combination of the five rules in the theorem with weights s, $(n-1)t$, $(n-1)u$, $(n-1)v$, and $(n^2-3n+2)w$, respectively. □

Remark 3. When there are only two agents ($n = 2$), the almost-diagonal, almost-column, and almost-row rules are all deterministic and the almost-off-diagonal rule does not exist. Thus, for $n = 2$, the rules satisfying the three axioms in the theorem are of the following form: there exist weights $s, t, u, v \in [0,1]$ with $s + t + u + v = 1$ such that for each $P \in \mathcal{P}$ and all $i, j \in N$ with $i \neq j$, $f_i(P)$ is the weighted average of P_{ii}, P_{ij}, P_{ji}, and P_{jj}, with the associated weights s, t, u, and v, respectively. This characterization for the two-agent case follows from the proof of the above theorem (the only difference is that Step 1 is no longer needed). △

Interesting among the five rules in Theorem 3 are the liberal and almost-column rules. To identify agent i's membership, the liberal rule puts full weight on his self-opinion; and the almost-column rule puts full weight on the others' opinions about i, or "social consent" on i's membership, but the weight is evenly distributed to the $n-1$ opinions. Each symmetrized agentwise weighted-average rule is a convex combination of the liberal and almost-column rules and can therefore be viewed as a compromise between two principles in determining individual identity: liberalism and social consent. Imposing weak agentwise change, we can characterize the symmetrized agentwise weighted-average rules.

Corollary 3. *Assume that there are at least three agents ($n \geq 3$). A rule satisfies independence of irrelevant opinions, deterministic full range, weak agentwise change, and symmetry if and only if it is a symmetrized agentwise weighted-average rule.*

It also follows from Corollaries 2 and 3 that the symmetrized agentwise weighted-average rules are the only rules satisfying independence of irrelevant opinions, agentwise unanimity, and symmetry.

The characterization of the liberal rule in the deterministic setup (Cho and Ju, 2017) obtains as a corollary to Theorem 3. Except the liberal rule, the four rules in the theorem are all fractional. Thus, a deterministic rule satisfying independence of irrelevant opinions, non-degeneracy, and symmetry, should put zero weight on the four rules, implying that it is the liberal rule.

Corollary 4 (Cho and Ju, 2017)**.** *Assume that there are at least three agents ($n \geq 3$). A deterministic rule satisfies independence of irrelevant opinions, non-degeneracy, and symmetry if and only if it is the liberal rule.*

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