Multidimensional Second-Price and English Auctions*

Seungwon (Eugene) Jeong†

May 21, 2017

Abstract

This paper presents novel graph-based mechanisms, the "multidimensional second-price" (MSP) auction and the "multidimensional English" auction. In high-stakes auctions externalities are prevalent, where an agent's type is multidimensional. While the Vickrey-Clarke-Groves mechanism may require losers to pay, widely used one-dimensional auctions also have many problems including inefficiency, regret of not bidding high or low enough, and the following new "group winner regret" problem: when there are three bidders, two bidders might compete against each other unnecessarily and have worse payoffs than if they had lost to the third bidder. In contrast, MSP is a unique direct mechanism that is free of a loser's payment, pairwise stable, and has good incentive properties, including no group winner regret. In MSP, the winner cannot win at any lower price, and losers cannot be better off winning by misreporting. MSP is strategyproof for a bidder who does not suffer externalities, and it reduces to the second-price auction when there are no externalities. Simulations suggest that MSP outperforms the second-price auction in terms of both revenue and efficiency.

JEL Classification: D44, D62, D47, C78

Keywords: externalities, group winner regret, collusion, monotonicity, core, graph

---

*Please check `http://tinyurl.com/MSPpaper` for the most up-to-date version.

†Department of Economics, University of Bristol, The Priory Road Complex, Bristol, BS8 1TU, UK. eugene.jeong@gmail.com. This is based on the first chapter of my PhD dissertation. I am deeply indebted to my advisors Fuhito Kojima and Peter Glynn, and to my dissertation committee members Jeremy Bulow, and Ilya Segal, for invaluable advice, discussions, encouragement, and guidance. I am grateful to Peter Biro, Gabriel Carroll, Itay Fainmesser, Eun Jeong Heo, Timo Hiller, Matthew Jackson, Philippe Jehiel, Jinwoo Kim, Vijay Krishna, SangMok Lee, Jonathan Levin, Paul Milgrom, Joshua Mollner, Stephen Nei, Michael Ostrovsky, Bobak Pakzad-Hurson, Svetlana Pevnitskaya, Alvin Roth, Shigehiro Serizawa, Fanqi Shi, Andy Skrzypacz, Rakesh Vohra, Siyang Xiong, and seminar participants at Bilkent, Columbia, Durham, Ipek, ITU, KAIST BTM, KAIST KCB, KDI/KDI School, KERI, POSTECH, SNU TEMEP, UCL, UCSC, U of Virginia, and Vanderbilt. This research was supported by the Korea Foundation for Advanced Studies.

1 Introduction

The sale of Toshiba's NAND flash memory unit in 2017 drew many large tech companies' attention around the world.[1] As in Jehiel et al. (1996), the seminal paper on auctions with externalities, (identity-dependent) externalities existed. Not only companies running their own large semiconductor unit (e.g., Broadcom, Western Digital, Micron Technology, SK Hynix) but also other big tech companies (e.g., Apple, Amazon, Google) participated in bidding. Notably, Apple's participation led to the consideration of participation by other companies such as Samsung, since Apple is not only their strongest competitor in the smartphone industry but also a giant customer of their semiconductor unit. That is, each bidder had a different willingness to pay in order to beat each competitor, i.e., their type is multidimensional. Likewise, the seller, i.e., Toshiba (and the Japanese government), also had externalities imposed by buyers. Toshiba hoped to sell it to Japanese companies but not to Chinese companies even if they offer a substantially higher offer (about $30 billion), fearing for China's fast growing tech industries, which may eventually threaten Japanese industries.[2]

Externalities are prevalent when buyers are commercial organizations due to different degrees of competition depending on competitors, e.g., M&A deals, as in the Toshiba's case; and spectrum license sales, where externalities can depend on the competitor's coverage map. In addition, when participants care about externalities, it is likely a high-stakes deal. These externalities often cause a delay in negotiation (Jehiel and Moldovanu, 1995) because bidders' willingness to pay or even participate depends on who will win when they lose. Due to externalities, even losing buyers may want to pay in order to avoid certain undesirable allocations (Jehiel et al., 1996); however, in many cases *loser's payment-free* (LPF) mechanisms are used, as in the Toshiba and Kepco examples. What is a "good" LPF mechanism when externalities exist? There has been no practical multidimensional mechanism thus far, which leads to many problems. This paper is the first to present graph-based LPF multidimensional mechanisms, which also naturally generalize standard one-dimensional mechanisms.[3]

[1] See `http://www.reuters.com/article/us-toshiba-accounting-chip-sale-idUSKBN17335J`, link to "Apple, Amazon, Google Join Bidding for Toshiba Chip Unit" (accessed April 21, 2017). The NAND flash is a semiconductor device that is used for USB flash drive, solid-state drives, etc.

[2] See `https://www.bloomberg.com/news/articles/2017-04-10/foreign-bidders-said-more-aggressive-in-toshiba-chip-unit-sale`, link to "Foreign Bidders Prove More Aggressive in Toshiba Chip Sale" (accessed April 21, 2017). A Chinese company, Hon Hai, also bid much more aggressively to face down Japan government opposition to win control of another Japanese company, Sharp, in 2016.

[3] Jehiel et al. (1996) present a multidimensional optimal but not LPF auction. Jehiel et al. (1999) show the conditions for incentive compatible and individually rational multidimensional auctions, but not present any specific multidimensional auction, and then focus on one-dimensional LPF auctions. The menu auction (Bernheim and Whinston, 1986) is also not LPF when externalities exist. See also Jehiel and Moldovanu (1996), Caillaud and Jehiel (1998), Jehiel and Moldovanu (1999), Segal (1999), Jehiel and Moldovanu (2000), Varma (2002), Aseff and Chade (2008), Figueroa and Skreta (2009), Rhee (2010), Figueroa and Skreta (2011),

Even when externalities exist, one-dimensional bid auctions, e.g., a *scalar* bid in sealed-bid auctions and *one* "button" (for either "stay" or "drop out") against all competitors in English auctions (Milgrom and Weber, 1982), are often used. Finding a one-dimensional bidding strategy is quite difficult,[4] and one-dimensional auctions have many other problems. First, one-dimensional mechanisms are inefficient. In addition, bidders may regret not bidding high or low enough. For instance, in the Kepco's auction, suppose LG also participated and Hyundai was willing to pay up to \$10 billion against Samsung but only up to \$5 billion against all others including LG. Hyundai does not know who will win if they lose, but their Bayesian optimal bid is between \$5 and \$10 billion, say \$7 billion, and they bid it. However, if Samsung wins with a bid lower than \$10 billion, Hyundai will regret not bidding high enough, but if LG is the second-highest bidder with a bid higher than \$5 billion, Hyundai will regret bidding too high. Note that these regrets are not due to the first-price auction's "bid-shading." The same problems happen in the second-price auction.

Moreover, a group of bidders may have "*group winner regret*" (GWR), which has not been studied in the auction literature to the best of my knowledge.[5] Suppose three companies Hyundai, Samsung, and LG participate in an English auction, assuming Hyundai and Samsung are competing against each other intensely but not against LG. Then, even if Hyundai and Samsung no longer want to compete with LG above a certain price, Hyundai and Samsung may compete against each other unnecessarily and eventually receive worse payoffs than if they had dropped out together before LG. However, if only Hyundai drops out, then Samsung may be better off staying until LG drops out, and Hyundai will regret dropping out alone. This regret therefore cannot be solved individually. That is, although Hyundai and Samsung competed with each other before, they have incentives to collude by dropping out together at a certain price, as "yesterday's enemy is today's friend." Thus, it is desirable to make a mechanism GWR-free, since it removes the collusion incentive to avoid GWR and also tends to increase efficiency.[6]

Hu et al. (2013), Jeziorski and Segal (2015), and Chen and Micali (2016).

[4] See De Castro and Karney (2012), for instance. One-dimensional mechanisms lack closed-form solutions in general even for three-bidder cases; thus, the literature only studies three-bidder cases with some restrictions on the externality structure (e.g., symmetric or circular externalities, externality imposed by only one bidder, fixed-size externalities, etc). Jehiel et al. (1999) show the unique symmetric equilibrium of the second-price auctions (which will be used for simulations in Appendix A) for symmetric cases, i.e., every bidder's type is drawn from the same distribution.

[5] The following example may seem intuitive, but the formal definition of group winner regret is nontrivial since we cannot say Hyundai and Samsung regret not losing to LG unless the winning of LG can be "justified" in some sense. Jehiel and Moldovanu (1996) show two equilibria in the first-price auction where Hyundai and Samsung either "race" or "coordinate," but they neither define the regret nor present a mechanism free of the regret. See also Laffont and Martimort (1997, 2000); Che and Kim (2006); Rhee (2007) on collusions.

[6] The intuition is that if many bidders prefer another allocation, it is likely that the current allocation is inefficient, which is also supported by simulations in Appendix A. In some models, including the example in

I present a novel multidimensional direct mechanism, the "*multidimensional second-price*" (MSP) auction, and its open ascending version, the "*multidimensional English*" (ME) auction.[7] In MSP, the winner cannot win at a price lower than some bidder's bid against the winner. That is, MSP is *pairwise stable* (Theorem 2), i.e., one bidder and the auctioneer cannot block the outcome; thus, it does not suffer the "low revenue problem." MSP has good incentive properties, including no GWR. The winner cannot win at any lower price by any misreport (*no overpay regret*), and losing bidders cannot be better off winning by any misreport (*no overturn regret*) (Theorem 3). Furthermore, MSP is strategyproof for a bidder who does not suffer externalities. When there are no externalities at all, MSP and ME reduce to the standard one-dimensional second-price (SP) and English auctions, respectively (Theorem 4). Therefore, the new mechanisms not only have good properties, but can also be seen as natural multidimensional extensions of the standard mechanisms, which makes them easily understandable by and acceptable to both bidders and the seller.[8] Note that extensions of SP and English auctions into multiple dimensions are not trivial. Comparing bids pairwise to determine the winner can induce a cycle, e.g., bidder 1 beats bidder 2, and bidder 2 beats bidder 3, but bidder 3 beats bidder 1. Thus, it is nontrivial to define the winner and the second-price. In addition, simulations (in Appendix A) suggest that MSP outperforms SP in terms of both revenue and efficiency. The intuition behind this is that GWR-freeness tends to increase efficiency, and pairwise stability tends to increase both revenue and efficiency. While MSP does not satisfy some desirable properties in mechanism design, those properties are impossible in any mechanisms (Theorem 1), and MSP satisfies "next-best" alternatives. Furthermore, MSP is a unique direct mechanism that satisfies certain good properties (Theorem 5). MSP is also the first to use a (network) graph, called the *bid graph*, for auctions mechanisms; thus, this paper also provides a new approach to auction mechanism design.[9]

In addition, I define *group winner regret-free monotonicity* (GWRF-monotonicity). For a nontrivial mechanism to have good incentive properties, the allocation rule normally need

this paragraph, GWR implies inefficiency, but this is not true in general. However, achieving exact efficiency leads to other problems, as will be explained in the remainder of the introduction.

[7] In MSP, Hyundai, in the Kepco example, now can submit separate bids of \$10 and \$5 billion against Samsung and LG, respectively (and a bid against the seller). The seller can also be treated as a bidder. In the Toshiba example, the seller, Toshiba, can bid higher against Chinese companies than Japanese companies.

[8] "Easiness to understand" is practically important, e.g., the generalize second-price (GSP) auction (Edelman et al., 2007; Varian, 2007) is widely used for online advertising auctions due to its simplicity (i.e., prices are natural generalization of SP) despite its lack of incentive compatibility (IC). In contrast, the Vickrey-Clarke-Groves mechanism is IC but difficult to explain the payment rule to advertisers. See additional discussions in the conclusion.

[9] The bid graph approach is different from another good use of graphs in mechanism design, i.e., the network *flow* approach (e.g., Rochet (1987); Heydenreich et al. (2009); Vohra (2011); Che et al. (2013)).

to be "monotone" in some sense, and this monotonicity is also useful to characterize mechanisms.[10] GWRF-monotonicity is a sufficient (but not necessary) condition for GWR-freeness. In addition, the "revenue equivalence result" holds for GWRF-monotone mechanisms. In particular, as in the Kepco example, despite its lack of incentive compatibility, the first-price auction is much more widely used than the second-price auction, possibly due to several reasons: risk-aversity (Holt, 1980), budget constraints (Che and Gale, 1998), privacy and cheating (Rothkopf et al., 1990), and asymmetry (Maskin and Riley, 2000). Thus, I also introduce the "*multidimensional first-price*" (MFP) auction, which still has several good properties and also reduces to the standard first-price auction when no externalities exist.

This paper follows the spirit of the "Wilson doctrine" (Wilson, 1985, 1987), i.e., designing *detail-free* mechanisms is practically important.[11] Not to mention the difficulty in finding optimal Bayesian strategies, in high-stakes auctions, which are likely rare events and where bidders are normally employees of companies, *ex-post* properties can be more important than properties *in expectation* because what will be eventually seen by their CEO is an ex-post outcome. For instance, recall the hypothetical Kepco's first-price auction where Hyundai's maximum willingness to pay against Samsung and LG was $10 and $5 billion. Even if Hyundai's $7 billion bid was optimal in expectation, if Samsung wins at some lower price than $10 billion, Hyundai's CEO may not be satisfied because he already allowed his employees to spend up to $10 billion. That is, bidders normally do not want a competitor to win at a lower price than their willingness to pay against that competitor. What makes bidding complicated, however, is a winner wants to win as cheap as possible, and losers would also rather win if they could at a good price. MSP resolves all these problems.

Desirable properties in auctions with externalities The remainder of this introduction explains the problems of existing mechanisms and the importance of the properties that MSP satisfies in detail. Some usual desirable properties in mechanism design are efficiency, incentive compatibility (IC), individual rationality (IR), and some stability and "fairness" property (e.g., the core property, pairwise stability). While the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) is efficient, IC, and IR, a VCG outcome may not be in the core, as in package auctions without externalities (Ausubel and Milgrom, 2006; Rothkopf, 2007).[12]

[10] See Rockafellar (1970); Dasgupta et al. (1979); Myerson (1981); Rochet (1987); Mailath (1987); Milgrom and Shannon (1994); Maskin (1999); Athey (2001); Milgrom and Segal (2002); Bikhchandani et al. (2006); Ashlagi et al. (2010); Kojima and Manea (2010a). In our model, Jehiel et al. (1999) show the condition for incentive compatibility, and Krishna and Maenner (2001) show the condition for the revenue equivalence.

[11] See also Hurwicz (1972), Dasgupta and Maskin (2000), and Bergemann and Morris (2005).

[12] While single-item auctions *with* externalities and package-auctions *without* externalities are similar in the sense that their bids are multidimensional, some fundamental differences are the existence of a loser's

Other problems with VCG arise due to externalities. First, it may require losing bidders to pay. In reality, even when externalities exist, LPF auctions (e.g., the first-price, SP, and English) are widely used, as in the Toshiba and Kepco examples, and this paper considers "loser's payment"-free (LPF) mechanisms.[13] Second, the low revenue problem is more serious. VCG may result in a negative payment (i.e., subsidy, see Example 2-(5)) as Myerson and Satterthwaite (1983) indicate. In addition, reserve prices cannot be simply implemented by treating the auctioneer as a bidder with a bid of reserve prices. Third, the "shill bidding problem" (i.e., a false-name bid) (Yokoo et al., 2004) is also more serious: there exists a shill bidding strategy that weakly dominates truthful bidding, and VCG may result in an undefined outcome (Jeong, 2016).

When VCG fails to produce a core outcome, one possible solution is a core-selecting mechanism, as in core-selecting package auctions (Parkes and Ungar, 2000; Day and Raghavan, 2007; Day and Milgrom, 2008; Day and Cramton, 2012; Erdil and Klemperer, 2010). However, Jehiel and Moldovanu (1996) show that the core can be empty in general unless externalities are "negligible." In contrast, Jeong (2017) shows that the core is nonempty if some "unrealistic" deviations are not allowed. In particular, he shows that the core is nonempty if bidders cannot refuse to pay, i.e., there always exists an outcome where no group of bidders is willing to pay together more than the winning price, i.e., "no justified envy." Unfortunately, Jeong (2017) also shows that there is no LPF core-selecting mechanism even with no payment refusals. Nevertheless, the core property might be less important than it is in package auctions. In package auctions, each bidder in a blocking coalition is willing to pay to receive some item. In contrast, when externalities exist, some bidders need to pay not to receive the item, but to decrease negative externalities by changing the allocation. Thus, for the same reasons why a loser's payment might be undesirable, bidders in the blocking coalition might be willing to accept the non-core outcome.[14]

However, a mechanism should have a certain degree of stability and "fairness." A well-known alternative to the core is pairwise stability (PS), where a pair consists of one bidder

payment and a nonempty core, which will be explained later in the introduction.

[13] I do not argue that a loser's payment is undesirable in all cases, but I suggest some possible reasons why some markets may prefer no loser's payment. Of course, losing bidders may want to pay when there exist huge negative externalities such as nuclear weapons (Jehiel et al., 1996). However, externalities are normally more difficult to estimate than the own valuation because they can change in response to the future environment. For instance, a losing bidder may want to improve the relationship with the winner to decrease the externality. In addition, the winner might suffer from the *winner's curse*, or might not use the item for a long time, in which case it might be less threatening than expected. That is, a loser's payment can be seen as an *immediate payment* for an *uncertain future*.

[14] One possible way to indirectly see the desirability of a loser's payment or the core property is a hybrid approach. For instance, let the auctioneer run MSP first as a default outcome and then run a core-selecting mechanism with the same bids. If the outcomes of two auctions are different, then let bidders in a blocking coalition have a chance to block the MSP outcome.

and the auctioneer.[15] If an outcome is not PS, the winning price is less than some losing bidder's bid against the winner, which might be too disputable or easily deviated by one bidder and the seller. In addition, PS enables the seller to simply employ bidder-specific reserve prices with its bid, which might be useful, as in the Toshiba example. Thus, I impose LPF and, as the second requirement, PS. Then, unfortunately, neither IC nor IR is possible even with inefficiency (Theorem 1).

Even when there are no externalities, IC is often impossible with other desirable properties, whereas IR is easy to achieve. Nevertheless, several non-IC mechanisms are widely used, and they have some good incentive properties, e.g., minimum-revenue core-selecting package auctions minimize the sum of incentives to deviate, and the generalized second-price (GSP) auction (Edelman et al. 2007; Varian 2007) has a "locally envy-free" equilibrium, i.e., each bidder does not envy another bidder who is either right above or right below. Likewise, MSP is free of certain regrets (e.g., winners never regret about the price; losers never regret not winning; no GWR), and it is strategyproof for a bidder without externalities imposed by others. These ex-post incentive properties can be more important than Bayesian IC (which is impossible, anyway) for bidders who are "regret-averse," which is plausible in high-stakes auctions, which are rare events.

While IR is also impossible, a certain degree of incentive to participate should better to be satisfied, since we cannot simply make the winner arbitrarily overpay to achieve other desirable properties. In fact, IR might be too strong a requirement, especially when it is difficult to predict the outcome of nonparticipation. Weak IR, i.e., the payoff of participation is at least as large as the worst payoff of nonparticipation, might be sufficient and desirable since it is what the bidders are guaranteed to receive with nonparticipation. An outcome that is not weakly IR implies that the winner needs to pay more than the maximum bid against all bidders, e.g., in the hypothetical Kepco example, Hyundai with bids of \$10 and \$5 billion may need to pay \$18 billion, which may be undesirable and discourage participation. MSP is weakly IR (Lemma 2). IR implies weak IR, and weak IR is also a well-known alternative to IR in the core with externalities literature (see footnote 26).

Unfortunately, however, even weak IR cannot be attained with efficiency. Although efficiency (of bidders and the auctioneer) is normally desired in the literature, there may exist an inefficient outcome that has higher bidder surplus than an efficient outcome. That is, efficiency may come from the sacrifice of bidders. Thus, if we consider the welfare of bidders only (or put more weight on this), or consider other surplus (e.g., consumer's), efficiency might be less important.[16] In particular, when an MSP outcome is inefficient, the

[15] Jeong (2017) shows that in auctions where a loser's payment and payment refusals are not allowed, pairwise stability becomes the core property. Thus, MSP is a core-selecting mechanism in this class.

[16] Note also that if we allow resales (but without commitment) efficiency is impossible even with a loser's

new winner for an efficient allocation always prefers to lose (i.e., the MSP outcome) in any LPF and PS mechanisms. Thus, even if she is provided with a second chance to win, she never prefers to win.

The remainder of this paper is organized as follows. Section 2 illustrates GWR and MSP with a motivating example. Section 3 introduces the model and the impossibility results with LPF and PS. Section 4 introduces MSP, ME, MFP, and GWRF-monotonicity, and shows the main results. Section 5 introduces the formal definition of GWR. Section 6 concludes the paper. Appendix A shows the simulation results. Any omitted proofs are provided in Appendix C.

2 Illustrative motivating example

I will informally illustrate group winner regret (GWR) and the new mechanism, MSP, with the motivating example mentioned in the introduction. First, I use the English auction to illustrate GWR and then describe MSP, which is free of GWR.[17]

The auctioneer sells one item to three bidders (with independent private values). For simplicity, the seller has no value on the item and does not impose nor suffer externalities. The type profile of bidders is $T = \begin{bmatrix} 16 & -6 & 0 \\ -12 & 20 & 0 \\ 0 & 0 & 21 \end{bmatrix}$ (or $B = \begin{bmatrix} 0 & 26 & 21 \\ 28 & 0 & 21 \\ 16 & 20 & 0 \end{bmatrix}$), where the j-th column vector is the type of bidder j, e.g., bidder 1's valuation on the item is 16, and bidder 1 suffers negative externality of 12 if bidder 2 wins. Then, j's maximum willingness to pay in order to beat i is $b_{ij} \equiv t_{jj} - t_{ij}$, e.g., bidder 1 is willing to pay up to $28 = 16 - (-12)$ to beat bidder 2. Note that bidder 3's winning is the efficient allocation, since $\arg\max_i \sum_j t_{ij} = 3$. Bidder j's payoff is $u_j \equiv t_{ij} - p_j$ when her payment is p_j and bidder i wins, i.e., quasilinear.

In the English auction, for simplicity, let the bidding function of bidder j be

$$\hat{\beta}(\mathbf{b}_j, R) \equiv \max_{i \in R \setminus \{j\}} \{b_{ij}\}, \tag{1}$$

where R is the set of remaining bidders at the current price, i.e., each bidder stays until the maximum bid against all remaining competitors.[18] As in footnote 17, the English auction

payment (Jehiel and Moldovanu, 1999). See also footnote 14.

[17] I define GWR with direct mechanisms, and MSP is a direct mechanism. For an intuitive explanation, however, I use a dynamic auction, the English auction, to explain GWR because what *drop* and *remaining bidders* mean are clear. Note that a dynamic mechanism with some strategies can be implemented as a direct mechanism that runs a dynamic auction "internally" after receiving bids.

[18] This is not an equilibrium. However, there is no closed-form equilibrium in English auctions in general. Even in a simple model where an equilibrium (which is unnecessarily complicated for an intuitive explanation)

with $\hat{\beta}$ can be implemented as a direct mechanism, which is $\hat{\varphi}$ in Algorithm 4 (which will be used to define *winner's regret* (Definition 12), the individual version of GWR).

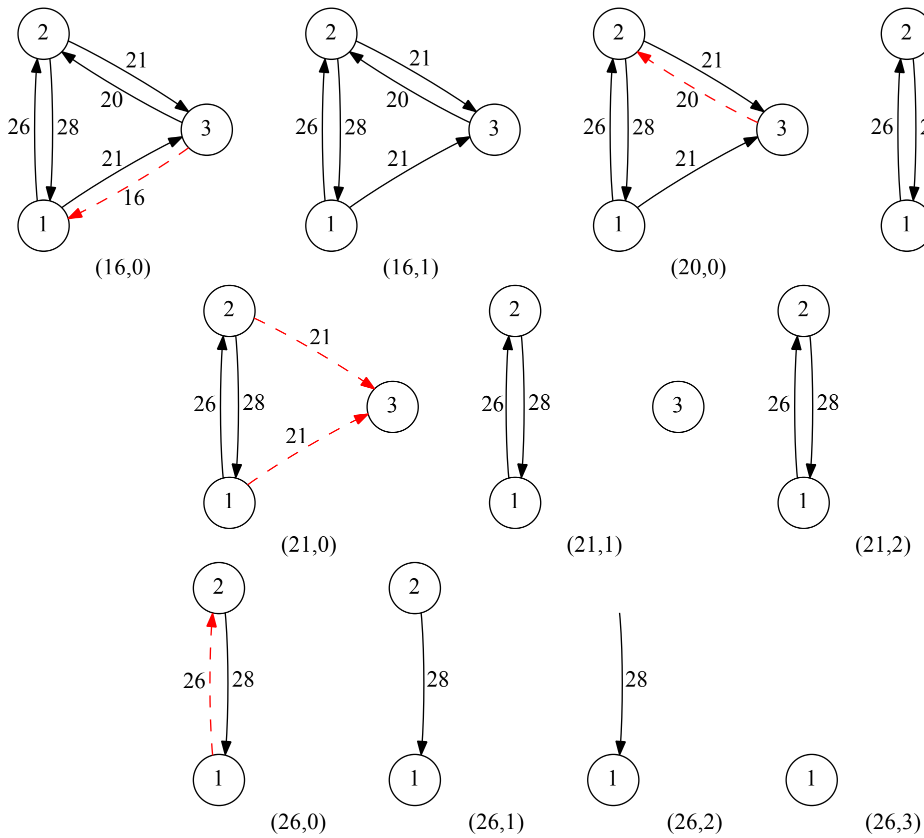

Figure 1: (q, k)-bid graph of the motivating example

A *bid matrix*, B, is useful (compared to a type matrix, T) to interpret an auction as updating a *bid graph*, a graph representation of a bid matrix. In a bid graph, each *node* represents a *bidder*, and each *edge* represents a *bid*. Figure 1 shows the (q, k)-*bid graph*, which may be simply referred to as (q, k)-graph or (q, k), where q is the current price, and k is the step (starting from 0) at the same price q. Note that in English auctions, one bidder's drop out can lead to another bidder's drop out at the same price; thus, k is used to distinguish each step at the same price. For illustrative purposes, in each $(q, 0)$-graph, the edge that will be removed in the next $(q, 1)$-graph, is shown with a dashed arrow (colored in red).

The English auction starts from price $q = 0$. At $q = 16$, bidder 1 no longer needs to bid against bidder 3, but she still needs to stay due to bidder 2, which is shown in $(16, 1)$. Likewise, at $q = 20$, bidder 2 no longer needs to bid against bidder 3, but she still needs to stay due to bidder 1. At $q = 21$, bidder 3 no longer needs to bid against either bidder 1 or

exists, GWR can occur. Thus, for simplicity, $\hat{\beta}$ is used.

2, i.e., bidder 3's staying any longer is dominated. Thus, bidder 3 drops out. Then, only two bidders are left, which in fact makes $\hat{\beta}$ a dominant strategy for bidders 1 and 2. Thus, bidder 2 drops out at $q = 26$, and bidder 1, the only remaining bidder, wins at $p = 26$. Then, bidders' payoffs are $\mathbf{u} = (-10, -6, 0)$. However, the payoffs when bidder 3 wins at some price p are $\mathbf{u} = (0, 0, 21 - p)$. Thus, bidders 1 and 2 are better off losing to bidder 3, and they could have lost to bidder 3 if they had dropped out together earlier than bidder 3. That is, bidders 1 and 2 have GWR. Moreover, the outcome is inefficient.

Roughly, GWR consists of the following three conditions: (i) a group of bidders is better off losing to any of the "remaining" bidders; (ii) the group could have lost to some remaining bidder; and (iii) each bidder in the group can be worse off if they lost to some bidder in the group. The first condition not only enables some mechanism (e.g., MSP) to be free of the regret, but also makes the group regret "more" in the following sense. Suppose the group regrets not losing to *one specific remaining bidder*, as opposed to *any of the remaining bidders*. Then, it is impossible to make a mechanism free of this regret. Moreover, even if there is such regret, the group might regret "less" because it is difficult for bidders to forecast who will win. However, they might regret "more" if they could have been better off no matter which remaining bidder had won. The second condition "justifies" the regret, i.e., even if they are better off losing to some bidder, they cannot say they "regret" if there is no way for them to lose to the bidder. The third condition is the reason why GWR cannot be solved individually. Depending on the bids of others, the bidder who drops out alone to avoid GWR can be worse off if some competitor in the group wins.

I will now show that the three conditions are satisfied in the above example. Figure 2 shows the (q, k)-*group bid graph*, which now shows "group." A thicker arrow (colored in blue) denotes a bid across groups. Until $(20, 0)$, there is only one group, but at $(20, 1)$, there are two groups G1 and G2. G1 has bidder 3, and G2 has bidders 1 and 2. The two members in G2 bid only against each other, not against G1, while G1 still bids against G2. This is "unnecessary" internal competition in G2 because they are better off losing to any bidder in G1 (in this example, bidder 3, since it is the only bidder) at any price $p > 20$ (i). That is, their staying above 20 is dominated by dropping out *together* at 20 (ii). However, suppose bidder 1 drops out alone first, but if $b_{32} > b_{23}$, then bidder 2 eventually wins, where $u_1 = -12$, which is worse for bidder 1 than winning at 26, where $u_1 = -10$ (iii).

In MSP, as shown in Figure 2, G1 drops out at $(20, 1)$ and bidder 3 wins at $(20, 3)$, i.e., the price is 20. This is the main idea behind GWR and MSP. MSP finds groups whose members are in unnecessary internal competition and drops such groups. A group and its unnecessary internal competition can be intuitively recognized by the connectedness of a bid graph. A *strongly connected component* (SCC) is a maximal subgraph that has a bidirectional path

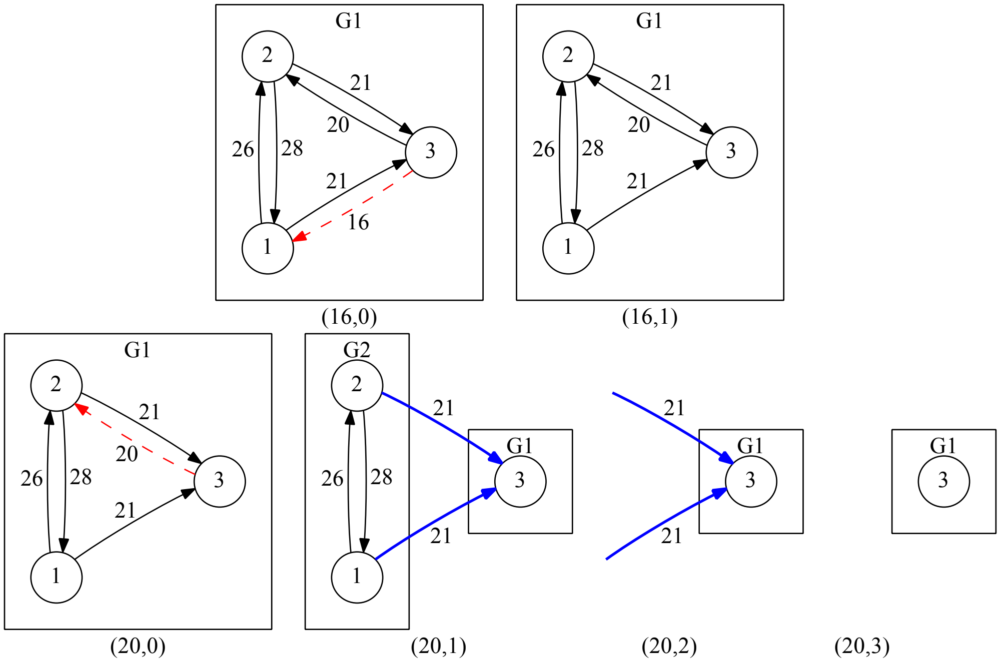

Figure 2: (q, k)-group bid graph of the motivating example

between each pair of nodes. An SCC can then be interpreted as a group. In general, there can be more than two groups, and any group can have more than one bidder.

MSP, is a "groupwise" version of $\hat{\varphi}$. That is, MSP is a direct mechanism version of the English auction with the following bidding function for any bidder in group S,

$$\beta^*(\mathbf{b}_S, R) \equiv \max_{i \in R \setminus S, j \in S} \{b_{ij}\}, \tag{2}$$

where R is the set of remaining bidders at the current price, i.e., each bidder in a group stays until the maximum bid against all remaining competitors outside the group.

Roughly, MSP (Algorithm 1) works as follows (see Example 2 for more examples). Each bidder j submits $\mathbf{b}_j$. MSP increases the current price q to find a bid of this price that has not been removed (or *unblocked*). Let b_{cm} denote this bid. Then, MSP unblocks b_{cm}, which implies that m (**m**e) no longer bids against c (**c**ompetitor) above price q. If the unblocking still results in only one group, then MSP finds the next unblocking. Otherwise, i.e., if multiple groups exist, the "path" (called here the *externality chain*) of the groups is always well-defined (Lemma 1, the Chain Lemma), i.e., it always starts at the group including m (who just gave up bidding) and ends at the group including c (the competitor whom m just gave up bidding against), shown with a thicker arrow (colored in blue), even when there are multiple paths, as in Example 2-(3) which shows multiple paths in $(2, 1)$-group graph in Figure 4. Then MSP drops all bidders in the first group, i.e., the group including m, since this group does not bid against any other groups, and then drops all bidders in the next group (if any, i.e., if more than one group is still remaining) since now this group does not bid against any other groups, and so on, by following the externality chain sequentially, until the last group including c is left. (In fact, this well-defined externality chain enables the simplification of MSP, called s-MSP (Algorithm 2). That is, s-MSP simply drops (all the bidders in) all groups except for the group including c (i.e., not by following the externality chain) since no matter how many groups and externality chains exist, any externality chain always ends at the group including c.) This process is repeated until one bidder, the winner (the last c), is left. Only the winner pays (i.e., LPF), and the price is the last $q = b_{cm}$, the last bid that is unblocked, which is defined as the *second-price*. By its nature, MSP can be easily implemented as a dynamic open ascending auction, ME (in Algorithm 3).

3 The Model

The auctioneer (denoted by 0) is selling one indivisible item to the set of bidders, $N = \{1, 2, ..., n\}$. Let $N^0 \equiv N \cup 0$.[19] Each bidder only knows her own valuation of the item and the externalities imposed by others. That is, the type of bidder j is denoted by a column vector $\mathbf{t}_j = (t_{ij})_{i \in N^0}$, $t_{ij} \in \mathbb{R}$, where t_{ij} is the (negative if $t_{ij} < 0$) externality imposed on bidder j when bidder $i \neq j$ wins the item, and t_{jj} is the bidder j's own valuation of the item. The type profile is denoted by a $(n+1)$-by-$(n+1)$ *type matrix*, $T = (t_{ij})_{i,j \in N^0}$, and each $\mathbf{t}_j$ is independent of each other. The payoff of $j \in N^0$ is $u_j^T \equiv t_{ij} - p_j$ when her payment is p_j and $i \in N^0$ wins, i.e., quasilinear. Each bidder's outside option is determined by the outcome of an auction, i.e., even if bidder j does not participate in an auction, $u_j^T = t_{ij}$ if bidder i wins. As usual, $T_S \equiv (t_{ij})_{i,j \in S}$ for some $S \subseteq N^0$ and $T_{-S} \equiv T_{S^c}$, where $S^c \equiv N^0 \backslash S$.

Note that $b_{ij} \equiv t_{jj} - t_{ij}$ is bidder j's maximum willingness to pay in order to beat bidder i, which is called *j's bid against i*. Likewise, $\mathbf{b}_j = (b_{ij})_{i \in N^0}$ is called *j's bid*, and $B = (b_{ij})_{i,j \in N^0}$ is called a *bid matrix*. Each $\mathbf{b}_j \in \mathcal{B}_j \equiv [\underline{b}, \overline{b}]^{n+1}$, where $\underline{b}, \overline{b} \in \mathbb{R}$ are fixed bounds,[20] i.e., $\mathcal{B}_j$ is the set of all possible $\mathbf{b}_j$, and let $\mathcal{B} \equiv \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$. Likewise, $\mathcal{T}_j \equiv \{\mathbf{t}_j : \mathbf{b}_j \in \mathcal{B}_j\}$ and $\mathcal{T} \equiv \mathcal{T}_0 \times \cdots \times \mathcal{T}_n$. Let $\underline{\mathbf{b}} = (\underline{b}, ..., \underline{b})'$ denote the *lowest bid*, i.e., $\mathbf{b}_j \geq \underline{\mathbf{b}}$ for all $\mathbf{b}_j \in \mathcal{B}_j$, where "$\geq$" is component-wise. Likewise, $\mathbf{t}_j$ is said to be a *lowest type* if $\mathbf{b}_j = \underline{\mathbf{b}}$.

Each $\mathbf{b}_j$ can also be interpreted as the normalization such that the valuation is zero, $b_{jj} = 0$, and each b_{ij} is the size of negative externality. The normalized payoff of j is $u_j^B \equiv -b_{wj} - p_j$ when w wins. For simplicity, u_j will be used instead of u_j^T (unnormalized)

[19] Abusing the notation, $\{j\}$ can be written as j.

[20] This is for simplicity of exposition. Each $\mathcal{B}_j$ can be asymmetric. In general, $\underline{b}$ can be negative, and t_{jj} itself can be negative as well, e.g., auctioning off garbage, which might have a negative value for many bidders, but positive for recycling companies.

or u_j^B (normalized) when the reference type is unambiguous. Unless otherwise specified, $t_{i0} = t_{0j} = 0$ for all $i, j \in N^0$, i.e., the auctioneer neither suffers nor imposes externalities. Then, T induces a unique B. Thus, without loss of generality, we can consider direct mechanisms in which each bidder j submit its bid $\mathbf{b}_j$ instead of its type $\mathbf{t}_j$.[21]

A direct auction mechanism is a pair of functions $\varphi = (x, \rho)$, $\varphi : \mathcal{T} \to (N^0, \mathbb{R}^n)$,[22] where x is the winner-determination rule (with some tie-breaking rule) and ρ is the payment rule. $w \equiv x(T)$ is the winner (can be the auctioneer, i.e., no-sale), $\mathbf{p} \equiv \rho(T) \in \mathbb{R}^n$ is the payment of all bidders, and $p \equiv \sum_{j \in N} p_j$ is revenue. Let $X : \mathcal{T} \to 2^{N^0}$ be a correspondence such that $W \equiv X(T)$ is the set of all possible winners due to ties.

A mechanism φ is said to be *loser's payment-free* (LPF)[23] if $\rho_j(T) = 0$ for all $j \neq w$ and $T \in \mathcal{T}$. An auction outcome is said to be *efficient* if $w \in \arg\max_{i \in N^0} \sum_{j \in N^0} t_{ij}$. A mechanism φ is said to be efficient if the outcome is efficient for all $T \in \mathcal{T}$. Let $u_j(T'_S; T) = u_j(T'_S; T, \varphi) = t_{x(T'_S),j} - \rho_j(T'_S)$ denote the payoff of j when $S \subseteq N^0$, $0 \in S$, participates and submits T'_S to $\varphi = (x, \rho)$, while the true type of N^0 is T.

Definition 1. A mechanism φ is *incentive compatible* (IC)[24] if for all $j \in N$,

$$u_j((\mathbf{t}_j, \mathbf{t}_{-j}); T) \geq u_j((\mathbf{t}'_j, \mathbf{t}_{-j}); T) \text{ for all } T \in \mathcal{T}, \mathbf{t}'_j \in \mathcal{T}_j.$$

[21] A bid matrix is notationally simpler and easier for payoff comparisons when there is no loser's payment, e.g., if j wins at price p, then j is better off losing to i if "$p > b_{ij}$," which is equivalent to "$t_{jj} - p < t_{ij}$" using a type matrix. However, a type matrix is the usual way to describe externalities in the literature. Thus, T is used for preliminary discussions and VCG, and from Section 3.2 on, B is used.

[22] When only $S \subseteq N^0$ participates, φ runs an auction with subtype T_S, where φ is implicitly defined in the subspace $T = T_S$. Note that $\varphi : \mathcal{B} \to (N^0, \mathbb{R}^n)$ when B is used instead of T. Abusing the notation, even when only S participates, $\mathbf{p} \in \mathbb{R}^n$. When no bidders participate, the auctioneer keeps the item, and $\mathbf{p} = \mathbf{0}$.

[23] The suffix form, "-free," is introduced to use the same acronym for both noun and adjective forms, e.g., LPF can stand for both "loser's payment-free" and "loser's payment-freeness." The same rule applies to any regret, i.e., "XYZ regret-free" and "free of XYZ regret"; "XYZ regret-freeness" and "no XYZ regret."

[24] Throughout the definition, *ex-post* is assumed. Bayesian (or *ex-interim*) versions can be defined similarly. Note that ex-post IC and strategyproof are the same due to the independent private value assumption. In the literature, due to the impossibility result in Myerson and Satterthwaite (1983), φ is said to be IC (or IR) if it is IC (or IR, respectively) only for all bidders $j \in N$, not including the auctioneer. However, weak IR can be satisfied for all bidders and the auctioneer, e.g., MSP.

A mechanism φ is *individually rational* (IR)[25] if for all $j \in N$,

$$u_j(T;T) \geq u_j(T_{-j};T) \text{ for all } T \in \mathcal{T}.$$

A mechanism φ is *weakly individually rational* (weakly IR)[26] if for all $j \in N^0$,

$$u_j(T;T) \geq \inf_{T'_{-j}} u_j(T'_{-j};T) \text{ for all } T'_{-j} \in \mathcal{T}_{-j}, T \in \mathcal{T}.$$

IR requires that the payoff of participation is at least the payoff of nonparticipation. Note that due to externalities, the right-hand side of the inequality is not zero. In contrast, weak IR requires that the payoff of participation is at least the worst possible payoff of nonparticipation, i.e., $t_{wj} - p_j \geq \min_{i \neq j}\{t_{ij}\}$. Thus, weak IR is satisfied for losing bidders when there is no loser's payment. For the winner, weak IR requires $p_w \leq \max_{i \neq w}\{b_{iw}\}$, i.e., the payment is (weakly) less than its maximum bid against all other bidders. As its name suggests, IR implies weak IR, but not vice versa.

The *core* is the set of payoff profiles that are not blocked by any coalition. In other words, the core is the set of payoff profile where *group rationality* for any $S \subseteq N^0$ holds.[27] A mechanism is said to be *core-selecting* (or said to have the *core property*) if the outcome is in the core for any $T \in \mathcal{T}$. Although the core property is desirable, due to the impossibility result (Theorem 1), the following pairwise stability will be used as the alternative stability concept in this paper.

Definition 2. An auction outcome is said to be *pairwise stable* (PS) when there exists no blocking coalition that consists of the auctioneer and one bidder. That is, an outcome is pairwise stable if and only if $p \equiv \sum_{j \in N} p_j \geq b_{wj} + p_j$ for all $j \in N$.[28] An auction mechanism is said to be pairwise stable if the outcome is pairwise stable for any $T \in \mathcal{T}$.

[25] IR is used only for the impossibility result (Theorem 1); thus, this simple yet practically reasonable (i.e., easy to check) definition is used for simplicity, but the result holds with other alternative definitions. For instance, as in Jehiel and Moldovanu (1996), when externalities exist, if bidder j does not participate, there is no guarantee that all the other bidders participate. In addition, if φ is not IC, there is no guarantee that participating bidders bid truthfully. Thus, a more reasonable definition of IR might be "$u_j(\sigma(T);T) \geq u_j(\sigma(T_{-j});T)$" where σ is a subgame perfect Nash equilibrium (SPNE) in a two-stage (i.e., the participation decision stage and the auction stage) game (while finding an SPNE is often quite difficult). In fact, when externalities exist, there can be various definitions of IR depending on which strategies the outside players, $N^0 \backslash j$, play, which is a concept of *effectiveness* (e.g., Aumann and Peleg (1960); Shapley and Shubik (1969); Chander and Tulkens (1997); Jeong (2017)).

[26] When j does not participate, $N^0 \backslash j$ chooses the worst outcome (i.e., allocation) for j, which is the concept of α-effectiveness (Aumann and Peleg, 1960). That is, weak IR is IR by α-*effectiveness*.

[27] A mechanism φ is *group rational* for $S \subseteq N^0$ if $\sum_{j \in S} u_j(T;T) \geq \sum_{j \in S} u_j(T_{-S};T)$ for all $T \in \mathcal{T}$. The core is used only for the impossibility result (Proposition 1). See Jeong (2017) for the more general definition of the core of auctions with externalities. See also footnote 25.

[28] Interestingly, a blocking by $j = w$ (and the seller) is possible, i.e., $p < b_{ww} + p_w = p_w$ is possible. As the inequality shows, this can happen if $p_i < 0$ for some $i \neq w$.

In particular, an LPF outcome is PS if and only if $p \geq b_{wj}$ for all $j \neq w$, i.e., when winner w wins at price p, no other bidder's bid against w is larger than p. By definition, if an outcome is not PS, it is not in the core. As an example of multidimensional direct auction mechanisms, the Vickrey-Clarke-Groves (VCG) mechanism is given.

Definition 3. For a reported type profile T, the VCG auction mechanism is $\varphi = (x, \rho)$, where $x(T) \in \arg\max_{i \in N^0} \sum_{j \in N^0} t_{ij}$ and $\rho_j(T) = \sum_{k \in N^0 \setminus j} t_{x(T_{-j}),k} - \sum_{k \in N^0 \setminus j} t_{x(T),k}, \forall j \in N$.

Example 1. For $T_N = \begin{bmatrix} 9 & -3 & 0 \\ 0 & 7 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, $w = 1$, $\mathbf{p} = (8, 0, 1)$, and $p = 9$. There is a loser's payment. The outcome is not in the core, and is not even PS for bidder 2 (and the auctioneer) since $b_{12} = 10 > 9 = p$.

3.1 Impossibility results

Hereafter, loser's payment-freeness (LPF) is given as the first requirement of the mechanism design, as discussed in the introduction. For stability and "fairness," the core property is usually desirable since a blocking coalition (including the auctioneer) means "justified" envy of both bidders (i.e., they are willing to pay more) and the auctioneer (i.e., low revenue), and the outcome will be unstable if the coalition actually deviates. Jehiel and Moldovanu (1996) show that the core can be empty in general unless externalities are "negligible" compared to valuations. Some deviations, however, may be "unrealistic," and Jeong (2017) shows that the core is nonempty if bidders cannot refuse to pay. Nevertheless, he also shows that there is no LPF core-selecting mechanism even without no payment refusals. The intuition is that the winner has to pay excessively to prevent a blocking, violating even weak IR.

Proposition 1 (Jeong 2017)**.** *There is no loser's payment-free core-selecting mechanism.*[29]

For instance, as shown in Example 1, VCG is neither core-selecting, PS, nor LPF. A mechanism, however, should still have a certain degree of stability and "fairness," and a well-known alternative to the core is pairwise stability (PS). If an outcome is not PS, it might be too disputable or easily deviated by the auctioneer and one bidder. In addition, as the core property resolves the low revenue problem, PS at least ensures the revenue of $\max_{j \neq w}\{b_{wj}\}$. Also, b_{j0} can be used as a reserve price for bidder j, which may be useful for government auctions. That is, the government can impose different reserve prices for each bidder (e.g., company) to mitigate monopoly or increase social welfare.

[29] The result holds even when bidders cannot refuse to pay. As mentioned in footnote 25, the definition of the core also depends on which *effectiveness* concept is used, but this negative result holds with the least sharp core, i.e., the α-core, which means if the α-core is empty, all other cores are also empty.

However, if we impose LPF and PS, neither IC nor IR is possible, even with inefficient mechanisms, as shown below. Again, the intuition behind this is the winner has to overpay. Note that PS implies that the auctioneer should sell when there exists $b_{0j} > 0$ for some bidder j. Thus, a trivial no-sale mechanism in inefficient cases cannot be a counterexample.

Theorem 1 (Impossibility). *There is no mechanism that is loser's payment-free, pairwise stable, and also any of the following:*
(i) *incentive compatible,*[30]
(ii) *individually rational,*
(iii) *efficient and weakly individually rational.*

The core should be efficient, IR, and PS; thus, (ii) implies that a loser's payment-free core-selecting mechanism is impossible, as shown in Proposition 1. Moreover, (iii) shows that efficiency cannot be achieved if at least weak IR is required. That is, to achieve efficiency, the winner may need to pay even more than its maximum bid against all competitors.

These impossibility results may appear disappointing. When externalities exist, however, IR might be too strong a requirement, especially when predicting the outcome of nonparticipation is difficult. Nevertheless, auction mechanisms should have a certain degree of incentive to participate since we cannot simply make the winner arbitrarily overpay to achieve other desirable properties. In particular, it may be desirable that the payoff of participation is at least the worst possible payoff of nonparticipation since this payoff is what bidders are guaranteed to receive with nonparticipation, which is weak IR. Fortunately, there are mechanisms that are LPF, PS, and weak IR, e.g., MSP.

3.2 No ex-post regret as incentive properties

IC is impossible with LPF and PS. However, as in core-selecting package auctions or the generalized second-price (GSP) auctions, which are not IC but widely used, mechanisms should have certain good incentive properties. Under incomplete information, each bidder is typically assumed to maximize the expected payoff. However, in the spirit of the "Wilson doctrine," designing *detail-free* mechanisms is practically important. Not to mention the difficulty in finding optimal Bayesian strategies due to externalities, in a high-stakes auction, which is a rare event, *ex-post* properties might be more important than properties *in expectation*. Since bidders are normally employees of companies, they might want to avoid certain undesirable outcomes that can be easily recognized by their CEO or board of directors.

[30] For inefficient mechanisms, the following assumption is needed: the lowest type bidder cannot win unless every bidder reports the lowest type.

Obviously, due to the impossibility result on IC, it is impossible for a mechanism to guarantee no regrets of any kind. Thus, we need to define the subset of regrets that does not occur, and then no (ex-post)[31] regret of a certain kind means strategyproofness with certain restrictions. For instance, as in the first-price auction, the winner w would regret not bidding $\mathbf{b}'_w$ if w could still have won at a lower price.

Definition 4. For a mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, the winner w has *overpay regret* if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$ such that $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$ and $\rho_w(\mathbf{b}'_w, \mathbf{b}_{-w}) < p_w$.[32]

In other words, w has no overpay regret if $u_w(\mathbf{b}_w, \mathbf{b}_{-w}) \geq u_w(\mathbf{b}'_w, \mathbf{b}_{-w})$ for all $\mathbf{b}'_w$ and B when $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$. Note that without the restriction, "when $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$," the statement implies strategyproofness for the winner, which is impossible. However, if we impose this restriction, there is a mechanism that is free of overpay regret, e.g., MSP. Other regrets will be defined when they are introduced.

4 Multidimensional second-price and English auctions

4.1 Bid graph

I will first introduce the *bid graph*, which is essential for the MSP mechanism itself and the definition of group winner regret (GWR). To the best of my knowledge, using a graph for the auction mechanism itself is a new approach, which can also be used for standard one-dimensional auction mechanisms (e.g., Theorem 4).

Another advantage of a bid matrix is it can be interpreted as an adjacency matrix for a directed weighted graph, a *bid graph*; i.e., each *node* represents a *bidder*, and each directed weighted *edge* represents a *bid*. Then, an auction can be interpreted as updating a bid graph.

Definition 5. For a bid matrix $\widetilde{B} = (\tilde{b}_{ij})_{i,j \in S \subseteq N^0}$, a directed weighted graph $G = G(\widetilde{B}) = (V, E, f)$ induced by $\widetilde{B}$ as an adjacency matrix is called a *bid graph* (or *social bid network*), where $V = \{j \in S : \tilde{b}_{ij} \neq 0 \text{ for some } i\}$ is a vertex set, $E = \{(i, j) \in S \times S : \tilde{b}_{ij} \neq 0\}$ is a directed edge set,[33] and $f(e) = \tilde{b}_{ij}$ for $e = (i, j) \in E$ is a weight function.

[31] Again, every regret in this paper is *ex-post*; thus, ex-post will be omitted.

[32] $X(\cdot)$ (instead of $x(\cdot)$) is used to handle a tie. Also, for simplicity, let $\rho_w(\mathbf{b}'_w, \mathbf{b}_{-w})$ denote the payment of w when w wins by the tie-breaking. Note also that depending on the characteristics of $\mathbf{b}'_w$, regret about the price can be defined with different degrees, as in Definition 9.

[33] In graph theory, for an adjacency matrix B to have full information of E and f, typically $b_{ij} = 0$ means no edge between i and j. Thus, for notational simplicity and alignment with notational conventions in graph theory, $\underline{b} > 0$ is assumed without loss of generality by Lemma 8 (linearity). Note also that "$(i, j) \in S \times S$" is used instead of "$(i, j) \in V \times V$," the standard definition in graph theory. This enables to describe the moment that some bidder i has just been "dropped," i.e., $i \notin V$, while some bidder j is bidding against i, i.e., $(i, j) \in E$. Such i is called a *phantom node*. For instance, bidder 2 at $(26, 2)$ in Figure 1 or bidders 1 and 2 at $(20, 2)$ in Figure 2 are phantom nodes.

Notation. In the algorithms, initially $\widetilde{B} = B$, then $\widetilde{B}$ will be updated, i.e., $B = (b_{ij})$ denotes the original bid matrix, and $\widetilde{B} = (\tilde{b}_{ij})$ denotes an updated bid matrix. ":=" means "set" or "update," e.g., $\widetilde{B} \coloneqq B$ and $k \coloneqq k+1$. $H \subseteq G$ denotes that H is a subgraph of G. For simplicity, $i \in G$ means $i \in V(G)$, and $V(B) = V(G(B))$.

While MSP is a direct mechanism, it works as a dynamic auction "internally" (or "proxy" dynamic auction); thus, for an intuitive explanation, the term "drop" will also be used for the direct mechanisms. In particular, "*drop* bidder j" means "set $\tilde{b}_{ij} = 0$ for all i." Likewise, "*unblock* $\tilde{b}_{ij}$" means "set $\tilde{b}_{ij} = 0$," and "j is blocking i at q" means "$\tilde{b}_{ij} > q$."

The reason why GWR occurs can be intuitively explained by the connectedness of a bid graph. To do this, I first introduce some definitions and notations, which were roughly explained in Section 2. For a directed graph, a *path* is a sequence of nodes (including phantom nodes in footnote 33) that is connected by a sequence of edges. The length of a path is defined by the number of edges in the path. A *direct* path is a path that has a unit length, and is denoted by the "arrow" notation, e.g., $i \to j$.

A *strongly connected component* (SCC) of G is a maximal subgraph H of G such that for any $i, j \in H$ with $i \neq j$, there exists a *bi-directional* path. $\mathcal{G}(G)$ (or simply $\mathcal{G}$) is the set of SCCs of G, i.e., $\mathcal{G} \equiv \{H \subseteq G : H \text{ is an SCC of } G\}$. A *weakly connected component* (WCC) of G is a maximal subgraph H of G such that for any $i, j \in H$ with $i \neq j$, there exists *at least* a *unidirectional* path. G is said to be *connected* if G has only one WCC, i.e., G itself is a WCC. G is said to be *strongly connected* if G has only one SCC, i.e., G itself is an SCC.

Note that each *SCC* can be interpreted as a "*group*." For instance, at $(20, 1)$ in Figure 2, G_2 no longer bids against G_1 while G_1 still bids against G_2, which can be interpreted as the "unnecessary" internal competition of G_2. This is the main idea behind GWR and MSP.

4.2 Multidimensional Second-Price Auctions

Now I introduce the Multidimensional Second-Price (MSP) auction mechanism. As illustrated in Section 2, MSP is a direct mechanism version of the English auction with the bidding function β^* (Equation 2). $B^{(q,k)}$ induces (q, k)-*group bid graph* that describes the auction status at price q and step k. Note that (q_j^*, k_j^*) is the price and the step at which bidder j is dropped or won, and it will be used to prove several results. For an auction outcome itself (i.e., w and p), k, $B^{(q,k)}$, and (q_j^*, k_j^*) are not necessary. In addition to the motivating example in 2, see also Example 2 for additional examples.

Algorithm 1. The Multidimensional Second-Price (MSP) Auction

1. Each bidder j submits $\mathbf{b}_j$. $\widetilde{B} \coloneqq B$. $\tilde{b}_{jj} \coloneqq 1, \forall j$. Let $G(\widetilde{B}) = (V, E, f)$.[34]

[34] For illustrative purposes, $\tilde{b}_{jj}$ is used to differentiate "right before" and "right after (i.e., phantom node)"

2. $q \coloneqq \min\{\tilde{b}_{ij} : (i,j) \in E, i \neq j\}$. $k \coloneqq 0$ and $B^{(q,k)} \coloneqq \widetilde{B}$.
3. $M \coloneqq \{j \in V : \tilde{b}_{ij} = q\}$. If $M = \emptyset$, go to step 2. Otherwise, choose a bidder $m \in M$.[35]
4. $C \coloneqq \{i \in V : \tilde{b}_{im} = q\}$. If $C = \emptyset$, go to step 3. Otherwise, choose a bidder $c \in C$.
5. (*unblock*) $\tilde{b}_{cm} \coloneqq 0$. $k \coloneqq k+1$ and $B^{(q,k)} \coloneqq \widetilde{B}$.
6. $\mathcal{G} \coloneqq \{H \subseteq G : H$ is an SCC of $G\}$. If $|\mathcal{G}| = 1$, go to step 4.
7. (*sequential drop*) $\tilde{b}_{ij} \coloneqq 0$ and $(q_j^*, k_j^*) \coloneqq (q,k), \forall i \in V, \forall j \in D$, where $D = \cup_{H\in\mathcal{D}} V(H)$ and $\mathcal{D} \coloneqq \{H \in \mathcal{G} : \tilde{b}_{ij} = 0, \forall j \in H, \forall i \in K \in \mathcal{G}, K \neq H\}$. $k \coloneqq k+1$ and $B^{(q,k)} \coloneqq \widetilde{B}$.
8. (*sequential unblock*) $\tilde{b}_{ij} \coloneqq 0, \forall (i,j) \in E$ with $i \in D$. $k \coloneqq k+1$ and $B^{(q,k)} \coloneqq \widetilde{B}$.
9. If $|V| > 1$, go to step 6. Otherwise, $w = c$ wins at price $p = q = b_{cm}$. $(q_w^*, k_w^*) \coloneqq (q,k)$.

As explained in Section 2, MSP works as follows. Each bidder submits $\mathbf{b}_j$. In steps 2-4, the mechanism increases the current price q to find $\tilde{b}_{cm} = q$. In step 5 (unblock step), it unblocks $\tilde{b}_{cm} = q$, which implies that m no longer bids against c above price q. In step 6, if only one SCC (or group) exists, then it goes back to step 4 to find the next unblocking (if there is no one, go further back to step 2 to increase the price). Otherwise, i.e., if multiple groups exist, then in step 7 (sequential drop step), it finds groups that do not bid against some other group, i.e., $\mathcal{D}$ is the set of these groups. Then, it drops all the bidders in these groups, i.e., D is the set of these bidders. If bidder j drops, then in step 8 (sequential unblock step), the bidders who bid against j no longer bid against j. If there still exist multiple groups, this leads to some other group's drop in step 7 (sequential drop step). That is, sequential unblock and drop can be repeated without the increase of q. This process is repeated until one bidder, the winner (the last c), is left. Only the winner pays (i.e., loser's payment free (LPF)), and the winning price is the last $q = b_{cm}$, the last bid that is unblocked in step 5, which is defined as the *second-price*. The owner of this bid (the last m) is called the *threshold bidder*, who made the auction end.

MSP in Algorithm 1 is useful to show the *sequential* drop of groups that compete with its own group only. However, when an unblocking leads to multiple groups, there exists a unique starting group and a unique ending group (Chain Lemma, Lemma 1), which not only shows that the path of sequential drop of groups is well-defined, but also enables the simplified version of MSP, called "s-MSP" in Algorithm 2. To describe a path between groups, I first

a drop, e.g., $(26,1)$ and $(26,2)$ in Figure 1, and it does not create a self-loop in a bid graph. That is, even if $\tilde{b}_{ij} = 0$ for all $i \neq j$, bidder j is not dropped if $\tilde{b}_{jj} = 1$, which is also useful to make the winner undropped. For simplicity, updating $\widetilde{B}$ in any subsequent steps implies updating G, i.e., let $G(\widetilde{B}) = (V, E, f)$ again.

[35] M stands for "Me" and C stands for "Competitor." "$M = \emptyset$" is possible when coming back from step 4. Likewise, in step 4, "$C = \emptyset$" is possible when coming back from step 6. Choosing $m \in M$ or $c \in C$ is done by a tie-breaking rule if needed. Note that steps 3 and 4 are separated to break all ties in C first once m is chosen so that MSP can reduce to the second-price when there are no externalities (Theorem 4). Note also that for constructing (q,k)-group bid graph, step 2 is separated from step 3 only to reset the step counter k whenever q changes. Thus, steps 2 and 3 are combined in s-MSP in Algorithm 2.

introduce some definitions.

A *component path* is a sequence of SCCs such that there exists a path of nodes from each component to the next component. For two distinct SCCs $G_1, G_2 \in \mathcal{G}$, $G_1 \to G_2$ denotes a direct component path, i.e., there exists a direct path $i \to j$ for some $i \in G_1$ and $j \in G_2$. For SCCs, a subscript denotes any general sequence of indices, but a superscript denotes the SCC that contains a certain bidder, e.g., G^j is the SCC that includes bidder j, i.e., $j \in V(G^j)$, which is simply denoted by $j \in G^j$. A *start* (or *source*) component is an SCC $H \in \mathcal{G}$ such that $H \to J$ for some $J \in \mathcal{G}$ with $J \neq H$, but $K \nrightarrow H$ for all $K \in \mathcal{G}$ with $K \neq H$. An *end* (or *sink*) component is an SCC $H \in \mathcal{G}$ such that $J \to H$ for some $J \in \mathcal{G}$ with $J \neq H$, but $H \nrightarrow K$ for all $K \in \mathcal{G}$ with $K \neq H$. G is said to be a *chain* if G, $|\mathcal{G}| \geq 2$, is connected and has a unique start component and a unique end component.[36]

Now I present s-MSP. As explained in MSP (and footnote 35), k, $B^{(q,k)}$, (q_j^*, k_j^*) are not necessary for the auction outcome (i.e., w and p). Thus, they are omitted in s-MSP.

Algorithm 2. The simplified MSP (s-MSP)

1. Each bidder j submits $\mathbf{b}_j$. $\widetilde{B} := B$. $\tilde{b}_{jj} := 1, \forall j$. Let $G(\widetilde{B}) = (V, E, f)$.
2. $q := \min\{\tilde{b}_{ij} : (i,j) \in E\}$ and $M := \{j \in V : \tilde{b}_{ij} = q\}$, then choose a bidder $m \in M$.
3. $C := \{i \in V : \tilde{b}_{im} = q\}$. If $C = \emptyset$, go to step 2. Otherwise, choose a bidder $c \in C$.
4. (unblock step) $\tilde{b}_{cm} := 0$.
5. $\mathcal{G} := \{H \subseteq G : H \text{ is an SCC of } G\}$. If $|\mathcal{G}| = 1$, go to step 3.
6. (drop step) (drop all bidders in all SCCs except for G^c)
   $\tilde{b}_{ij} := 0$ and $\tilde{b}_{ji} := 0, \forall i \in V, \forall j \in D$, where $D = \cup_{H \in \mathcal{G} \setminus G^c} V(H)$.
7. If $|V| > 1$, go to step 2. Otherwise, winner $w = c$ wins at price $p = q = b_{cm}$.

As shown in the following Chain Lemma, when unblocking b_{cm} induces multiple groups in the *unblock step*, G is a chain such that $G^m = G_1 \to G_2 \to \cdots \to G_k = G^c$, which will be called an *externality chain*. Moreover, even if multiple component paths exist, every component path starts at G^m (m: who just gave up bidding) and ends at G^c (c: whom m just gave up bidding against). Recall that MSP drops G^m first, and G_2 (and G_2' if another component path $G^m \to G_2' \to \cdots \to G^c$ exists) needs to unblock and drop (sequential unblock and sequential drop), and so on, until only G^c is left. That is, MSP drops groups in the sequence of the externality chain. However, by the Chain Lemma, s-MSP can simply drop all the groups in the externality chain except for G^c, since the chain ends at G^c anyway. That is, the *drop step* combines the *sequential drop* and *sequential unblock* steps of MSP. This not only increases computational efficiency, but also simplifies the proofs.

[36] By definition, the start component and the end component are distinct. Note that there can exist multiple component paths from the start component to the end component. For instance, at $(2,1)$ in Figure 4, there are two component paths from 5 to 1, which are $5 \to 3 \to 2 \to 1$ and $5 \to 4 \to 1$.

Lemma 1 (Chain)**.** *In the s-MSP, if unblocking* $\tilde{b}_{cm}$ *leads to* $|\mathcal{G}| > 1$ *(i.e., in the beginning of the drop step), G is a chain that starts at* G^m *and ends at* G^c*.*

Corollary 1. *The mechanisms MSP and s-MSP are equivalent, i.e., both produce the same winner and the price up to ties.*

By Corollary 1, the names MSP and s-MSP will be used interchangeably. (In addition, Lemmas 1 (chain), 4 (connectedness), and 9 (stop in finite time) show that MSP is well-defined.) I present below three main theorems: pairwise stability (PS), incentive properties, and generalization of SP; and one lemma about participation incentive (weak IR). The characterization (Theorem 5) will be shown in Section 4.6 after defining GWR.

First, a mechanism should have some good stability and "fairness" property, but the core-property is impossible with LPF (Proposition 1). MSP satisfies a well-known alternative, pairwise stability (PS). In other words, if an outcome is not PS, then the price is even lower than some bidder's bid against the winner, which might be highly disputable or unstable.

Theorem 2 (Pairwise stability)**.** *The MSP is pairwise stable.*

PS of MSP is not obvious (as opposed to PS of $\hat{\varphi}$), since bidder j can be about to be dropped at q even though j is still bidding against some bidder i. However, by the Chain Lemma, we know that all such i is still being blocked by j when a chain appears, i.e., either $G^i = G^j$ or $G^i \to G^j$. Thus, all such i must be dropped earlier than or together with j, which is formalized in the Generalized Pairwise Stability (GPS) Lemma (Lemma 6 in Appendix C). Then, PS is a corollary of the GPS Lemma.

Unfortunately, IC is impossible with LPF and PS (Theorem 1). In fact, IC is often impossible with some other good properties even when there are no externalities. However, there are several non-IC mechanisms (e.g., core-selecting package auctions, the generalized second-price (GSP) auction) that are widely used, and they have some good incentive properties. Likewise, MSP has the following good ex-post incentive properties. When participants care about externalities, it is more likely a high-stakes auction which is also a rare event. In such auctions, good ex-post properties might be more important than some good Bayesian properties, where finding an optimal Bayesian strategy is often quite difficult. What bidders mostly care about may be as follows: winners wonder if they could have won at a lower price by misreporting. Losers wonder if they could have won at a good price by misreporting. MSP has no such regrets.[37]

[37] As in the case of minimum-revenue core-selecting package auctions, several good incentive properties and multidimensionality seem to make manipulating MSP difficult. Kojima and Pathak (2009) show the asymptotic strategyproofness in many-to-one matching markets (see also Roth and Peranson (1999), Immorlica and Mahdian (2005), Bulow and Levin (2006), Che and Kojima (2010), Kojima and Manea (2010b),

Theorem 3 (Incentive properties)**.** *The MSP has the following incentive properties.*

(i) *("no overpay regret" or "one price for one bidder") The winner cannot win at a different (lower or higher) price by misreporting.*

(ii) *("no overturn regret") A loser cannot be better off winning by misreporting.*

No overpay regret is equivalent to "*one price for one bidder*." That is, the winner cannot also win at a higher price by misreporting $\mathbf{b}'_w$, since if this is possible, then it means overpay regret of a bidder with true bid $\mathbf{b}'_w$ by misreporting $\mathbf{b}_w$. Note that no overpay regret or no overturn regret does not imply IC for the winner or losers, respectively. In some cases, interestingly, the winner can be better off losing to some bidder by misreporting. Likewise, a loser can be better off losing to another bidder than to the current winner by misreporting. Of course, any manipulation is impossible in some cases, e.g., in the motivating example in Section 2, truthful reporting is a dominant strategy for every bidder.

Now I explain some intuition behind the proofs. First, no overpay regret might seem surprising and difficult to prove, considering the fact that a bid is multidimensional. Perhaps surprisingly, however, it can be easily proven by PS and the fact that $p = b_{wh}$, i.e., the price is some bidder's bid against the winner. Thus, w cannot win at any lower price than p by PS. Second, no overturn regret means $p_j \geq b_{wj}$, where p_j denotes the winning price for a loser j by misreporting. This is nontrivial and it needs to be emphasized that LPF and PS do not imply no overturn regret. LPF and PS imply $p \geq b_{wj}$; thus, it is true that j cannot be better off winning by blocking the current outcome, i.e., paying more than p. However, j does not need to block the current outcome to win, since no overturn regret means j cannot win at any lower price than b_{wj} by any misreporting, not blocking the current outcome. No overturn regret, i.e., $p_j \geq b_{wj}$, can be shown by two steps: $p_j \geq q^*_j$ and $q^*_j \geq b_{wj}$. Note that the second inequality holds by the GPS Lemma (Lemma 6). The first inequality can be shown by the Blocking Lemma (Lemma 7), which implies that when bidder j is dropped at q^*_j, there exists some bidder i such that $b_{ji} \geq q^*_j$, i.e., i is blocking j until j is dropped. Thus, j cannot win at any price lower than q^*_j.

IR is also impossible with LPF and PS (Theorem 1), which might not be too disappointing when it is quite difficult to forecast an outcome with nonparticipation. Nevertheless, it may be desirable that every participant is guaranteed to receive at least the worst-case payoff of nonparticipation, which is weak IR. In particular, if an outcome is not weakly IR for the winner, then the price is even higher than her maximum bid against all other bidders, which might be quite undesirable. Weak IR of MSP is easy to show. For losers, it trivially

Kojima et al. (2013), Liu and Pycia (2013), Lee and Yariv (2014), Azevedo and Budish (2015), Azevedo and Leshno (2016), and Lee (2017)). Similarly, MSP might be asymptotically strategyproof in large markets. To prove or disprove this conjecture would be a good direction for future research.

holds due to LPF, and for the winner, by the nature of the algorithm, the winner is the last bidder who has some bid that has not been unblocked yet; thus the payment is at most the maximum bid against some bidder.

Lemma 2. *The MSP is weakly individually rational.*

In practice, mechanisms should be easily understandable by participants. For instance, despite its lack of IC, one reason why GSP is more widely used than VCG in online advertising auctions is it naturally generalizes SP so that bidders can easily understand (see footnote 8 and the conclusion). Likewise, the properties of MSP so far (i.e., PS (Theorem 2), incentive properties (Theorem 3), and weak IR (Lemma 2)) also hold in SP. Furthermore, the next theorem shows that MSP also naturally generalizes SP. In particular, (i) is also an additional good incentive property.

Theorem 4 (Generalization of SP)**.** *The MSP satisfies the following.*

(i) *The MSP is strategyproof for a bidder without externalities imposed by others.*

(ii) *The MSP reduces to the second-price auction when there are no externalities.*

First, (i) can be easily shown by using Theorem 3. For the winner, there is no overpay regret, i.e., one price for the winner, and it is easy to check losing is not profitable. For a loser, losing to another bidder has the same payoff, and winning is never profitable by no overturn regret. Second, (ii) can be shown by (i) and the Green-Laffont-Holmstrom Theorem. Alternatively, by the nature of the algorithm itself, MSP drops bidders in the same sequence as in the English auction whose direct mechanism version is SP.

Example 2. Some examples of MSP are provided with comparisons to other mechanisms. In all examples, $t_{jj} = \min_{i \in N^0}\{b_{ij}\}$, so T can be derived from B and therefore omitted. For the second-price (SP) auction, a unique symmetric Bayesian Nash equilibrium $\mathbf{b}_j^* = \overline{b_{ij}}$ (average) for $i \neq j$ is used (Jehiel et al. 1999, Proposition 4). As explained in the introduction, for a nonempty core in general, payment refusals are not allowed (Jehiel and Moldovanu, 1996; Jeong, 2017). In other words, the core outcome here means that no group of bidders are willing to pay together more than the current price, i.e., "no justified envy." (Bidders may still be able to deviate by refusing to pay although it might be unrealistic in practice.) MRC denotes a minimum-revenue core outcome where the winner pays as much as possible (i.e., up to weak IR holds) to prevent a loser's payment. Thus, whenever an MRC outcome is not LPF, there is no LPF core outcome.

(1) Motivating example (Figure 2)

$$B_N = \begin{bmatrix} 0 & 26 & 21 \\ 28 & 0 & 21 \\ 16 & 20 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	3	$(0,0,20)$	Y	Y	Y
VCG	3	$(0,0,10)$	N	N	Y
SP	2	$(0,22,0)$	N	N	Y
MRC	3	$(0,0,20)$	Y	Y	Y

At $(q,k) = (20,1)$, G_1 and G_2 are weakly but not strongly connected and $G_2 \to G_1$.

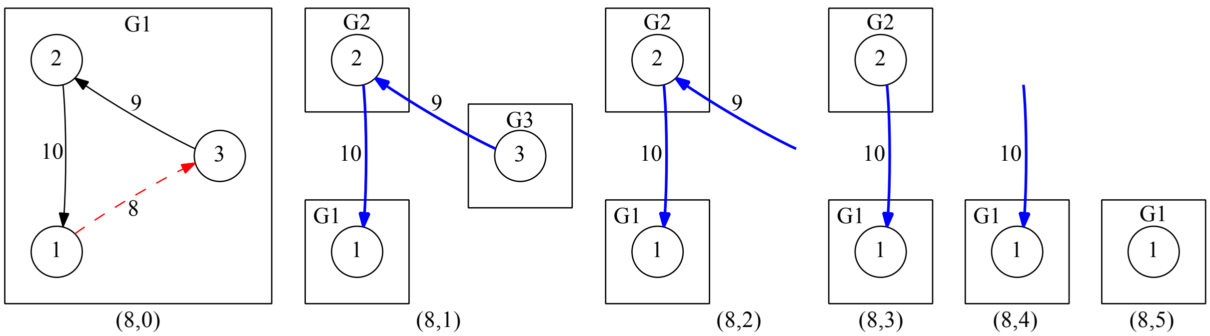

Figure 3: (q,k)-group bid graph of the "cycle" example

(2) Cycle (Figure 3)

$$B_N = \begin{bmatrix} 0 & 5 & 8 \\ 10 & 0 & 5 \\ 5 & 9 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	1	$(8,0,0)$	N	Y	Y
VCG	1	$(8,3,0)$	Y	Y	N
SP	1	$(7,0,0)$	N	N	Y
MRC	1	$(8,3,0)$	Y	Y	N

This example shows that the pairwise comparisons of bids can induce a cycle. Bidder 1 beats bidder 2 by $b_{21} > b_{12}$, and bidder 2 beats bidder 3 by $b_{32} > b_{23}$, but bidder 3 beats bidder 1 by $b_{13} > b_{31}$. Note that a "pairwise cycle" does not contradict pairwise stability. While it is true that bidder 3 beats bidder 1 by comparing bids pairwise, bidder 1 is willing to pay up to 10 against bidder 2. In fact, in this example, the only possible other outcome for bidder 1 by misreporting is bidder 2's winning where bidder 1 is worse off.

At $(q,k) = (8,1)$ in MSP, unblocking $\tilde{b}_{cm} = \tilde{b}_{13}$ leads to a chain: $G^m \equiv G^3 \to G^2 \to G^1 \equiv G^c$, as shown in the Chain Lemma. Thus, whereas MSP drops G^3 first and then G^2, s-MSP drops them all at once. That is, in s-MSP, the next step of $(8,1)$ is $(8,5)$.

(3) Multiple paths (Figure 4)

$$B_N = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 3 & 0 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & 3 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	1	$(2,0,0,0,0)$	N	Y	Y
VCG	1	$(2,1,0,0,0)$	N	Y	N
SP	1	$(1.5,0,0,0,0)$	N	N	Y
MRC	1	$(2,1,0.5,0.5,0)$	Y	Y	N

At $(2,1)$, unblocking $\tilde{b}_{15}$ leads to a chain with two paths: $5 \to 3 \to 2 \to 1$ and $5 \to 4 \to 1$. As shown in the Chain Lemma, no matter which path is chosen, the end group, G^1, is the same. Therefore, we can simply drop all the other groups except for the end group to find the winner and the price. That is, in s-MSP, the next step of $(2,1)$ is $(2,7)$.

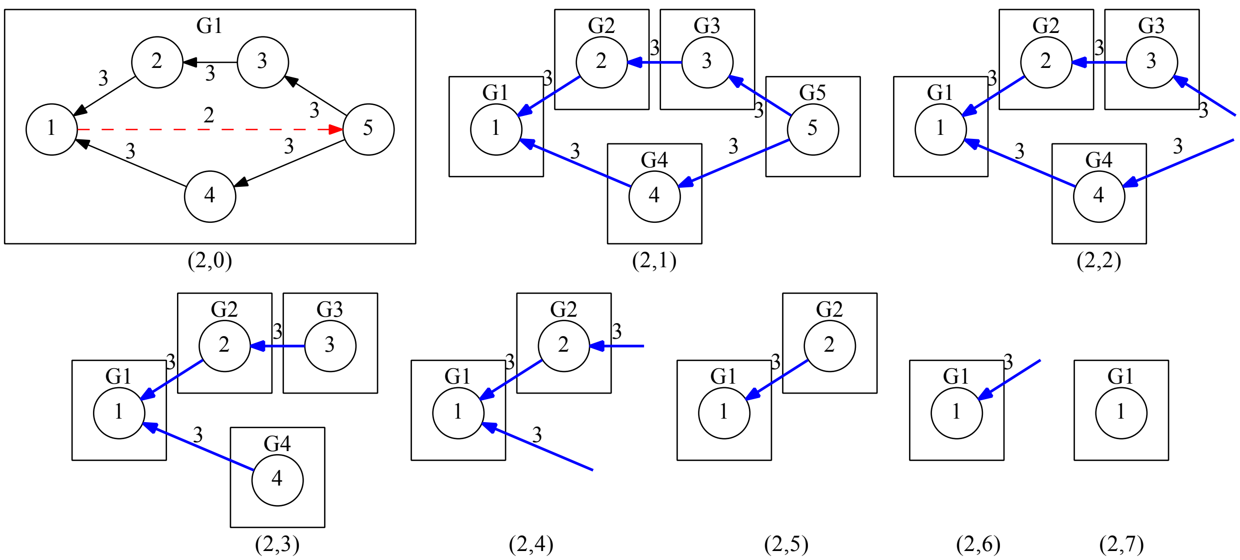

Figure 4: (q,k)-group bid graph of the "multiple paths" example

(4) Problems of VCG in Example 1

$$B_N = \begin{bmatrix} 0 & 10 & 1 \\ 9 & 0 & 3 \\ 9 & 7 & 0 \end{bmatrix}.$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	2	$(0,9,0)$	N	Y	Y
VCG	1	$(8,0,1)$	N	N	N
SP	1	$(8.5,0,0)$	N	N	Y
MRC	1	$(9,0,1)$	Y	Y	N

As already explained, VCG is not PS nor LPF.

(5) Another problem of VCG (subsidy)

$$T = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 8 & 0 & 0 \\ 0 & 2 & 10 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right] \text{ or } B = \left[\begin{array}{c|ccc} 0 & 8 & 10 & 8 \\ \hline 0 & 0 & 10 & 8 \\ 0 & 6 & 0 & 1 \\ 0 & 8 & 10 & 0 \end{array}\right].$$

	w	$\mathbf{p}$	core	PS	LPF
MSP	2	$(0, 6, 0)$	Y	Y	Y
VCG	2	$(0, -1, 0)$	N	N	Y
SP	2	$(0, 7, 0)$	Y	Y	Y
MRC	2	$(0, 6, 0)$	Y	Y	Y

Now the auctioneer, bidder 0, is included to show a subsidy explicitly. Note that even though positive externalities exist, $b_{ij} > 0$ for all $i \neq j$ and $j \neq 0$. That is, even when all bids are positive, a subsidy occurs in VCG. Moreover, $t_{00} = 0$ cannot work as a reserve price. Thus, this is another example showing that VCG is prone to shill bidding.

4.3 Multidimensional English Auctions

MSP is a direct mechanism; however, it "internally" runs a dynamic auction after receiving bids. Thus, it can be easily implemented as a dynamic open ascending auction, the multidimensional English (ME) auction. If we interpret an English auction as the so-called button auction model, the one-dimensional English auction provides only one button to each bidder. In contrast, ME provides each bidder separate buttons for each competitor. Initially, all buttons are pressed, i.e., $\tilde{b}_{ij} = 1$. If the current price q reaches b_{ij}, then j releases button against i, i.e., $\tilde{b}_{ij} = 0$. In the following, "submitting an unblocking bid C" implies "releasing buttons for all $i \in C$." From this information, the bid graph at a specific time can be constructed, which enables a dynamic open ascending auction.

Algorithm 3. The Multidimensional English (ME) Auction

1. Initialize $\widetilde{B}$ with $\tilde{b}_{ij} := 1, \forall i, j$. Let $G(\widetilde{B}) = (V, E, f)$.[38] $q := 0$.
2. Wait for an "unblocking bid" C for a given time.[39] If no bidders submit C, then increase q by a given increment, and go to step 2. Otherwise, let m be the bidder who submitted C first.
3. If $C = \emptyset$, go to step 2. Otherwise, choose a bidder $c \in C$ and $C := C \backslash \{c\}$.
4. (unblock step) $\tilde{b}_{cm} := 0$.
5. $\mathcal{G} := \{H \subseteq G : H \text{ is an SCC of } G\}$. If $|\mathcal{G}| = 1$, go to step 3.
6. (drop step) $\tilde{b}_{ij} := 0$ and $\tilde{b}_{ji} := 0, \forall i \in V, \forall j \in D$, where $D = \cup_{H \in \mathcal{G} \backslash G^c} V(H)$.

[38] In ME, a directed graph $G = (V, E)$ is sufficient since each $\tilde{b}_{ij} \in \{0, 1\}$. See also footnotes 34 and 35.

[39] An *unblocking bid* denotes a set of bidders that a bidder no longer bids against. Thus, the *truthful unblocking bid* C_j of bidder j is $C_j = \{i \in V \backslash j : b_{ij} \leq q\}$. When multiple bidders submit unblocking bids, one bid is chosen by a tie-breaking rule. Bidders may have different strategies depending on the details of an implementation, e.g., what bidders can observe (e.g., existence itself of an unblocking bid, identity (owner) of it, contents (competitors of the owner) of it), how to handle a tie (e.g., whether an unblocking bid that was not chosen is still valid after the chosen unblocking bid is processed).

7. If $|V| > 1$, go to step 2. Otherwise, winner $w = c$ wins at price $p = q$.

The following corollary (of Theorem 4) shows that ME generalizes the English auction.

Corollary 2 (Generalization of English). *The ME satisfies the following.*

(i) *The ME is strategyproof for a bidder without externalities imposed by others.*[40]

(ii) *The ME reduces to the English auction when there are no externalities.*

4.4 Group winner regret-free monotonicity

For a mechanism to have good incentive properties, the allocation rule normally need to be "monotone" in some sense, and this monotonicity is also useful to characterize mechanisms. For instance, in one-dimensional auctions, the allocation rule is said to be "monotone" (or even "standard" since this is the most natural allocation rule) if a higher bid beats a lower bid, i.e. the highest bid wins, or more generally, a higher bid has a higher probability of winning. In addition, this monotonicity is necessary condition for IC. In our model, Jehiel et al. (1999) show a monotonicity condition for IC, which cannot be applied due to the impossibility result (Theorem 1). I define *group winner regret-free monotonicity* (GWRF-monotonicity) and derive the revenue equivalence result in GWRF-monotone mechanisms.

Definition 6. An allocation rule is *group winner regret-free monotone* (GWRF-monotone) and denoted by x^* if it is the same as the allocation rule of MSP.

GWRF-monotone allocation (other monotone) rule is an allocation rule of a direct mechanism, i.e., it directly gives the winner or winning probability. Due to its monotonicity, however, it can also be used in open ascending dynamic auctions, e.g., ME. That is, the GWRF-monotone allocation rule drops a group sequentially as the price increases. Any GWRF-monotone mechanism is determined by the payment rule. For instance, the payment rule of MSP is only the winner pays (i.e., LPF), and the winning price is q_w^*, where q_j^* is the price at which bidder j is dropped or won by the GWRF-monotone allocation rule x^*. It should be emphasized that q_j^* including q_w^* is solely determined by the allocation rule x^* (not by the payment rule). In particular, q_w^* can also be thought as the price when w is "dropped" as the last bidder by x^*. Note also that when there are no externalities (i.e., in one-dimensional auctions), the GWRF-monotone allocation rule reduces to the aforementioned "*standard*" allocation rule.

As the name suggests, there is a relationship between GWRF-monotonicity of allocation rules and GWR-freeness of mechanisms. Since the formal definition of GWR (Definition 15)

[40] In ME, bidder j is IC (or strategyproof) if the payoff when j uses the truthful unblocking bid (footnote 39) all the time is at least as large as the payoff when j uses any different unblocking bid.

is nontrivial, the following simplified "definition" (i.e., necessary and sufficient condition) is used instead for the remainder of this section. (Remind that X is the correspondence version of x when a tie exists.)

Proposition 2. *For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, a group of bidders S with $w \in S$ has group winner regret in φ if and only if $w \notin X^*(B)$, $p_w > q_S^*$, and $S = \{j : q_j^* = q_w^*$ and $k_j^* = k_w^*\}$.*

Corollary 3. *The MSP is group winner regret-free.*

That is, S including the winner w has group winner regret if and only w is different from the winner by the GWRF-monotone allocation rule x^* and the price is higher than q_S^*, which is the price at which S is dropped by x^*. Thus, a GWRF-monotone mechanism is GWR-free. If w, however, wins at $p < q_S^*$ in some mechanism φ, then it is still GWR-free, which shows GWR-monotonicity is only a sufficient but not necessary condition for GWR-freeness. The intuition why it is still GWR-free if w wins at $p < q_S^*$ is that she can be worse off losing to some "remaining bidders" since w wins at a low price, Note that, however, w's winning at $p < q_S^*$ violates at least one of PS and LPF, as will be explained in the characterization result (Theorem 5).

For GWRF-monotone mechanisms, the classical payoff (and thereby revenue) equivalence result (e.g., Myerson (1981)) holds as follows. Note that since a GWRF-monotone mechanism is not IC, we need to apply the revelation principle twice and then apply the general payoff equivalence result of Krishna and Maenner (2001).

Proposition 3. *Let φ' and φ'' be any two GWRF-monotone mechanisms that are loser's payment-free. Then, there exists an equilibrium of φ' and an equilibrium of φ'', respectively, such that the expected payoffs of any fixed bidder in φ' and φ'' are the same in equilibrium.*

4.5 Multidimensional First-Price Auctions

While incentive compatibility is desirable, as in the Kepco example, in practice, the first-price auction is much more widely used than the second-price auction. There are several possible explanations from the literature: risk-aversity (Holt, 1980), budget constraints (Che and Gale, 1998), privacy and cheating (e.g., a fake second-highest bid by the seller) (Rothkopf et al., 1990), and asymmetry (Maskin and Riley, 2000). Thus, considering the first-price version of MSP would be of interest both in theory and practice.

Unlike the standard first-price auction, due to multidimensionality, there can be many possible choices for the "first-price." Of course, there is no reason why any particular bid (against some bidder) of the winner cannot be chosen as the first-price, but the choice obviously affects the incentives of bidders and other properties of the mechanism. That is, even though the expected payoff of a bidder or revenue of the seller is equivalent, for certain good properties, the price should be chosen carefully. For instance, for PS, the only candidates for the price are b_{iw} for all i where $b_{iw} \geq q^*_w$. However, for no overturn regret, it should be chosen from $\{b_{iw} : i \in D$ at step 9 in MSP$\}$, i.e., bids against the remaining bidders right before the winning by GWRF-monotone allocation rule, since those bidders are the reason why the winner still stayed.

While there are still many choices, I choose the maximum from this set as the *first-price* of the *multidimensional first-price* (MFP) auction mechanism. MFP (or with any payment rule of $\rho(B) \in \{b_{iw} : i \in D$ at step 9 in MSP$\}$) also has several good properties: LPF and PS (by the payment rule); GWR-freeness (by the GWRF-monotone allocation rule); and notably, no overturn regret. However, due to a potentially strong incentive to underbid to avoid overpay regret, as in the first-price auction, especially no overturn regret needs to be carefully interpreted.

Definition 7. The multidimensional first-price (MFP) auction mechanism is (x^*, ρ), where $\rho(B) = \max_i\{b_{iw} : i \in D$ at step 9 in MSP$\}$ and $w = x^*(B)$.

Corollary 4. *The MFP is pairwise stable and weakly individually rational, and free of the following: loser's payment, group winner regret, and overturn regret. Furthermore, the MFP reduces to the first-price auction when there are no externalities.*

Proof. By Theorems 2, 3, and 4, and Lemma 2. □

4.6 Characterizations

In this section, I characterize MSP and also provide additional characterizations without GWR-freeness. MSP is free of overpay regret (Theorem 3), but for the characterization result, a weaker version of no overpay regret is sufficient. Depending on the characteristics of misreporting, regret about the price can be defined with different degrees as follows.

Definition 8. For a given $\mathbf{b}_j$ and $q \in [\underline{b}, \overline{b}]$, a *q-capped bid* is defined as $\mathbf{b}^{\overline{q}}_j \equiv (b'_{ij})$, where $b'_{ij} = b_{ij}$ if $b_{ij} \leq q$, otherwise $b'_{ij} = q$.

Definition 9. For a mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, the winner w has

- *overpay regret* if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$
- *underbid overpay regret* if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$ with $\mathbf{b}'_w \leq \mathbf{b}_w$
- *capped-bid overpay regret* if there exists $\mathbf{b}'_w \neq \mathbf{b}_w$ with $\mathbf{b}'_w = \mathbf{b}^{\overline{q}}_w$ for some $q \in [\underline{b}, \overline{b}]$

such that $w \in X(\mathbf{b}'_w, \mathbf{b}_{-w})$ and $\rho_w(\mathbf{b}'_w, \mathbf{b}_{-w}) < p_w$.

Overpay regret means the winner could have won at a lower price by any misreport, and the others have some restriction on misreports. By construction, overpay regret includes underbid overpay regret, and underbid overpay regret includes capped-bid overpay regret. For the characterization in Theorem 5, the weakest property, capped-bid overpay regret-freeness, is sufficient, while MSP has the strongest property–overpay regret-freeness. Note that this extension is not obvious since a bid is multidimensional. Now I characterize MSP and provide some intuition.

Theorem 5 (Characterization). *The MSP is a unique (up to ties) direct mechanism that is*

(i) *loser's payment-free (LPF),*
(ii) *pairwise stable (PS),*
(iii) *weakly individually rational (weakly IR),*
(iv) *group winner regret-free (GWRF), and*
(v) *capped-bid overpay regret-free (CORF).*

It has been shown that MSP satisfies each axiom (Theorems 2 and 3, Corollary 3, and Lemma 2), and it can also be shown that the axioms (i)-(v) in the above characterization are independent. Here is some intuition of the uniqueness. First, for the allocation rule, while GWRF-monotone allocation rule is a sufficient condition of GWRF (of a mechanism), only GWRF is needed for the characterization. Thus, by GWRF only, it may have a different winner w' (remind that if the price is lower than $q^*_{w'}$, then it is still GWRF). However, then by the Blocking Lemma (Lemma 7, i.e., w' cannot win any lower price than $q^*_{w'}$ in any LPF and PS mechanism), it violates at least one of PS and LPF. That is, GWRF, LPF, and PS determines the allocation rule. Second, for the payment rule, LPF and PS provide a lower bound, and weak IR provides an upper bound, and then CORF pins down the payment of the winner. Finally, for independence, here are counterexamples for each case. (i) LPF: some losing bidder pays while weak IR for that bidder is preserved; (ii) PS: a "third-price"; (iii) Weak IR: the winner pays $\bar{b}$, which interestingly does not violate CORF; (iv) GWRF: $\hat{\varphi}$; and (v) CORF: MFP. Note also that we can characterize MSP as follows.

Corollary 5 (Alternative characterization). *Of all mechanisms that satisfy (i)-(iv), MSP is a unique (up to ties) minimum-revenue direct mechanism.*

As minimum-revenue core-selecting package auctions minimize the sum of individual incentives (to deviate by misreporting) in the class of core-selecting mechanisms, MSP also minimizes the sum of individual incentives in the class of the direct mechanism that is LPF, PS, weakly IR, and GWRF. One may want to further minimize the sum of individual incentives while sacrificing some properties of MSP. Thus, I show two additional characterizations with fewer axioms, where the existence of the two mechanisms is guaranteed by MSP.

Proposition 4. *Up to ties, the following holds.*

(i) *Of all mechanisms satisfying LPF and PS, a unique minimum-revenue direct mechanism is* $\varphi^{LP} = (x, \rho)$, *where* $x(B) \in \arg\min_i \max_{j \neq i}\{b_{ij}\}$, $\rho_w(B) = \max_{j \neq w}\{b_{wj}\}$, *and* $\rho_i(B) = 0$ *for all* $i \neq w$.

(ii) *Of all mechanisms satisfying LPF, PS, and weak IR, a unique minimum-revenue direct mechanism is* $\varphi^{LPW} = (x, \rho)$, *where* $x(B) \in \arg\min_i \max_{j \neq i}\{b_{ij}\}$ *such that* $\max_j\{b_{ij}\} \leq \max_j\{b_{ji}\}$, $\rho_w(B) = \max_{j \neq w}\{b_{wj}\}$, *and* $\rho_i(B) = 0$ *for all* $i \neq w$.

Note that $p^{LP} \leq p^{LPW}$, where p^{LP} and p^{LPW} are the revenues of the two mechanisms, respectively. The inequality can hold strictly for some B. Likewise, $p^{LP} \leq p^{LPW} \leq p^{MSP}$, and the last inequality can hold strictly. Since $p^{LP} \leq p^{LPW} \leq p^{MSP}$, the sum of incentives to deviate by *individual* misreporting is smaller in φ^{LPW} than in MSP, and in turn is smaller in φ^{LP} than in φ^{LPW}. However, neither of φ^{LP} and φ^{LPW} is free of GWR, i.e., *groupwise* deviation incentives to avoid GWR exist. Once the bidders collude, they might want to further collude, e.g., bid very low and share profits. Thus, the deviation incentives should be read carefully.

Despite the impossibility theorem (Theorem 1), another plausible interest may be *constrained efficiency maximization*, i.e., more efficient allocation while sacrificing some properties of MSP. This can be easily done in a similar way to Proposition 4. That is, from the most efficient allocation, check if required axioms are satisfied, and move to the second most efficient allocation if not, and so on.

5 Group winner regret

This section formally defines *group winner regret* (GWR) in *direct* mechanisms. Due to the novelty and nontriviality, I will first define its individual version, *winner's regret* (WR). To the best of my knowledge, group winner regret has not been studied in the auction literature. Its individual version, winner's regret, is also a novel approach that handles the case when one bidder imposes externalities on multiple bidders in a unified manner.[41]

For intuition, in open ascending *dynamic* (i.e., not *direct*) mechanisms where "drop" and "remaining" are well-defined, the intuition behind *winner's regret* is that the winner says, "I regret not dropping out at that price. No matter which remaining bidder had won at what price, I would have been better off." That is, the intuition behind *avoiding* winner's regret

[41] For instance, Varma (2002) assumes that there is no such case, and Hu et al. (2013) use simple models with three bidders. This paper allows the completely general externality structure and any finite number of bidders, and defines winner's regret that can be avoided by truthful report.

is, “I don’t know who will win at what price if I drop out now. But I’m sure that I will be better off.”

The direct analogue for *group winner regret* might seem to be that a group of bidders says, “We regret not dropping out together at that price. No matter which remaining nonmember had won at what price, *all of us* would have been better off.” However, this “all of us” is too strong to be satisfied, and the intuition behind group winner regret is, “We regret not dropping out together at that price. No matter which remaining nonmember had won at what price, each member would have been better off than if that member or some other member had won.” In other words, each member says, “Losing to any remaining nonmember is better than my winning, so I want to drop out. But I can’t due to another member that I don’t want to lose to,” which induces “unnecessary” internal competition. When the group size is 2, all the members are better off losing to any remaining nonmember since the two members are all the members. In addition, there are realistic conditions on externalities (i.e., correlated externalities in Definition 14) where even for the group whose size is larger than 2, all the members are better off losing to any remaining nonmember.

Now we need to interpret what “drop” and “remaining” mean in direct mechanisms. In direct mechanisms, a “drop” strategy can be implemented by a *capped* bid (Definition 8), as in the following obvious lemma. “Remaining” bidders can be interpreted as “undominated” bidders in direct mechanisms, which will be defined by (q,k)-*bid graph* for WR and (q,k)-*group bid graph* for GWR.

Lemma 3. *For given B and a mechanism φ that is weakly IR, let $\varphi(\mathbf{b}_j^{\overline{q}}, \mathbf{b}_{-j}) = (w, \mathbf{p})$ for some $q \in [\underline{b}, \overline{b}]$. Then, it is impossible that $w = j$ with $p_w > q$. If φ is also LPF and PS, and there exists $i \neq j$ such that $b_{ji} > q$, then it is impossible that $w = j$ with $p_w \geq q$.*

5.1 Winner’s regret

As illustrated in Section 2, $\hat{\varphi}$ is a direct mechanism version of the English auction with the bidding function $\hat{\beta}$ (Equation 1).

Algorithm 4. Mechanism $\hat{\varphi}$

1. Each bidder j submits $\mathbf{b}_j \in \mathcal{B}_j$. $\widetilde{B} := B$. $\tilde{b}_{jj} := 1$ for all j. Let $G(\widetilde{B}) = (V, E, f)$.
2. $q := \min\{\tilde{b}_{ij} : (i,j) \in E, i \neq j\}$. $k := 0$ and $B_{(q,k)} := \widetilde{B}$.
3. (unblock) $\tilde{b}_{ij} := 0$ for all $(i,j) \in E$ with $\tilde{b}_{ij} = q$. $k := k+1$ and $B_{(q,k)} := \widetilde{B}$.
4. $D := \{j \in V : \tilde{b}_{ij} = 0 \text{ for all } i \in V, i \neq j\}$. If $D = \emptyset$, go to step 2.
5. (sequential drop) $\tilde{b}_{jj} := 0$ and $(\hat{q}_j, \hat{k}_j) := (q,k)$ for all $j \in D$. $k := k+1$ and $B_{(q,k)} := \widetilde{B}$. If $|V| = 0$, go to step 7.
6. (sequential unblock) $\tilde{b}_{ij} := 0$ for all $(i,j) \in E$ with $i \in D$. $k := k+1$ and $B_{(q,k)} := \widetilde{B}$.

7. If $|V| > 1$, go to step 4. Otherwise, winner w wins at price $p = q$, where $w \in V$ if $|V| = 1$, or w is chosen out of D by the tie-breaking rule if $|V| = 0$. $(\hat{q}_w, \hat{w}_j) \coloneqq (q, k)$

Definition 10. An allocation rule is *winner regret-free monotone* (WRF-monotone) and denoted by $\hat{x}$ if it is the same as the allocation rule of $\hat{\varphi}$.

Definition 11. $B_{(q,k)}$ is called a (q,k)-*(undominated) bid matrix*, and its graph $G(B_{(q,k)})$ is called a (q,k)-*bid graph.* A bidder j is said to be *undominated* at (q,k) if $j \in V(B_{(q,k)})$, otherwise *dominated* at (q,k). That is, $V(B_{(q,k)})$ is the *set of undominated bidders* at (q,k).

As shown in Figure 1, the (q,k)-bid graph describes the status of an auction at price q and step k, and $(\hat{q}_j, \hat{k}_j)$ denotes the price and the step when bidder j is dropped or won. When B needs to be specified, $(\hat{q}_j, \hat{k}_j)$ will be written as $(\hat{q}_w(B), \hat{w}_j(B))$. In fact, if j is dropped at $(\hat{q}_j, \hat{k}_j)$, then j's staying any longer is dominated (in English auctions), assuming no one has played dominated strategies. That is, losing to any bidder $i \in V(B_{(\hat{q}_j, \hat{k}_j)}) \backslash j$ is weakly better off than if j wins at any price $p \geq \hat{q}_j$ (strictly better off if $p > \hat{q}_j$). $\hat{\varphi}$ drops bidders who are dominated as prices increase. Now I define winner's regret in direct mechanisms.

Definition 12. For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, winner w has *winner's regret* if there exists $q < p_w$ such that

$b_{iw} \leq q$ for all $i \in V(B'_{(\hat{q}_w(B'), \hat{w}_j(B'))}) \backslash w \neq \emptyset$, where $B' = (\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w})$.[42]

The above definition interprets the intuition behind winner's regret in terms of direct mechanisms: if w submits $\mathbf{b}_w^{\overline{q}}$ ("drop"), then w will be better off no matter which bidder $i \in R = V(B'_{(\hat{q}_w(B'), \hat{w}_j(B'))}) \backslash w$ (R: "remaining bidders") will win at any price $q' \geq q$ since $b_{iw} \leq q < p_w$.

If w has winner's regret in φ, then q (satisfying the condition) should exist. Thus, when q needs to be mentioned explicitly, it will be written as "w has winner's regret at q in φ" (the same notation applies to group winner regret). The infimum of such q's is $\hat{q}_w(B)$ as in the following proposition, which can be used to determine whether w has winner's regret or not by comparing with p_w; and this proposition will also be used to prove the relationship between winner's regret and group winner regret in Corollary 8.

Proposition 5. *For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, winner w has winner's regret in φ if and only if $w \neq \hat{x}(B)$ and $p_w > \hat{q}_w$.*

[42] "$V(\cdot)\backslash w \neq \emptyset$" excludes the case that w still wins with B' at some $q' \leq q$, which is overpay regret. Strictly speaking, the definition should say "$V(\cdot)\backslash w \neq \emptyset$ for all outcomes by the tie-breaking" in order to exclude the existence of winner's regret solely due to tie-breaking. For instance, suppose $B = [0,7,5;7,0,5;9,5,0]$ and $\varphi(B) = (1,8)$. Then, depending on the tie-breaking, it is possible that $V(\cdot)\backslash w = \{2\}$ for $q = 7$, but $b_{21} \leq 7$. Note also that p_w (instead of p) is used to include non-LPF mechanisms for the independence of the axioms in the characterization (Theorem 5). This footnote also applies to group winner regret.

Corollary 6. *If w has winner's regret,* $\inf_q\{q : w \text{ has winner's regret at } q\} = \hat{q}_w$.[43]

When the infimum needs to be mentioned explicitly, it will be written as "w has winner's regret at $\underline{\hat{q}_w}$ in φ" (the same notation applies to group winner regret). Note that Proposition 5 is the counterpart of Proposition 2, which was introduced as an alternative "definition" of group winner regret. As GWRF-monotonicity is a sufficient (but not necessary) condition for GWR-freeness of a mechanism, WRF-monotonicity is a sufficient (but not necessary) condition for WR-freeness of a mechanism. For instance, $\hat{\varphi}$ is winner's regret-free.

One might claim that the definition of winner's regret in φ should not use an *external* reference mechanism, $\hat{\varphi}$, i.e., the set of the "remaining" bidders, R, should not be defined by $\hat{\varphi}$. This may seem a reasonable concern. However, note that we first need some restriction on R since it is impossible to have no regret against all other bidders. Thus, R should be some subset of $N^0\backslash w$. The choice of $R = V(B'_{(\hat{q}_w(B'),\hat{w}_j(B'))})\backslash w$ is reasonable for direct mechanisms for the following three reasons, which can also be applied to group winner regret similarly. First, there are mechanisms free of the regret, e.g., $\hat{\varphi}$ and MSP. Second, it has a well-founded interpretation in English auctions, i.e., R is the set of undominated bidders since $\hat{\varphi}$ drops dominated bidders successively as prices increase. Thus, while some $i \in R$ will definitely win at some $q' \geq q$ when w drops at q, any $i \in D = R^c\backslash w$ cannot win since it has been dropped already. That is, there is no reason that w worries if some $i \in D$ might win when w drops at q even when $b_{iw} > q$.

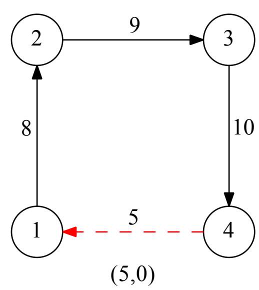

Figure 5: "remaining" bidders in direct mechanisms

Likewise, in direct mechanisms, if some $i \in D$ wins at some $q' \geq q$ when w submits $\mathbf{b}_w^{\overline{q}}$, the following problem occurs, which is the third justification for R. The problem is that now i has "winner's regret," i.e., with respect to the outcome for $B' = (\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w})$, i regrets not using a q''-capped bid, where $\hat{q}_w < q'' < q'$. In Figure 5, for instance, bidder 4 wins at 5 in $\hat{\varphi}$, and $\hat{q}_j = 5$ for all j. Suppose bidder 3 wins at 9 in φ, which is winner's regret at q, where $\hat{q}_3 < q < 9$. If bidder 3 submits $\mathbf{b}_3^{\overline{7}}$, i.e., $B' = (\mathbf{b}_3^{\overline{7}}, \mathbf{b}_{-3})$, then $R = \{4\}$ and $D = \{1, 2\}$.

[43] The infimum is necessary, i.e., the minimum may not exist. For instance, in Figure 5, if bidder 3 wins at some p such that $5 = \hat{q}_3 < p \leq 9$, bidder 3 has winner's regret at $\underline{5}$. However, if it submits $\mathbf{b}_3^{\overline{q}}$ for $q = 5$, then $R = \{1, 2, 4\}$, not $R = \{4\}$ for $5 < q < p$. The same applies to group winner regret.

Now, if bidder $i \in D$ wins at $q' \geq 7$ for B', the following problem occurs. First, bidder 1 cannot win at $q' \geq 7$ due to weak IR. However, if bidder 2 wins at $q' \geq 7$ (which is weakly IR), then bidder 2 regrets not losing to bidder 3 or 4 by using $\mathbf{b}_2^{\overline{q''}}$, where $\hat{q}_2 < q'' < q'$, e.g., $q'' = 6$, since for $B'' = (\mathbf{b}_3^{\overline{7}}, \mathbf{b}_2^{\overline{6}}, \mathbf{b}_{\{1,4\}})$, bidder 1 cannot win at $q''' \geq 6$ due to weak IR. That is, winner's regret can be recursively defined without $\hat{\varphi}$ as follows:

w has winner's regret if there exists $q < p_w$ such that $b_{iw} \leq q$ for all i that can win at q, and there does not exist $q' < q$ such that $b_{i'i} \leq q'$ for all i' that can win at q', and there does not exist $q'' < q'$ such that $b_{i''i'} \leq q'$ for all i'' that can win at q'', ...

Note that the part "*all i that can win at q, ...*" restricts R, which makes $R = V(B'_{(\hat{q}_w(B'),\hat{w}_j(B'))})\backslash w$ in Definition 12; e.g., in the above simple example in Figure 5, the recursion of the definition ends at the second line since there is no such i'.

5.2 Group winner regret

Now I formally define the group bid graph and then group winner regret in direct mechanisms.

Definition 13. $B^{(q,k)}$ is called a (q,k)-*group (undominated) bid matrix*, and its graph $G(B^{(q,k)})$ is called (q,k)-*group bid graph*. A bidder j is said to be *group undominated* at (q,k) if $j \in V(B^{(q,k)})$, otherwise *group dominated* at (q,k). That is, $V(B^{(q,k)})$ is the *set of group undominated bidders* at (q,k).

As shown in Figure 2, the (q,k)-group bid graph denotes the auction status of MSP at price q and step k. Note that, however, (q,k)-group bid graph is independent of the payment rule, e.g., MSP and MFP has the same group bid graph. To emphasize this fact, let φ^* denote any mechanism that has the same group bid graph with MSP. Then, φ^* drops a group of bidders who are group dominated as prices increase. In particular, (q_j^*, k_j^*) denotes the price and the step when bidder j is dropped or won. (q_S^*, k_S^*) denotes the "earliest" price and step among (q_j^*, k_j^*) for all $j \in S$, i.e., $(q_S^*, k_S^*) = (\min_{j\in S}\{q_j^*\}, \min_{j'\in \arg\min_{j\in S}\{q_j^*\}}\{k_{j'}^*\})$. Note that if $S = V(H)$ for some $H \in \mathcal{G}$, then $(q_j^*, k_j^*) = (q_{j'}^*, k_{j'}^*)$ for all $j, j' \in S$.

As in winner's regret, one might expect that if S drops at (q_S^*, k_S^*), then S's staying any longer is dominated (in English auctions, assuming no one has played dominated strategies), i.e., every member $j \in S$ is (weakly) better off losing to any nonmember $i \in R = V(B^{(q_S^*,k_S^*)})\backslash S$ than if some member $j' \in S$ wins at any price $q' \geq q_S^*$ (strictly better off if $q' > q_S^*$). Unfortunately, however, this may not be true when $|S| > 2$. This is because an SCC does not necessarily imply a *direct* bidirectional path between each pair of nodes, i.e., it can be a *indirect* path. Thus, if $b_{j'j} < b_{ij}$, then j can be better off losing to j' than losing to i. When $|S| = 2$, however, strong connectedness means a bidirectional path between the two nodes. Thus, every member $j \in S$ prefers to lose to any nonmember

$i \in R$, which is also trivially true when $|S| = 1$. When $|S| > 2$, to make every member $j \in S$ prefer to lose to any nonmember $i \in R$, some condition on externalities is needed, and one reasonable sufficient condition that is also often assumed in the literature is shown below.

Definition 14. B is said to have *correlated externalities* if in $G(B^{(q,k)})$ for some (q, k), the fact that there is a path from j' to j but no path from i to j implies $b_{j'j} \geq b_{ij}$.

The correlated externalities means that at some price q, if bidders j' is at least an indirect competitor of j (i.e., at least an indirect path exists), but i is not even an indirect competitor of j, then j suffers greater externality due to j' than due to i. Roughly, this implies "if you are my competitor, your competitor is my competitor as well," which is plausible when externalities mainly come from local competition. For instance, consider an auction for a Major League Baseball posting, where two leagues, the American and the National, exist. Each team is more likely to bid higher against teams in the same league than the other.

Note that the following commonly used externality structure in the literature also satisfies the correlated externalities: each bidder has only one "strong" competitor, i.e., each bidder suffers a negative externality only due to one bidder. Now I define group winner regret.

Definition 15. For a given mechanism φ and its outcome $\varphi(B) = (w, \mathbf{p})$, a group of bidders $S \subset N^0$ with $w \in S$ has *group winner regret* if there exists $q < p_w$ such that

(i) $b_{ij} \leq q$ for all $j \in S$ and $i \in V(B'^{(q^*_S(B'),k^*_S(B'))}) \backslash S \neq \emptyset$, where $B' = (\mathbf{b}^{\overline{q}}_S, \mathbf{b}_{-S})$, and

(ii) for all $j, j' \in S$ with $j \neq j'$, there exists a sequence of distinct bidders $(j_n)^k_{n=1}$ such that $j_1 = j$, $j_k = j'$, $b_{j_{n+1},j_n} \geq q$ for all $1 \leq n \leq k-1$, and $b_{1,k} \geq q$.[44]

The above definition interprets the intuition behind group winner regret in terms of direct mechanisms: (i) if S submits $\mathbf{b}^{\overline{q}}_S$ ("drop"), then every member $j \in S$ will be better off, no matter which nonmember $i \in R = V(B'^{(q^*_S(B'),k^*_S(B'))}) \backslash S$ ("remaining bidders") wins at any price $q' > q$, than if j wins. However, (ii) each member j_1 is better off winning (at q) than losing to some other member j_2, and j_2 is better of winning than losing to j_3, ..., and j_k is better of winning than losing to j_1. That is, (ii) is the reason for the "unnecessary" internal competition. Also, (i) and (ii) imply that $b_{k_j,j} \geq b_{ij}$, which in turn means that any member j is better off losing to any nonmember i than to some other member k_j. Note also that (i) and (ii) also implies the three conditions (now denoted by (i'), (ii'), (iii')) in Section 2. That is, (i) implies (i') and (ii'), and (ii) implies (iii').

The counterpart of Proposition 5 (necessary and sufficient condition of WR) is already introduced in Proposition 2 (necessary and sufficient condition of GWR), as an alternative "definition" of GWR, which also implies the following corollary.

[44] Note that (ii) ensures H, where $S = V(H)$, is strongly connected, and (i) ensures that H is a maximal subgraph that has this property. That is, (i) and (ii) together make H is an SCC. See also footnote 42.

Corollary 7. *If S has group winner regret,* $\inf_q\{q : S$ *has group winner regret at* $q\} = q^*_S$.

Example 3 (Motivating example continued)**.** For B_N in Section 2, $S = \{1, 2\}$ has GWR at $\underline{20}$ in $\hat{\varphi}$ since bidder 1 wins at $26 > q^*_S = 20$. S has GWR at $\underline{20}$ in SP with a unique symmetric Bayesian Nash equilibrium (Jehiel et al. 1999, Proposition 4), $b^*_j = \overline{b_{ij}}$ for $i \neq j$. Thus, $\mathbf{b}^* = (16 + 12/2, 20 + 6/2, 21) = (22, 23, 21)$, and bidder 2 wins at $22 > q^*_S$.

Note that the winners in English and SP auctions are different, but both outcomes are inefficient. In any case, GWR cannot be solved individually depending on other's bid. For instance, suppose $\mathbf{t}_2 = (0, 22, 0)'$. If bidder 1 drops out before bidder 3 to avoid GWR, then bidder 2 wins at 21, where bidder 1 regrets not bidding high enough.

As its name suggest, group winner regret includes winner's regret, i.e., if there exists winner's regret, then there exists group winner regret, but not always vice versa. I will show this as a corollary of a more general result. One main difference between $\hat{\varphi}$ and φ^* is that $\hat{\varphi}$ drops bidders *individually* while φ^* drops bidders *groupwise*. Thus, although it may not necessarily be obvious, we may expect that φ^* drops each bidder no later than $\hat{\varphi}$, which is in fact true.

Proposition 6. *For any* B, $\hat{\mathbf{q}}(B) \geq \mathbf{q}^*(B)$. *That is, each bidder is never dropped at a higher price in* φ^* *than in* $\hat{\varphi}$.

Corollary 8. *For a given mechanism* φ *and its outcome* $\varphi(B) = (w, \mathbf{p})$, *if* $w \neq x^*(B)$ *has winner's regret at* q *in* φ, *then there exists* $S \subset N^0$ *with* $w \in S$ *such that* S *has group winner regret at* q' *in* φ *with* $q' \leq q < p_w$.

Proof. By Propositions 2, 5, and 6. □

6 Conclusion

> We[45] thought very seriously about changing the GSP auction to a VCG auction during the summer of 2002. There were three problems: (i) the existing GSP auction was growing very rapidly and required a lot of engineering attention, making it difficult to develop a new auction; (ii) the VCG auction was harder to explain to advertisers; and (iii) the VCG auction required advertisers to raise their bids above those they had become accustomed to in the GSP auction. The combination of these issues led to shelving the VCG auction in 2002.
>
> Varian and Harris (2014)

[45] Google. This is about online advertising auctions, but the desiderata can be applied to other auctions.

Despite the lack of IC, GSP is widely used for online advertising auctions due to its simplicity in two ways: (i) "easy to implement": MSP is easy to implement and "fast" (i.e., polynomial time algorithm) and (ii) "easy to understand": GSP naturally generalizes SP, i.e., each winner only needs to pay the minimum amount that she can still win that position. In contrast, VCG is IC but difficult to explain.[46] MSP also naturally generalizes SP. The price is the minimum amount that the winner can still win. Losers never regret not winning. MSP is strategyproof for a bidder who does not suffer externalities. Furthermore, MSP reduces to SP when there are no externalities. Finally, (iii) "price matters": In MSP, there is no regret about the price. For losers, there is no "good" price, i.e., no regret about not winning. For the seller, simulations suggest MSP outperforms SP in terms of both revenue and efficiency.

Externalities are prevalent in auctions where commercial bidders participate. Such auctions are likely high-stakes rare events where good ex-post properties may be more important than Bayesian properties. Thus far, one-dimensional auctions have been widely used where finding bidding strategies is not only difficult but also prone to various kinds of regrets. I believe one of the reasons for this is that no practical multidimensional mechanism has been available. This paper presents multidimensional mechanisms that not only have good properties but also naturally generalize the standard mechanisms. Some usual desirable properties that MSP does not satisfy are impossible in other mechanisms, and MSP satisfies well-known alternative properties. Furthermore, MSP is a unique direct mechanism that satisfies certain good properties. In addition, this paper is the first to introduce and resolve group winner regret, and MSP is the first mechanism that uses a network graph for an auction mechanism, which may inspire further research.

A Simulations

One might expect that group winner regret-freeness (GWRF) may increase efficiency since GWR implies that a group of bidders prefers another outcome. Likewise, pairwise stability (PS) may increase both revenue and efficiency. Due to multidimensionality, the analytic comparison of MSP and other mechanisms is difficult. In general, for instance, neither MSP nor SP dominates the other in terms of either revenue or efficiency. Thus, I turn to simulations, which support the above intuition. Simulations suggest that MSP outperforms SP in terms of both revenue and efficiency.

For all simulations, the number of iterations is 5,000, and the following bidding strategies are used. For VCG, a truthful bid is used because it is IC. For SP, a unique symmetric

[46] An explanation, "you only need to pay the amount of externalities you impose" might be beautiful in theory, but in practice advertisers might react, "externalities that I impose? Am I doing something wrong?"

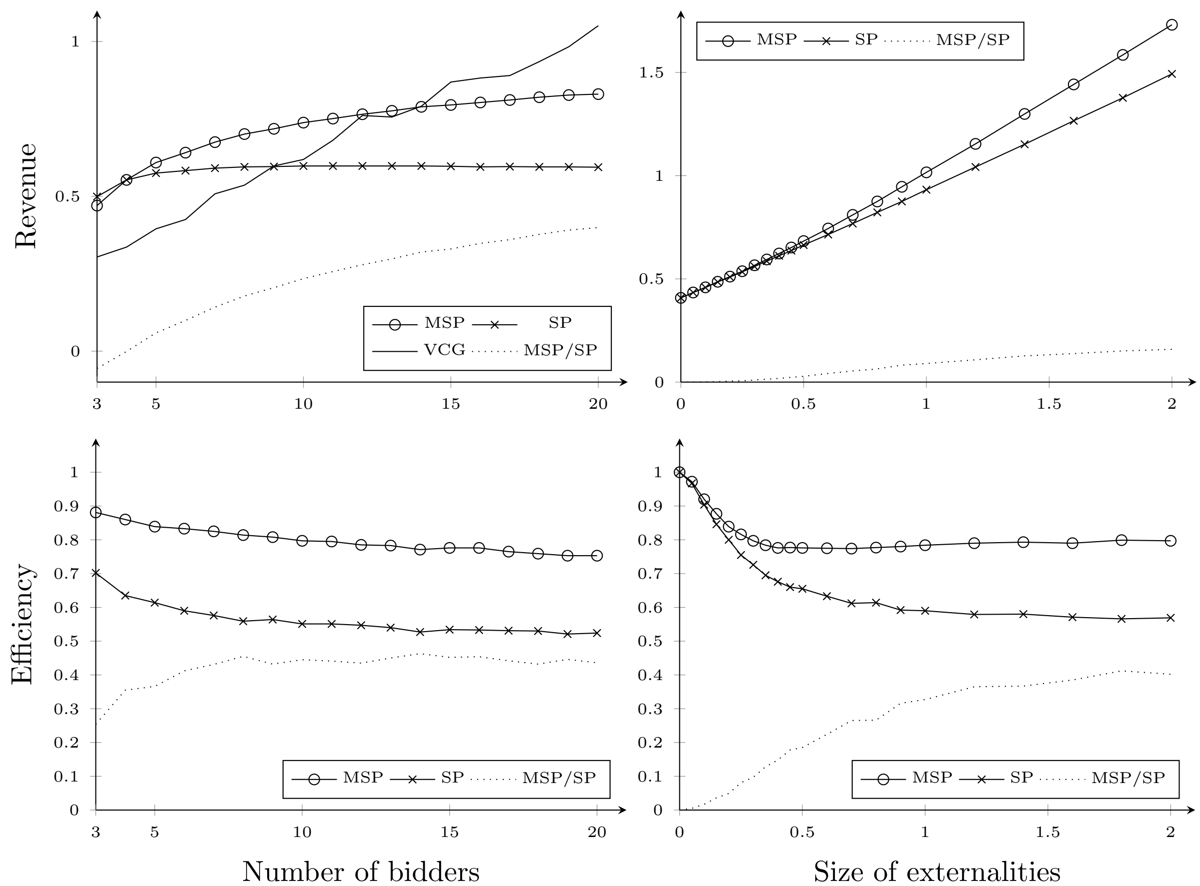

Figure 6: Main result on revenue and efficiency

Bayesian Nash equilibrium (Jehiel et al. 1999, Proposition 4), $b_j^* = \overline{b_{ij}}$ for $i \neq j$, is used. For MSP, a truthful bid is used. As in minimum-revenue core-selecting package auctions, due to several good incentive properties and multidimensionality, it seems quite difficult to manipulate MSP. For instance, a profitable manipulation $\mathbf{b}'_j$ can be neither underbidding $\mathbf{b}'_j \leq \mathbf{b}_j$ nor overbidding $\mathbf{b}'_j \geq \mathbf{b}_j$, which makes a manipulation more difficult.

While I show the comparison of revenue and efficiency, I also examine the effects of the number of bidders and the size of externalities and therefore use a different model for each purpose. The first model is used to show the effect of the number of bidders, n. Each bid is i.i.d. and drawn from a uniform distribution $U[0, 1]$ for $i \neq j$. The second model is used to show the effect of the size of externalities when $n = 10$. Each valuation is i.i.d. and drawn from a uniform distribution $U[0, 0.5]$. Each externality is i.i.d. and drawn from another independent uniform distribution $U[-e, 0]$ for $i \neq j$, where e is the upper bound of the size of negative externalities, which is the value of the x-axis. Note that this model approaches the previous model (with a different bound) as the size of externalities increases because the externalities eventually dominate the valuation. Thus, all ratio values approach the same

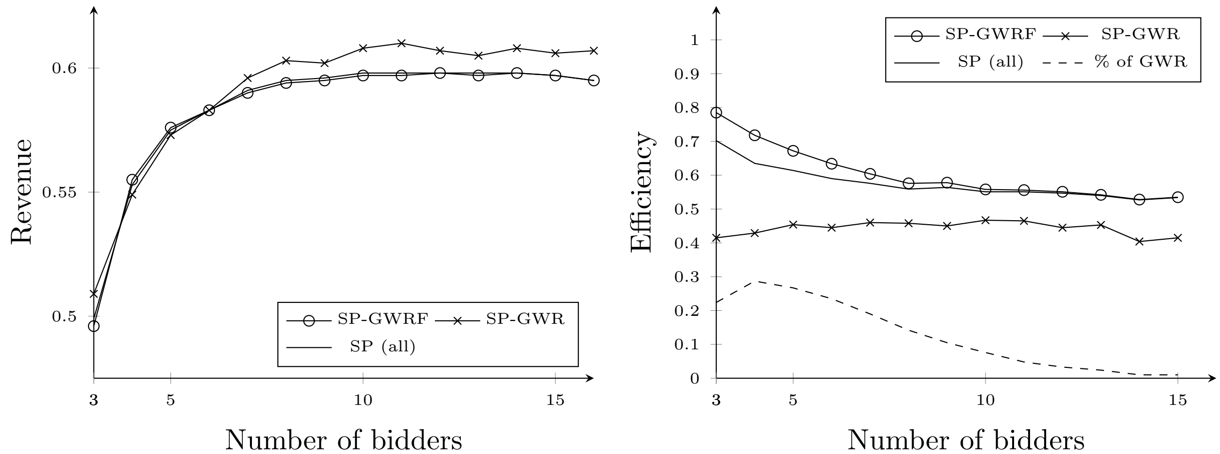

Figure 7: Effect of group winner regret-freeness (GWRF)

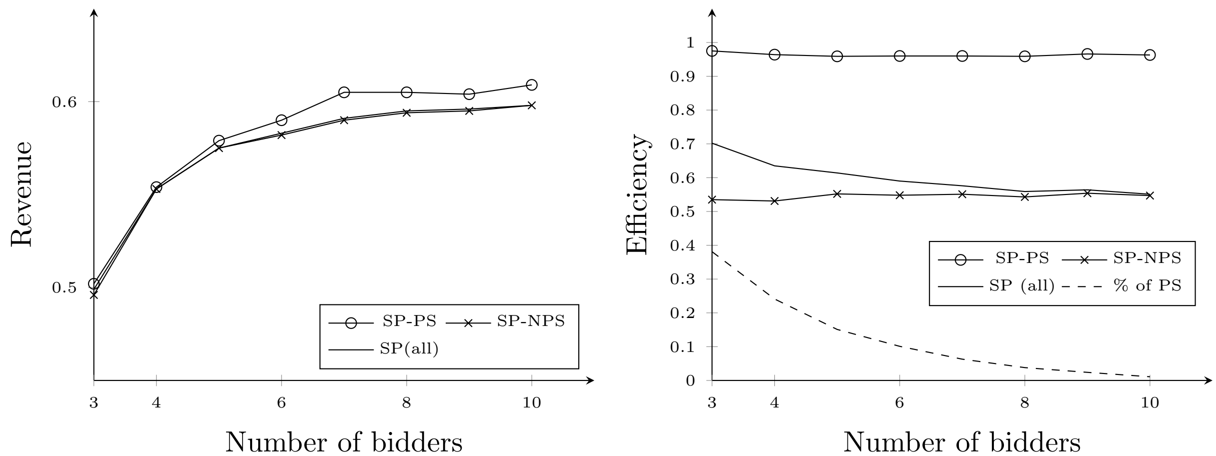

Figure 8: Effect of pairwise stability (PS)

values of the first model with $n = 10$.

Figure 6 shows the main finding. The upper pane shows that MSP has higher revenue than SP for $n \geq 5$ (on average). Interestingly, the lower pane shows that MSP also has higher efficiency than SP. That is, MSP outperforms SP in terms of both revenue and efficiency. "MSP/SP" denotes the gain, i.e., "MSP/SP (e.g., revenue) = (revenue of MSP / revenue of SP) - 1." The right pane shows that when there are no externalities, MSP and SP have the same outcome, as shown in Theorem 4. As externalities increase, MSP starts to outperform SP in terms of both revenue and efficiency, and the ratio values (i.e., efficiency and the gains of both revenue and efficiency) in the right pane approach the values in the left pane, as explained in the simulation model.

To examine the effect of GWRF or PS, I divide the outcomes. Figure 7 supports the intuition that GWRF tends to increase efficiency but may decrease revenue. "SP-GWRF"

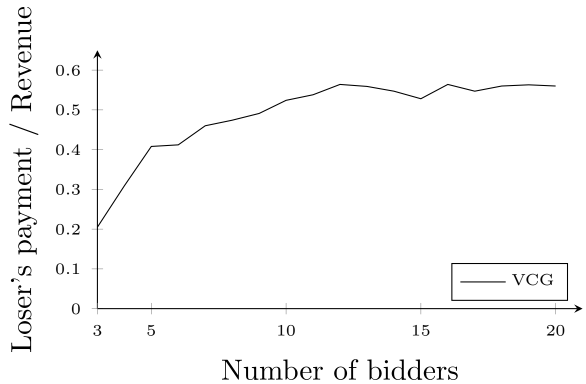

Figure 9: VCG's ratio of the loser's payment to revenue

denotes the SP outcomes that are GWRF, which can be found by Proposition 2. "SP (all)" denotes all SP outcomes. "% of GWRF" denotes the percentage of the outcomes that are GWRF. On the other hand, Figure 8 supports the intuition that PS tends to increase both revenue and efficiency. "SP-(N)PS" denotes the SP outcomes that are (not) PS, respectively. Thus, for $n = 3$ in Figure 6, SP has higher revenue than MSP because the effect of GWRF might be larger than PS. However, any results for small n should be interpreted cautiously since the manipulation possibility of each mechanism might not be ignorable.

Interestingly, Figure 6 (upper pane) shows that, as n increases, VCG starts to have higher revenue than MSP around $n = 15$. Figure 9, however, shows that more than half of the revenue comes from losing bidders, as n increases. In core-selecting mechanisms, the loser's payment ratio is higher than VCG. Thus, the loser's payment in VCG and core-selecting mechanisms might be a real concern. Note that Figures 6 (lower pane) and 9 together suggest a tradeoff between efficiency and no loser's payment.

B No loser's payment

C Proofs

C.1 Proof of Theorem 1 (Impossibility)

Although the definition of the mechanism (i.e., x is deterministic) and the theorem statement is written as ex-post (IC or IR) for simplicity, it also holds for ex-interim (Bayesian) (IC or IR). The proofs for ex-interim versions are obvious from ex-post versions since a simple distribution can be easily found such that the same counterexamples hold.

(i) Since the assumption in footnote 30, i.e., the lowest type bidder cannot win unless every bidder reports the lowest type, is needed only for inefficient mechanisms, I will show efficient and inefficient cases separately.

efficient mechanisms Let $T_N = \begin{bmatrix} 9 & -3 & 0 \\ 0 & 7 & -2 \\ 0 & 0 & 1 \end{bmatrix} \equiv T_N^1$ (Example 1). Since the payoff function is quasilinear, the only efficient and IC mechanism is *some* VCG mechanism (i.e., VCG in Definition 3 with an additive constant) by the Green-Laffont-Holmstrom Theorem (Green and Laffont 1977; Holmstrom 1979). In particular, since $\mathcal{T}$ is convex ($\mathcal{T}$ is closed and bounded in $\mathbb{R}^n$) and u_j is quasilinear, by Theorem 2 of Holmstrom (1979), the only efficient and IC mechanism is Groves' scheme.[47] Then, as shown in Example 1, VCG has a loser's payment, $\mathbf{p} = (8, 0, 1)$. Now adjusting the payment for the lowest type (e.g., $\mathbf{p} = (7, -1, 0)$ by lowering 1 for each bidder) cannot achieve LPF without a subsidy, and even if we allow a subsidy, it is still not PS since $p = 6 < 9 = b_{12} + p_2$.

inefficient mechanisms Let $T_N = \begin{bmatrix} 7 & -3 & -3 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \equiv T_N^2$. Without loss of generality, let $\underline{b} = 0$. Due to symmetry, we only need to check bidders 1 and 2. For bidder 1 to win, $p_1 \geq 8$ and then $u_1 \leq -1$. If bidder 1 submits a zero bid $(0, ..., 0)$ instead, either bidder 2 or 3 should win, and then the new payoff $u'_1 = 0 > u_1$. Thus, it is not IC for bidder 1. Now, for bidder 2 to win, $p_2 \geq 7$, and then $u_2 \leq -2$. If bidder 2 submits a zero bid instead, either bidder 1 or 3 should win. Note that bidder 1 can still win since neither efficiency nor IR is required. However, if bidder 1 wins, it is not IC for bidder 1 as before; therefore, bidder 3 should win, and $u'_2 = 0 > u_2$. Thus, it is not IC for bidder 2.

Note that the assumption that the lowest type bidder cannot win is necessary. Otherwise, there exists an (inefficient) IC mechanism, e.g., a mechanism that makes bidder 1 always win with the minimum price that satisfies PS.

(ii) Let $T_N = T_N^2$. Due to symmetry, we only need to check bidders 1 and 2. For bidder 1 to win, $p_1 \geq \max_{j \neq 1}\{b_{1j}\} = 8$, and then $u_1 \leq -1 < \min_{i \neq 1}\{t_{i1}\} = 0$. Thus, it is not weakly IR for bidder 1. Also, for bidder 2 to win, $p_2 \geq \max_{j \neq 2}\{b_{2j}\} = 7$, and then $u_2 = u_2(T; T) \leq -2$. If bidder 2 does not participate, either bidder 1 or 3 should win, but again bidder 1 cannot win since it violates weak IR for bidder 1. Therefore, bidder 3 should win, and then the new payoff $u'_2 = u_2(T_{-2}; T) = t_{32} = 0 > u_2$. Thus, it is not IR for bidder

[47] Alternatively, Theorem 1 of Holmstrom (1979) can be directly used for the proof by showing that $\mathcal{T}$ is *smoothly connected*, which means that for any $\mathbf{t}_j, \mathbf{t}'_j \in \mathcal{T}_j$, there exists a one-dimensional parametrized family of valuation functions $\{v_j(i; y_j) \in \mathcal{T}_j : y_j \in [0, 1]\}$, $i \in N$, $y_j \in [0, 1]$ such that $v_j(i; 0) = t_{ij}$, $v_j(i; 1) = t'_{ij}$, and $\partial v_j(i; y_j)/\partial y_j$ exists for all $y_j \in [0, 1]$. Note that $v_j(i; y_j) = t_{ij} + (t'_{ij} - t_{ij})y_j$ satisfies the condition.

2. Note that it is still weakly IR, and there exists a weakly IR mechanism, e.g., MSP and the optimal mechanism in Jehiel et al. (1996).

Note also that IR in Definition 1 is IR by δ-effectiveness in Jeong (2017), but this is just for simplicity, and the result holds much more generally with other effectiveness concepts. One very convincing one is IR in Jehiel and Moldovanu (1996). That is, first define a two-stage game that consists of the participation decision stage and the auction stage, i.e., bidders first simultaneously decide their participation and then only the participants run the auction. During this two stage game, players behave according to subgame perfect Nash equilibria (SPNE). Now we show IR is also impossible in this sense.[48] Again, consider bidder 2. When bidder 2 participates, participation of bidder 1 and nonparticipation of bidder 3 in the first stage, and truthful bidding in the second stage is an SPNE. Thus, bidder 2 wins and $u_2 \leq -2$ as before. Now if bidder 2 does not participate, participation of bidders 1 and 3 and then truthful bidding is an SPNE. Thus, bidder 3 wins, which makes $u_2' = 0 > u_2$.

Another convincing definition of IR might be IR by e-effectiveness in Jeong (2017), which assumes that outside players would have an efficient allocation among them. In our case, if bidder 2 does not participate, bidder 3 should win by efficiency among $\{0, 1, 3\}$; thus, as in other two previous cases, IR by e-effectiveness is also impossible.

(iii) Let $T_N = T_N^1$. By pairwise stability, $p_1 \geq \max_{j\neq 1}\{b_{1j}\} = 10$, then $u_1 \leq -1 < \min_{i\neq 1}\{t_{i1}\} = 0$. Thus, it is not weakly IR for bidder 1.

C.2 Proofs of main lemmas and Theorem 2

Here I introduce the Connectedness, Generalized Pairwise Stability, and Blocking Lemmas and prove these as well as the Chain Lemma (Lemma 1). These lemmas greatly simplify the proofs of main theorems. One fundamental difference between English (or $\hat{\varphi}$) and s-MSP auctions is that the bid graph of s-MSP is always connected as follows.

Lemma 4 (Connectedness)**.** *In the s-MSP, G is connected. In particular, G is strongly connected except for, possibly, step 5 (i.e., after the unblock step and before the drop step). That is, the unblock step results in either of the following two cases: (1) G remains strongly connected; (2) G is weakly but not strongly connected.*

Proof. G is strongly connected in the beginning of an auction. Also, each drop step leaves only one SCC, which means G is strongly connected. Thus, we only need to check the unblock step. Suppose G is disconnected after unblocking b_{cm}. Then there exists at least two SCCs that are disconnected from each other. Now, adding back only one edge b_{cm} to G

[48] As usual, we assume bidders participate if they are indifferent between participation and nonparticipation, e.g., losers still participate if participation does not hurt them.

cannot make G strongly connected, which contradicts the strong connectedness of G before any unblocking. Therefore, G is at least weakly connected. The existence of both cases (1) and (2) can be shown by examples. $\square$

To prove the Chain Lemma, I first prove the following simple lemma.

Lemma 5. *In the s-MSP, if unblocking b_{cm} leads to $|\mathcal{G}| > 1$, then $G^m \neq G^c$.*

Proof. Suppose $G^m = G^c$. Then there exists an SCC $H \in \mathcal{G}$ such that $H \neq G^m$, and H is weakly but not strongly connected to G^m by the Connectedness Lemma. Now adding back b_{cm} cannot make G strongly connected since $m, c \notin H$ and $m, c \in G^m = G^c$. That is, H is not strongly connected before the unblock step, which contradicts the Connectedness Lemma. $\square$

Proof of Lemma 1 (Chain). First, G is connected by the Connectedness Lemma. Second, by Lemma 5, $G^m \neq G^c$. Third, there should be a component path $G^m \to \cdots \to G^c$. Suppose there is no such path. Then adding back b_{cm} cannot make G strongly connected, a contradiction to the Connectedness Lemma. Note that there is no component path $G^c \to \cdots \to G^m$ because if it exists, then together with $G^m \to \cdots \to G^c$, G^m and G^c are strongly connected, a contradiction to $G^m \neq G^c$. Lastly, I will prove the uniqueness of the start component G^m and the end component G^c. Suppose there exists another start component $H \neq G^m$. Then, adding back b_{cm} cannot make H strongly connected with a path $G^m \to \cdots \to G^c$, a contradiction to the Connectedness Lemma (i.e., G is strongly connected right before the unblock step). Likewise, there cannot be another end component. Thus, G is a chain. $\square$

MSP is pairwise stable, but the following more general result holds (which is also used in other results). That is, if j is dropped at some q despite $b_{ij} > q$ for some i, then every such i has been already dropped earlier than or together with j.

Lemma 6 (Generalized Pairwise Stability (GPS))**.** *During any step at any price q in the MSP, there exist no two bidders i and j such that $i \in G$ and $j \notin G$ but $b_{ij} > q$.*

Proof. Suppose $b_{ij} > q_j^*$. At the beginning of the drop step, there are two cases: (i) $i \in G^j$. Since $G^i = G^j$, i is dropped together with j; thus, $i \in G$ but $j \notin G$ is impossible; (ii) $i \notin G^j$. If $i \notin G$, then the proof is complete. If $i \in G$, then due to $b_{ij} > q_j^*$, there exists $G^i \to \cdots \to G^j$ by the Chain Lemma, and G^i should be dropped earlier than (or together with in s-MSP) G^j. In any case, at the end of the drop step, both i and j have been dropped; thus, after the drop step, $i, j \notin G$. Before the drop step, $j \in G$. $\square$

Again, Theorem 2 (pairwise stability) is in fact a corollary of the GPS Lemma.

Proof of Theorem 2. When $i = w$ and $q = p$, the GPS Lemma implies PS. □

The following lemma, which means when bidder i is dropped in MSP, there is another bidder j who is blocking i, is useful to prove the main theorems.

Lemma 7 (Blocking)**.** *For every loser i, there exists $b_{ij} \geq q_i^*$ (the strict inequality holds if there is no tie in B) for some bidder j. Therefore, i cannot win at some price $p_i < q_i^*$ (the weak inequality holds if there is no tie) in any loser's payment-free and pairwise stable mechanism.*

Proof. By the Chain Lemma, in the beginning of the drop step where i is dropped, there exist SCCs G^i and H such that $G^i \to H$. If $|G^i| = 1$, then there exist such $j \in H$ due to $G^i \to H$. If $|G^i| > 1$, then there exist such $j \in G^i$ since G^i is an SCC (note that such j may also exist in H due to $G^i \to H$). Note also that LPF (in particular, no positive sum of payments of losers other than j) is needed for the second part, i.e., if $\sum_{k \neq w,j} p_k > 0$, PS can still hold even with $b_{ij} > p_i$. □

C.3 Proof of Theorem 3 (Incentive properties)

(i) Since $p = b_{wh}$, where h is the threshold bidder, winning at $p' < p$ is impossible by PS. Winning at $p' > p$ by some $\mathbf{b}'_w$ is impossible; otherwise, it implies winning at $p < p'$ is possible by $\mathbf{b}_w$ when the true type is $\mathbf{b}'_w$, a contradiction to the first statement.

(ii) We need to show $p_l \geq b_{wl}$ for any loser l, where p_l is the lowest price at which l can win by misreporting. By the GPS Lemma, $q_l^* \geq b_{wl}$ (otherwise, at q_l^*, $w \in G$ and $l \neq G$ but $b_{wl} > q_l^*$, which is impossible). Also, by the Blocking Lemma, l cannot win at some price lower than q_l^*, i.e., $p_l \geq q_l^*$. Thus, $p_l \geq q_l^* \geq b_{wl}$.

C.4 Proof of Theorem 4 (Generalization of SP)

(i) Let $\mathbf{b}_j = (b, b, ..., b) \in \mathcal{B}_j$. Suppose j wins at p by $\mathbf{b}_j$. Then no other winning price is possible by Theorem 3-(i), i.e., one price for one bidder. Also, losing by misreporting is unprofitable since $p \leq b$ by weak IR. Now suppose j loses by $\mathbf{b}_j$. Losing to another bidder by misreporting does not change the payoff. Winning by misreporting is unprofitable by Theorem 3-(ii), i.e., no overturn regret.

(ii) Without loss of generality, $\underline{b} = 0$ is assumed so that the payment of a bidder with the lowest type, i.e., the zero bid $(0, 0, ..., 0)$, is zero in SP. Let $\mathbf{b}_j = (b_j, b_j, ..., b_j) \in \mathcal{B}_j$ for all j. By (i), MSP is IC for all bidders. The winner for an efficient allocation is $w \in \arg\max_j \{b_j\}$.

By PS and weak IR, w wins in MSP, i.e., MSP is efficient. Also, weak IR becomes the same as IR since for each bidder j, $b_{ij} = b_j$ for all $i \neq j$. If bidder j with the lowest type is a losing bidder, then $p_j = 0$ by LPF. Even if bidder j with the lowest type is the winner (this is possible if every bidder submits the zero bid), then $p_j \leq 0$ by weak IR, and $p_j \geq 0$ by PS; therefore, $p_j = 0$. By the Green-Laffont-Holmstrom Theorem, VCG (in particular, the Groves' scheme) is the only mechanism that is efficient, IC, and IR. Therefore, when there are no externalities, MSP is VCG with zero payment for the lowest type bidder, which is SP.

Alternative proof Without loss of generality, assume $b_1 \geq b_2 \geq \cdots \geq b_n$ and ties (if any) are broken in this order. Even when ties exist both within a bidder and across bidders, MSP unblocks all bids within a bidder first (step 4), then starts unblocking the bids of other bidders (step 3). Thus, MSP drops bidders in the sequence of $n, n-1, ..., 1$; therefore, bidder 1 wins at price b_2. Also, both auctions have no loser's payment. Thus, MSP reduces to SP.

C.5 Proof of Theorem 5 (Characterization) and Corollary 5

Proof of Theorem 5. First, each property is satisfied as follows:

(i) (LPF) By Step 9, only the winner needs to pay.

(ii) (PS) By Theorem 2.

(iii) (Weak IR) By Lemma 2.

(iv) (GWRF) By Corollary 3.

(v) (CORF) By Theorem 3-(i), MSP is free of overpay regret, which implies it is free of capped-bid overpay regret (Definition 9).

Uniqueness Without loss of generality, assume there is no tie in B, i.e., any tie in B is broken by a tie-breaking rule in advance. Abusing the notation, "$\varphi(B) = (w, p)$" is used instead of "$\varphi(B) = (w, \mathbf{p})$" if φ is LPF, i.e., $p = p_w$. Let $\varphi(B) = (w, p)$ be the outcome of MSP (denoted by φ), and let φ' be a different mechanism. Then, $\varphi'(B)$ can be one of the following four cases.

(1) ($w' \neq w, p' > p$): Since w' lost in φ, $q^*_{w'} \leq p$ and $w \notin G^{w'}$ right before w' is dropped. Thus, by Proposition 2, $V(G^{w'})$ has GWR in φ'.

(2) ($w' \neq w, p' \leq p$): By the Blocking Lemma (Lemma 7), w' cannot win at $p' \leq q^*_{w'}$ in φ' since φ' is LPF and PS; thus, $p' > q^*_{w'}$. Then $V(G^{w'})$ has GWR by Proposition 2. Note that whether $p' \leq p$ is not used. Thus, this also proves case (1); however, a different proof without using the Blocking Lemma is provided there.

(3) ($w, p' > p$): Note that this does not immediately imply capped-bid overpay regret in φ' since it is a different mechanism. Let $\mathbf{b}^{\overline{q}}_w$ be a q-capped bid for some q with $p < q < p'$,

and let $B' = (\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w})$. If w can still win by $\mathbf{b}_w^{\overline{q}}$ in φ', i.e., $\varphi'(B') = (w, p'')$ for some p'', then w has capped-bid overpay regret in φ' since $p'' \leq q < p'$ by weak IR. Now suppose not, i.e., $\varphi'(B') = (w', p'')$ for $w' \neq w$ and some p'' (note that there is no reason that $p'' \geq q$ yet, which will not matter anyway). However, $\varphi(B') = (w, p)$ since $q > p$ and any $b_{iw} > p$ had never been unblocked. Thus, $\varphi'(B') = (w', p'')$ for any p'' is impossible by cases (1) and (2).

(4) $(w, p' < p)$: PS of φ' is violated by $b_{wh} = p > p'$, where h is the threshold bidder.

This completes the proof of the characterization, but I also show the independence of the axioms below.

Independence of the axioms Let φ be MSP. In each case, the existence of a new mechanism φ' will be shown by its outcome $\varphi'(B) = (w', \mathbf{p}')$. Before I show each counterexample, note first that if $w' = w$, GWR cannot exist by Proposition 2. Thus, GWRF holds for all cases except for (iv).

(i) (necessity of LPF) $w' = w$, $p'_l = \max_{i \neq l}\{b_{il}\}$ for some loser l with $p'_l > 0$ (which is possible for some bid profile), and $p'_j = p_j$ for all $j \neq l$.

φ' has a loser's payment, but satisfies all other axioms: PS holds since $p > p'$; i.e., more difficult to block by h (and the seller) if $l \neq h$, where h is the threshold bidder. While we can choose $l \neq h$ from the beginning in general, it is worth to mention that even if $l = h$, h cannot block, i.e., willing to pay additionally but the price is also increased by the same additional amount; weak IR clearly holds; and CORF holds since p'_w is independent of $\mathbf{b}_w$ if w still wins.

(ii) (necessity of PS) $w' = w$, $p'_w = \max_{j \neq h}\{b_{wj}\}$, and $p'_l = 0$ for all $l \neq w$. That is, the new winning price p'_w is the second highest b_{wj} for all $j \neq w$, denoted by $b_{wh'}$, and $p'_w < p_w$ is possible when $b_{wh'} < b_{wh}$.

φ' is not PS by $b_{wh} > b_{wh'}$. However, φ' satisfies all other axioms: LPF and weak IR clearly hold; and CORF holds, as in (i).

(iii) (necessity of weak IR) $w' = w$, $p'_w = \overline{b} > p$ (which is possible when $p = b_{wh} < \overline{b}$), and $p'_l = 0$ for all $l \neq w$.

φ' is not weak IR for w when $\max_{i \neq w}\{b_{iw}\} < \overline{b}$. However, φ' satisfies all other axioms: LPF clearly holds; PS holds since $p'_w > p$, i.e., "harder to block"; and CORF holds, as in (i).

(iv) (necessity of GWRF) $\varphi' \equiv \hat{\varphi}$ in Algorithm 4.

φ' is not GWRF, but satisfies all other axioms. LPF is obvious. As opposed to MSP, PS is also obvious since each bidder stays until the maximum bid against all remaining competitor. Weak IR and CORF can also be shown similarly as in Lemma 2 and Theorem 3, respectively.

(v) (necessity of CORF) MFP, or another example: $w' = w$, $p'_w = \max_{i \neq w}\{b_{iw}\}$, $p'_l = 0$

for all $l \neq w$, assuming that $p'_w > p$ (which is possible when $\max_{i \neq w}\{b_{iw}\} > p$).

φ' is not CORF since $\varphi'_1(\mathbf{b}^{\overline{q}}_w, \mathbf{b}_{-w}) = w$ but $\varphi'_2(\mathbf{b}^{\overline{q}}_w, \mathbf{b}_{-w}) = \max_{i \neq w}\{b^{\overline{q}}_{iw}\} = q < p'_w$ for q such that $p < q < p'_w$. However, φ' satisfies all other axioms: LPF clearly holds; PS holds, as in (iii); and weak IR holds since $p'_w \leq \max_{i \neq w}\{b_{iw}\}$. □

Proof of Corollary 5. CORF was not used in (2) and (4) of the uniqueness proof. □

C.6 Proof of other lemmas and propositions

Proof of Proposition 2. Let $R^* = V(B_{(q^*_w, k^*_w)}) \backslash S$ and $R(q) = V(B'_{(q^*_w(B'), k^*_w(B'))}) \backslash S$, where $B' = (\mathbf{b}^{\overline{q}}_S, \mathbf{b}_{-S})$. (necessity) $R(q) = R^*$ is the same for all $q > q^*_S$ since any bid actually capped by $\mathbf{b}^{\overline{q}}_S$, i.e., $b_{ij} > q$ for $j \in S$, has not been unblocked before S drops at step 7. At the beginning of step 7 (S is about to drop), $b_{ij} \leq q$ for all $i \in R^*$ and $j \in S$. Also, by strong connectedness of S, for each $j \in S$, there exists $k_j \in S$ with $k_j \neq j$ such that $b_{k_j,j} \geq q$. Thus, S has GWR at q such that $q^*_S < q < p_w$.

(sufficiency) I will show that GWR cannot exist if any of $w \neq \varphi^*_1(B)$, $p_w > q^*_S$, and $S = \{j : q^*_j = q^*_w \text{ and } k^*_j = k^*_w\}$ is not true. First, suppose $p_w \leq q^*_S$ (which will also be used for $w = \varphi^*_1(B)$ case). Then, for the existence of GWR (at q), $q < p_w \leq q^*_w$. But for any $q < q^*_S$, there exists $i \in R(q) \neq \emptyset$ such that $b_{ij} > q$ for some $j \in S$ since S drops earlier than i. Second, suppose $w = \varphi^*_1(B)$. Note that $S = \{w\}$ since w cannot win in φ^* otherwise. For $q > q^*_w$, GWR (at q) cannot exist since $R(q) = \emptyset$. For $q = q^*_w$, depending on the tie-breaking, no winner change can occur where $R(q) = R(q^*_w) = \emptyset$. Third, if $S \neq \{j : q^*_j = q^*_w \text{ and } k^*_j = k^*_w\}$, then S is not a node set of an SCC; thus, (ii) of GWR cannot be satisfied. □

Proof of Proposition 3. By the revelation principle (e.g., Myerson (1981)), for any symmetric and increasing[49] equilibrium of φ' and its outcome, there exists a mechanism $\tilde{\varphi}'$ where truthful bidding is an equilibrium and has the same outcome of φ'. The same applies to φ''. Thus, now for two IC mechanisms $\tilde{\varphi}'$ and $\tilde{\varphi}''$, by Proposition 1 of Krishna and Maenner (2001), the expected payoff of a bidder is determined by the allocation rule up to an additive constant, which is also the same due to no loser's payment. □

Proof of Proposition 4. Both in (i) and (ii), LPF clearly holds. Minimum revenue and uniqueness hold by construction: each mechanism finds b_{ij} that satisfies the axioms in ascending order of b_{ij}. Thus, any different outcome with lower revenue is impossible.

[49] A bidding function β_j of any direct mechanism φ is said to be *increasing* (or *monotone*) if $(\mathbf{b}'_j - \mathbf{b}_j) \cdot (\beta_j(\mathbf{b}'_j) - \beta_j(\mathbf{b}_j)) \geq 0$ for all $\mathbf{b}_j, \mathbf{b}'_j \in \mathcal{B}_j$.

(i) By PS, for i to win, $p_i \geq \max_{j \neq i}\{b_{ij}\} \equiv q_i$. Thus, the winner that makes the minimum revenue is $w \in \arg\min_i\{q_i\}$, and the price is $p_w = q_w$.

(ii) The constraint "$\max_j\{b_{ij}\} \leq \max_j\{b_{ji}\}$" means $q_i \leq \max_{j \neq i}\{b_{ji}\}$, which implies $q_w \leq \max_{j \neq w}\{b_{jw}\}$. Thus, φ^{LPW} is weakly IR for w. □

Proof of Proposition 5. Let $R^* = V(B_{(\hat{q}_w, \hat{w}_j)})\backslash w$ and $R(q) = V(B'_{(\hat{q}_w(B'), \hat{w}_j(B'))})\backslash w$, where $B' = (\mathbf{b}_w^{\overline{q}}, \mathbf{b}_{-w})$. (necessity) $R(q) = R^*$ is the same for all $q > \hat{q}_w$ since any bid actually capped by $\mathbf{b}_w^{\overline{q}}$, i.e., $b_{iw} > q$, has not been unblocked before w drops at step 5. At the beginning of step 5 (w is about to drop), $b_{iw} \leq q$ for all $i \in R^*$. Thus, w has WR at q such that $\hat{q}_w < q < p_w$.

(sufficiency) I will show that WR cannot exist if any of $w \neq \hat{x}(B)$ and $p_w > \hat{q}_w$ is not true. First, suppose $p_w \leq \hat{q}_w$ (which will also be used for $w = \hat{x}(B)$ case). Then, for the existence of WR (at q), $q < p_w \leq \hat{q}_w$. But for any $q < \hat{q}_w$, there exists $i \in R(q) \neq \emptyset$ such that $b_{iw} > q$ since w drops earlier than i. Second, suppose $w = \hat{x}(B)$. For $q < \hat{q}_w$, it is already done. For $q > \hat{q}_w$, WR (at q) cannot exist since $R(q) = \emptyset$. For $q = \hat{q}_w$, depending on the tie-breaking, no winner change can occur where $R(q) = R(\hat{q}_w) = \emptyset$ (see footnote 42). □

Proof of Proposition 6. Suppose not. Then there exists a bidder who is dropped at a lower price in $\hat{\varphi}$ than in φ^*. Let j be the bidder who is dropped first among those bidder(s), then $\hat{q}_j < q_j^*$. Let $\hat{R} = V(B_{(\hat{q}_j, \hat{k}_j)})$, i.e., the set of remaining bidders right before j is dropped. Also, let $R^* = V(B^{(\hat{q}_j, \infty)})$.[50] Then $R^* \subseteq \hat{R}$ (otherwise, it contradicts the fact that j is the first bidder who is dropped at a lower price in $\hat{\varphi}$). However, the fact that j is dropped at $\hat{q}_j$ in $\hat{\varphi}$ implies $b_{ij} \leq \hat{q}_j$ for all $i \in \hat{R}\backslash j$. Thus, $b_{ij} \leq \hat{q}_j$ for all $i \in R^*\backslash j$, which implies j has to be dropped at $\hat{q}_j$ or a lower price in φ^*, i.e., $q_j^* \leq \hat{q}_j$, a contradiction to $\hat{q}_j < q_j^*$. The possibility of $\hat{\mathbf{q}} \neq \mathbf{q}^*$ can be easily shown by an example that is free of WR, but not free of GWR, e.g., the motivating example. □

The following lemma is needed for the case $b_{ij} \leq 0$ as explained in footnote 33. The same result holds for all mechanisms in Algorithms 1 to 4.

Lemma 8 (Linearity)**.** *In the MSP, the following holds up to ties:* $x(B) = x(c_1 \cdot B) = x(B + c_2)$, $\rho(c_1 \cdot B) = c_1 \cdot \rho(B)$, *and* $\rho(B + c_2) = \rho(B) + c_2$ *for* $c_1 \in \mathbb{R}_{++}$ *and* $c_2 \in \mathbb{R}_+$.

Proof. Note that $B' = B + c_2$ should be written as $B' = (b'_{ij})$, where $b'_{ij} = b_{ij} + c_2$ if $i \neq j$, otherwise $b'_{ij} = 0$. However, since the diagonal is unused in MSP, $B + c_2$ is used for

[50] To show the status of an auction at any price and any step, for (q, k) that is not used in the algorithm, $B^{(q,k)}$, where $q \in [\underline{b}, \overline{b}]$ and $k \in \mathbb{Z}_+ \cup \{\infty\}$, can be defined as the lexicographically previous entry. For instance, if $B^{(2,6)}$ and $B^{(4,0)}$ are the two consecutive group bid matrices used in the algorithm, then $B^{(2,6)} = B^{(2,k)} = B^{(3,k')} = B^{(4,0)}$ for $k \in [6, \infty]$, $k' \in [0, \infty]$. Also, $B^{(\underline{b},0)} \equiv B$.

simplicity. Let $w = x(B)$, $p = \rho(B) = b_{wh}$, $B' = c_1 \cdot B$, and $B'' = B + c_2$, i.e., $b'_{ij} = c_1 \cdot b_{ij}$ and $b''_{ij} = b_{ij} + c_2$. Then b'_{ij}'s and b''_{ij}'s have the same order of b_{ij}'s since $f(x) = c_1 \cdot x$ and $g(x) = x + c_2$, $x \in \mathbb{R}$, are isotone. Thus, all three auctions with B, B', and B'' have the same winner and threshold bidder up to ties since the sequence of bids that are unblocked in the unblock step and the sequence of bidders that are dropped in the drop step are the same. Therefore, $x(B') = x(B'') = w$, $\rho(B') = b'_{wh} = c_1 \cdot b_{wh}$, and $\rho(B'') = b''_{wh} = b_{wh} + c_2$. □

The following lemma shows that MSP ends in finite time. Note that Lemmas 1, 8, and 9 show that MSP is well-defined.

Lemma 9. *The MSP ends in finite time, i.e., $|V| = 1$ happens in finite time.*

Proof. $|V|$ monotonically decreases at each drop step since there is no step where it increases. Suppose $|V|$ stops decreasing at some $k > 1$. However, for G to be strongly connected, $|E| \geq |V|$. Since one b_{ij} is unblocked at each unblock step, i.e., $|E|$ decreases by one, eventually $|E| < |V|$ must happen. Then, $|V|$ has to further decrease in the drop step, a contradiction. $|V| = 0$ is impossible since the drop step always leaves G^c undropped. □

References

Aseff, J. and H. Chade (2008). An optimal auction with identity-dependent externalities. *RAND Journal of Economics 39*(3), 731–746.

Ashlagi, I., M. Braverman, A. Hassidim, and D. Monderer (2010). Monotonicity and implementability. *Econometrica*, 1749–1772.

Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica 69*(4), 861–889.

Aumann, R. J. and B. Peleg (1960). Von Neumann-Morgenstern solutions to cooperative games without side payments. *Bulletin of the American Mathematical Society 66*(3), 173–179.

Ausubel, L. and P. Milgrom (2006). The lovely but lonely Vickrey auction. In *Combinatorial Auctions*, pp. 17–40. MIT Press.

Azevedo, E. and E. Budish (2015). Strategyproofness in the large. Working paper.

Azevedo, E. M. and J. D. Leshno (2016). A supply and demand framework for two-sided matching markets. *Journal of Political Economy 124*(5), 1235–1268.

Bergemann, D. and S. Morris (2005). Robust mechanism design. *Econometrica 73*(6), 1771–1813.

Bernheim, B. D. and M. D. Whinston (1986). Menu auctions, resource allocation, and economic influence. *Quarterly Journal of Economics 101*(1), 1–31.

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan, and A. Sen (2006). Weak monotonicity characterizes deterministic dominant-strategy implementation. *Econometrica 74*(4), 1109–1132.

Bulow, J. and J. Levin (2006). Matching and price competition. *American Economic Review 96*(3), 652–668.

Caillaud, B. and P. Jehiel (1998). Collusion in auctions with externalities. *The RAND Journal of Economics*, 680–702.

Chander, P. and H. Tulkens (1997). The core of an economy with multilateral environmental externalities. *International Journal of Game Theory 26*(3), 379–401.

Che, Y.-K. and I. Gale (1998). Standard auctions with financially constrained bidders. *The Review of Economic Studies 65*(1), 1–21.

Che, Y.-K. and J. Kim (2006). Robustly collusion-proof implementation. *Econometrica 74*(4), 1063–1107.

Che, Y.-K., J. Kim, and K. Mierendorff (2013). Generalized reduced-form auctions: A network-flow approach. *Econometrica 81*(6), 2487–2520.

Che, Y.-K. and F. Kojima (2010). Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica 78*(5), 1625–1672.

Chen, J. and S. Micali (2016). Auction revenue in the general spiteful-utility model. In *Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science*, pp. 201–211. ACM.

Clarke, E. (1971). Multipart pricing of public goods. *Public Choice 11*(1), 17–33.

Dasgupta, P., P. Hammond, and E. Maskin (1979). The implementation of social choice rules: Some general results on incentive compatibility. *The Review of Economic Studies 46*(2), 185–216.

Dasgupta, P. and E. Maskin (2000). Efficient auctions. *The Quarterly Journal of Economics 115*(2), 341–388.

Day, R. and P. Milgrom (2008). Core-selecting package auctions. *International Journal of Game Theory 36*(3-4), 393–407.

Day, R. W. and P. Cramton (2012). Quadratic core-selecting payment rules for combinatorial auctions. *Operations Research 60*(3), 588–603.

Day, R. W. and S. Raghavan (2007). Fair payments for efficient allocations in public sector combinatorial auctions. *Management Science 53*(9), 1389–1406.

De Castro, L. I. and D. H. Karney (2012). Equilibria existence and characterization in auctions: achievements and open questions. *Journal of Economic Surveys 26*(5), 911–932.

Edelman, B., M. Ostrovsky, and M. Schwarz (2007). Internet advertising and the generalized second price auction: Selling billions of dollars worth of keywords. *American Economic Review 97*(1), 242–259.

Erdil, A. and P. Klemperer (2010). A new payment rule for core-selecting package auctions. *Journal of the European Economic Association 8*(2-3), 537–547.

Figueroa, N. and V. Skreta (2009). The role of optimal threats in auction design. *Journal of Economic Theory 144*(2), 884–897.

Figueroa, N. and V. Skreta (2011). Optimal allocation mechanisms with single-dimensional private information. *Review of Economic Design 15*(3), 213–243.

Green, J. and J.-J. Laffont (1977). Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica 45*(2), 427–438.

Groves, T. (1973). Incentives in teams. *Econometrica 41*(4), 617–631.

Heydenreich, B., R. Müller, M. Uetz, and R. V. Vohra (2009). Characterization of revenue equivalence. *Econometrica 77*(1), 307–316.

Holmstrom, B. (1979). Groves' scheme on restricted domains. *Econometrica 47*(5), 1137–44.

Holt, C. A. (1980). Competitive bidding for contracts under alternative auction procedures. *Journal of political Economy 88*(3), 433–445.

Hu, Y., J. Kagel, X. Xu, and L. Ye (2013). Theoretical and experimental analysis of auctions with negative externalities. *Games and Economic Behavior 82*, 269–291.

Hurwicz, L. (1972). On informationally decentralized systems. *Decision and organization*.

Immorlica, N. and M. Mahdian (2005). Marriage, honesty, and stability. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pp. 53–62. Society for Industrial and Applied Mathematics.

Jehiel, P. and B. Moldovanu (1995). Negative externalities may cause delay in negotiation. *Econometrica*, 1321–1335.

Jehiel, P. and B. Moldovanu (1996). Strategic nonparticipation. *RAND Journal of Economics 27*(1), 84–98.

Jehiel, P. and B. Moldovanu (1999). Resale markets and the assignment of property rights.

Review of Economic Studies 66(4), 971–991.

Jehiel, P. and B. Moldovanu (2000). Auctions with downstream interaction among buyers. *RAND Journal of Economics 31*(4), 768–791.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1996). How (not) to sell nuclear weapons. *American Economic Review 86*(4), 814–829.

Jehiel, P., B. Moldovanu, and E. Stacchetti (1999). Multidimensional mechanism design for auctions with externalities. *Journal of Economic Theory 85*(2), 258–293.

Jeong, S. E. (2016). The Vickrey-Clarke-Groves with externalities: Shill bidding that dominates truthful bidding. Working paper.

Jeong, S. E. (2017). On the core of auctions with externalities: Stability and fairness. Working paper.

Jeziorski, P. and I. Segal (2015). What makes them click: Empirical analysis of consumer demand for search advertising. *American Economic Journal: Microeconomics 7*(3), 24–53.

Kojima, F. and M. Manea (2010a). Axioms for deferred acceptance. *Econometrica 78*(2), 633–653.

Kojima, F. and M. Manea (2010b). Incentives in the probabilistic serial mechanism. *Journal of Economic Theory 145*(1), 106–123.

Kojima, F. and P. A. Pathak (2009). Incentives and stability in large two-sided matching markets. *American Economic Review 99*(3), 608–627.

Kojima, F., P. A. Pathak, and A. E. Roth (2013). Matching with couples: Stability and incentives in large markets. *Quarterly Journal of Economics 128*(4), 1585–1632.

Krishna, V. and E. Maenner (2001). Convex potentials with an application to mechanism design. *Econometrica 69*(4), 1113–1119.

Laffont, J.-J. and D. Martimort (1997). Collusion under asymmetric information. *Econometrica*, 875–911.

Laffont, J.-J. and D. Martimort (2000). Mechanism design with collusion and correlation. *Econometrica 68*(2), 309–342.

Lee, S. (2017). Incentive compatibility of large centralized matching markets. *Review of Economic Studies 84*.

Lee, S. and L. Yariv (2014). On the efficiency of stable matchings in large markets. Working paper.

Liu, Q. and M. Pycia (2013). Ordinal efficiency, fairness, and incentives in large markets.

Working paper.

Mailath, G. J. (1987). Incentive compatibility in signaling games with a continuum of types. *Econometrica*, 1349–1365.

Maskin, E. (1999). Nash equilibrium and welfare optimality. *The Review of Economic Studies 66*(1), 23–38.

Maskin, E. and J. Riley (2000). Asymmetric auctions. *The Review of Economic Studies 67*(3), 413–438.

Milgrom, P. and I. Segal (2002). Envelope theorems for arbitrary choice sets. *Econometrica 70*(2), 583–601.

Milgrom, P. and C. Shannon (1994). Monotone comparative statics. *Econometrica*, 157–180.

Milgrom, P. R. and R. J. Weber (1982). A theory of auctions and competitive bidding. *Econometrica 50*(5), 1089–1122.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research 6*(1), 58–73.

Myerson, R. B. and M. A. Satterthwaite (1983). Efficient mechanisms for bilateral trading. *Journal of Economic Theory 29*(2), 265–281.

Parkes, D. C. and L. H. Ungar (2000). Iterative combinatorial auctions: Theory and practice. In *AAAI/IAAI*, pp. 74–81.

Rhee, K.-E. (2007). Collusion in the presence of externalities. *The Journal of Industrial Economics 55*(3), 475–497.

Rhee, K.-E. (2010). Repeated auctions with impatient bidders and externalities. *Southern Economic Journal 77*(2), 482–500.

Rochet, J.-C. (1987). A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of mathematical Economics 16*(2), 191–200.

Rockafellar, R. T. (1970). Convex analysis.

Roth, A. E. and E. R. Peranson (1999). The redesign of the matching market for American physicians: Some engineering aspects of economic design. *American Economic Review 89*(4), 748–780.

Rothkopf, M. H. (2007). Thirteen reasons why the Vickrey-Clarke-Groves process is not practical. *Operations Research 55*(2), 191–197.

Rothkopf, M. H., T. J. Teisberg, and E. P. Kahn (1990). Why are Vickrey auctions rare? *Journal of Political Economy 98*(1), 94–109.

Segal, I. (1999). Contracting with externalities. *Quarterly Journal of Economics 114*(2), 337–388.

Shapley, L. and M. Shubik (1969). On the core of an economic system with externalities. *American Economic Review 59*(4), 678–684.

Varian, H. (2007). Position auctions. *International Journal of Industrial Organization 25*(6), 1163–1178.

Varian, H. R. and C. Harris (2014). The VCG auction in theory and practice. *The American Economic Review 104*(5), 442–445.

Varma, G. (2002). Standard auctions with identity-dependent externalities. *RAND Journal of Economics 33*(4), 689–708.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance 16*(1), 8–37.

Vohra, R. V. (2011). *Mechanism design: a linear programming approach*, Volume 47. Cambridge University Press.

Wilson, R. (1985). Incentive efficiency of double auctions. *Econometrica*, 1101–1115.

Wilson, R. (1987). Game-theoretic analyses of trading processes. In T. Bewley (Ed.), *Advances in Economic Theory: Fifth World Congress*, Chapter 2, pp. 33–70. Cambridge, U.K.: Cambridge University Press.

Yokoo, M., Y. Sakurai, and S. Matsubara (2004). The effect of false-name bids in combinatorial auctions: New fraud in internet auctions. *Games and Economic Behavior 46*(1), 174–188.