Correlation in the frequency domain under misspecification of trends

Very Preliminary

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1. Introduction

Specification of trends in time series processes have been a long-standing topic since early 1980s. A lot of significant literatures which study trending variables have been accumulated since early 1980s(Nelson and Kang, 1981; Nelson and Plosser, 1982; Perron, 1989; Murray and Nelson, 2000; Perron and Wada, 2009, to name a few). For many macroeconomic time series variables, stochastic trends and deterministic trends are known as the two most typical types of trends. Thus, a good deal of detrending methods have also been used in practce, which includes time trend removal, first-differencing, fracional differencing, Hodrick-Prescott filtering and various smoothing methods. As correct detrending cannot be emphasized enough in practical researches, various testing procedures have been used to identify the trends in time series context. As is widely knwon in practice, a list of test statistics include conventional unit root tests of Augmented Dickey-Fuller(ADF) test, the LM test by Kwiatkowski et al(1992), variance ratio tests by Breitung(2001) and so on.

Earlier literature have mainly focused on the effects of misspecification of trends on the behavior of spectral densities, in terms of cyclical pattern of the trending time series variables. Among them, Nelson and Kang(1981) study the effect of false detrending on the sample spectrums in the wide range of frequencies and indicates that true dynamics of the variables could be distorted due to incorrect detrending. In this regard, many researches clarifies the effects of false detrending(e.g., Harvey and Jaeger, 1993; Canova, 1998, Fleissing and Strauss, 1999, Aadland, 2005). As a recent work, we note that Dagum and Giannerini(2006) investigate the effects of misspecification of trends on various hypothesis tests, including tests for stationarity and tests for linearity. Ashley and Verbrugge(2006) also study how the parameter estimates in linear models are vulnerable to the way of detrending. The above mentioned findings draws attention to the possibility of distortion of true dynamics of the time series variables due to incorrect detrending.

In this work, we bring attention to the correlation in the frequency domain, as a useful measure of comovement of stationary time series variables. Specifically, we employ the dynamic correlation proposed by Croux et al(2001), which is the real part of well-known coherency measure. This model-free comovement measure provides useful implications in the context of economic forecasting along with the analysis of pro- and counter- cyclical properties of major indicators and risk management for early warning system for economic risks. Given this, we study how the dynamic correlation measure is affected by detrending methods, which is not investigated in a formal manner. While we pay attention to stochastic and deterministic trends, two possible cases of incorrect detrending are analyzed theoretically and numerically. In the section 2, the limiting form of dynamic correlations under correct and incorrect detrending is provided. Then, in the section 3, some numerical studies are given to see the effects of detrending on the behavior of dynamic correlations.

2. Effects of detrending on Dynamic Correlations

The dynamic correlation(DC hereafter), proposed by Croux et al(2001) is the real part of the coherency between the two stationary time series variables. While the coherency or squared coherency has been used for long decades, the DC has its own advantage including computational merit. Simply put, the DC is a correlation measure defined in the frequency domain as follows,

$$\rho_{xy}(\lambda) = \frac{f_{xy}(\lambda)}{\sqrt{f_x(\lambda) f_y(\lambda)}}, \text{ for } \lambda \in [-\pi, \pi], \tag{1}$$

where $f_{xy}(\lambda)$ is the cospectrum, which equals to the real part of the cross spectral density, and $f_x(\lambda)$ and $f_y(\lambda)$ are the auto-spectral densities of covariance-stationary x and y, respectively(e.g., Priestley, 1981).

In our work, we consider trends-stochastic trends and deterministic trends- and the effect of trend specification on the behavior of DC. The effect of detrending on the DC is rarely covered in Croux et al(2001). Depending the type of detrending method, the DC is expected to generate different implications. Thus, we focus on the effect of detrending on the correlations in the frequency domain, which also draws attention in the area of practical researches using time series variables. To be concrete, we consider two cases of false detrending. The first one is to specify the trends as stochastic trends when the true process is trend stationary. In this case, incorrect detrending arises when one takes the first-difference the series. The second case is the converse, where one gets the time removal for the series when the true process contains the stochastic trends.

We first consider the case that true processes consist of stationary components around the deterministic time trend. Moreover, we allow stationary innovations to have a linear structures as in Phillips and Solo(1992). It helps construct the limiting forms of DC under correct and incorrect detrending. Also, richer setup for the data generating processes(DGP) is provided, compared to Dagum and Giannerini(2006). We formally put the following assumption,

Assumption 1: *Bivariate series $q_t = (z_t, w_t)'$ follows trend stationary processes,*

(*i*) $q_t = \alpha + \tau t + e_t$,

where $\alpha = (\alpha_1, \alpha_2)'$, $\tau = (\tau_1, \tau_2)'$, and $e_t = (e_{1t}, e_{2t})'$ is *a linear process given by*

$$\begin{aligned} (ii)\ e_t &= \phi(L)\varepsilon_t = \textstyle\sum_{k=0}^{\infty} \phi_k \varepsilon_{t-j} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-k} \\ \varepsilon_{2t-k} \end{pmatrix}, \\ \varepsilon_t &= (\varepsilon_{1t}, \varepsilon_{2t})' \text{ is } iid.(0, \sigma^2 I_2), \\ \textstyle\sum_{k=0}^{\infty} k^{\delta} ||\phi_k|| &< \infty, \text{ for } \delta \geq 1 \text{ and } ||\phi_k|| = [\textstyle\sum_k |\phi_k^{i,j}|^2]^{1/2}. \end{aligned}$$

The linear structure of innovations given in the assumption is standard where long-run and short-run components are compactly expressed through well-known Beveridge-Nelson decomposition techniques(Phillips and Solo(1992)).

In relation to the variance and covariance structures of linear processes given above, we first define the following quantities.

$$\begin{aligned} f_j^{11}(L) &= \textstyle\sum_{k=0}^{\infty} a_k a_{k-j} L^k,\ f_j^{12}(L) = \sum_{k=0}^{\infty} b_k b_{k-j} L^k, \\ f_j^{21}(L) &= \textstyle\sum_{k=0}^{\infty} c_k c_{k-j} L^k,\ f_j^{22}(L) = \sum_{k=0}^{\infty} d_k d_{k-j} L^k, \\ g_j(L) &= \textstyle\sum_{k=0}^{\infty} a_k c_{k-j} L^k,\ h_j(L) = \sum_{k=0}^{\infty} b_k d_{k-j} L^k. \end{aligned} \tag{2}$$

Under the linear structure of the innovations, we can derive the explicit form of the LDC, as follows. The following theorem states the effects of correct and incorrect detrending on the behavior of the DC.

Theorem 1: *Under the assumption 1, (i) the dynamic correlation for time-detrended* q_t *equals to*

$$\rho_{xy}^*(\lambda) = \frac{f_{12}^*(\lambda)}{\sqrt{f_1^*(\lambda) f_2^*(\lambda)}},$$

where

$$\begin{aligned} f_1^*(\lambda) &= \textstyle\sum_{j=-\infty}^{\infty}(f_j^{11}(1)+f_j^{12}(1))\cos(j\lambda), \\ f_2^*(\lambda) &= \textstyle\sum_{j=-\infty}^{\infty}(f_j^{21}(1)+f_j^{22}(1))\cos(j\lambda), \\ f_{12}^*(\lambda) &= \textstyle\sum_{j=-\infty}^{\infty}(g_j(1)+h_j(1))\cos(j\lambda). \end{aligned}$$

(ii) *the dynamic correlation for first-differenced* Δq_t *equals to*

$$\rho_{xy}(\lambda) = \frac{f_{12}(\lambda)}{\sqrt{f_1(\lambda)f_2(\lambda)}},$$

where

$$\begin{aligned} f_1(\lambda) &= \textstyle\sum_{j=-\infty}^{\infty} f_{1j}\cos(j\lambda), \\ f_2(\lambda) &= \textstyle\sum_{j=-\infty}^{\infty} f_{2j}\cos(j\lambda), \\ f_{12}(\lambda) &= \textstyle\sum_{j=-\infty}^{\infty} f_{12j}\cos(j\lambda), \end{aligned}$$

with

$$\begin{aligned} f_{1j} &= 2(f_j^{11}(1)+f_j^{12}(1)) - f_{j-1}^{11}(1) - f_{j+1}^{11}(1) - f_{j-1}^{12}(1) - f_{j+1}^{12}(1), \\ f_{2j} &= 2(f_j^{21}(1)+f_j^{22}(1)) - f_{j-1}^{21}(1) - f_{j+1}^{21}(1) - f_{j-1}^{22}(1) - f_{j+1}^{22}(1), \\ f_{12j} &= g_j(1)+h_j(1)+2(g_{j-1}(1)+h_{j-1}(1)). \end{aligned}$$

The part (i) refers to the correct specification of trend, whereas the part (ii) arises from misspecification of trend. We only prove the part(ii).

[proof of Theorem 1] The proof is based on Phillips and Solo(1992, eq.28) and Maynard and Shimotsu(2009). For the differenced processes, the auto-covariances are written as $E(\Delta\varepsilon_t\Delta\varepsilon_{t-r}) = -\sigma^2$, for $r \neq 0$, and for $|r| > 1$, $E(\Delta\varepsilon_t\Delta\varepsilon_{t-r}) = 0$. Then, we obtain

$$\begin{aligned} E(\Delta e_{1t}\Delta e_{1t-j}) &= f_j^{11}(1)E(\Delta\varepsilon_{1t}^2) + \textstyle\sum_{r=1}^{\infty}[f_{j-r}^{11}(1)E(\Delta\varepsilon_{1t}\Delta\varepsilon_{1t-r}) + f_{j+r}^{11}(1)E(\Delta\varepsilon_{1t}\Delta\varepsilon_{1t+r}) \\ &\quad + f_j^{12}(1)E(\Delta\varepsilon_{2t}^2) + \textstyle\sum_{r=1}^{\infty}[f_{j-r}^{12}(1)E(\Delta\varepsilon_{2t}\Delta\varepsilon_{2t-r}) + f_{j+r}^{12}(1)E(\Delta\varepsilon_{2t}\Delta\varepsilon_{2t+r}) \\ &= \sigma^2[2(f_j^{11}(1)+f_j^{12}(1)) - f_{j-1}^{11}(1) - f_{j+1}^{11}(1) - f_{j-1}^{12}(1) - f_{j+1}^{12}(1). \end{aligned}$$

Similarly, we get

$$E(\Delta e_{2t}\Delta e_{2t-j}) = \sigma^2[2(f_j^{21}(1)+f_j^{22}(1)) - f_{j-1}^{21}(1) - f_{j+1}^{21}(1) - f_{j-1}^{22}(1) - f_{j+1}^{22}(1).$$

For the cross covariance and the cospectrum, we employ the results in Maynard and Shimotsu(2009, lemma 13) to get

$$E(\Delta e_{1t}\Delta e_{2t-j}) = \sigma^2[g_j(1) + h_j(1) + 2(g_{j-1}(1) + h_{j-1}(1))].$$

The results of the part (ii) come from the case that trend stationary processes are first-differenced, which entail over-differenced series. This type of misspecification is also known as the moving average(MA) unit root(Saikkonen and Luukkonen, 1993). Since over-differenced series generate additional correlations, the forms of auto-spectrum and cospectrum become more complicated than correctly detrended case. As a result, the true value of DC $\rho^*_{xy}(\lambda)$ is not equal to the $\rho_{xy}(\lambda)$, given by incorrectly detrended variables.

To further get an insight for the MA unit root problem in terms of the DC in the frequency domain, consider a simple example that $a_0 = b_0 = 1$ and $a_k = b_k = 0$ for $k > 0$ in the process of e_t. It then follows that $E(\Delta e_{1t}^2) = 2\sigma^2$, and $E(\Delta e_{1t}\Delta e_{1t-1}) = E(\Delta e_{1t}\Delta e_{1t+1}) = -\sigma^2$. Then, the auto-spectral density of e_{1t} becomes

$$f_1(\lambda) = 2(1 - \cos(\lambda)). \tag{3}$$

It is noted that degeneracy arises at the zero frequency, i.e., $f_1(0) = 0$. Thus, false detrending invalidates the DC in the long-run (Lee(2017)).

3. Numerical Studies

[in progress]

In this section, we conduct a small set of numerical studies to see the effect of correct and incorrect detrending on the LDC. In doing so, we use kernel-based nonparametric estimation for the auto-spectral density and cospectrum. As is well-known in the econometrics context, the auto-spectral densities of covariance-stationary variable x is given by

$$\hat{f}_x(0) = \hat{R}_x(0) + 2\sum_{j=1}^{T-1} k(j/M)\hat{R}_x(j), \tag{4}$$

where k is a kernel function, M is the lag truncation number(bandwidth) and the sample variances are given by

$$\hat{R}_x(j) = T^{-1}\sum_{t=|j|+1}^{T}(x_t - \overline{x})(x_{t-|j|} - \overline{x}),$$

with $\overline{x}$ is the sample mean of x. Also, the cospectrum estimator is given by

$$\begin{aligned} \hat{f}_{xy}(0) &= \textstyle\sum_{j=1-T}^{T-1} k(j/M)\hat{R}_{xy}(j) \\ &= \hat{R}_{xy}(0) + \textstyle\sum_{j=1}^{T-1} k(j/M)\hat{R}_{xy}(j) + \sum_{j=1}^{T-1} k(j/M)\hat{R}_{xy}(-j), \end{aligned} \quad (5)$$

where the sample cross covariance equals to

$$\hat{R}_{xy}(j) = T^{-1} \textstyle\sum_{t=|j|+1}^{T} (x_t - \overline{x})(y_{t-|j|} - \overline{y}),$$

and cross covariance is not symmetric in j.

The bandwidth M is required to satisfy the condition that $M \to \infty$ and $M/T \to 0$, which guarantees consistency of the estimators(e.g., Priestley, 1981, Andrews 1991, Newey and West 1994). Both parametric and nonparametric bandwidth choice rules can be considered in practice, in line with the context of heteroskedasticity and autocorrelation consistent (HAC) covariance matrix estimation. We then avoid further discussion on the choice of bandwidths in this work.

Firstly, we consider a linear time trend process as the data generating process. Let $q_t = (z_t, w_t)'$ follow

$$\text{DGP 1:} \quad \begin{pmatrix} z_t \\ w_t \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \delta_1 t \\ \delta_2 t \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix},$$

where the parameters are set as

$$\mu_1 = \mu_2 = 0, \ \delta_1 = \delta_2 = 0.1, \ \theta_{11} = \theta_{22} = 0.2, \ \theta_{12} = 0.05.$$

We estimate correctly detrended and incorrectly first-differenced DCs. The plot is given in Figure 1. It is observed that at small frequencies of $0 \leq \lambda < 0.4$, the first-differenced DC gets smaller than the time-removed DC, implying that first-differencing under-estimates the long-run correlations. If the series are monthly observations, the frequency $\lambda = 0.4$ roughly corresponds to 15 months. On the other hand, after $\lambda > 2$, which is associated with short-run dynamics, first-differencing, unlike the case of long-run correlations, over-estimates the correlations between the two variables.

Next, in order to study the converse case, we consider unit processes for q_t,

$$\text{DGP 2:} \ q_t = D_t'\delta + q_{t-1} + v_t, \quad (6)$$

where $D_t = (1, t)'$, and $v_t = (v_{1t}, v_{2t})'$ with $E(v_{1t}) = E(v_{2t}) = 0, E(v_{1t}v_{2t}) = \omega_{12}$ and $E(v_{1t}v_{2s}) = 0$ for $t \neq s$.

The DGP is rather simple, as it does not incorporate sufficient number of serial correlations of innovation processes. Thus, it is worthwhile to extend to more sophisticated and realistic design and investigate the effect of detrending on the patterns of DC.

4. Empirical Studies

[in progress].

5. Conclusion

[in progress]

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Figure 1: Estimated Dynamic Correlations