# INDIVISIBILITY AND NON-NEUTRALITY OF MONEY *

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*In this paper, we study the real effects of different degrees of divisibility of money in a random matching model. When money is very indivisible, as it is seemingly true in most of the world before the 19th century, welfare increases as the divisibility of money increases (non-neutrality). However, when the degree of divisibility is sufficiently high, as it is seemingly true for the current U.S. coinage system, there would be little or no welfare loss from reducing the degree of divisibility, the elimination of the penny.*

## I. INTRODUCTION

This paper explores numerically the effects of different degrees of divisibility of money on steady states in a random matching model. There are both theoretical and substantive motivations for the work. The theoretical motivation arises from the works of Berentsen and Rocheteau(2002) and Zhu(2003). Berentsen and Rocheteau(2002) show that indivisibility of money causes inefficiency in production and consumption in search-theoretic models of money, where individuals are supposed to hold either one or zero unit of money. Zhu(2003) removes the restriction on the upper bound of individual money holdings and

proves that different degrees of divisibility have real effects in the matching model. (The degree of indivisibility is the ratio of the per capita stock of money to the size of the smallest unit, a unit-free measure.) However, there are no general analytical results about the nature of the real effects that accompany different degrees of divisibility. Hence, the numerical work would provide some results about the implications of the model. With a parameterized version of a random matching model, Lee and Wallace(2006) explore numerically the implications of model on the optimal divisibility. They focus on the trade-off between the cost and benefit of additional divisibility, where cost is the physical cost of providing and maintaining stock of money. Here we try to investigate the implications of the model without such a physical cost. Because the benefit of additional divisibility mainly results from the distributional effects of wealth, there might be an upper bound on divisibility approximately (upper bound in the sense that there is only negligible improvement in distribution beyond that magnitude).[1] The substantive motivation involves two applications of the model, one at a very low degree of divisibility and the other at a high degree of divisibility.

The low degree of divisibility application is motivated by historical episodes in which low-valued money seemed to be scarce or non-existent. That seems to be true in some periods of 16th-century Europe (see Sargent and Velde(2002), pages 202-203). That was also when Europe experienced large increases in its stock of money that originated in new-world specie discoveries. In this regard, David Hume, in *Of money*, after describing the sense in which the quantity of money does not matter except for the level of prices, goes on to say:

> Notwithstanding this [neutrality] conclusion,..., it is certain, that, since the discovery of the mines in America, industry has increased in all the nations of Europe,...; and this may justly be ascribed, amongst other reasons, to the increase of gold and silver. (Hume (1987), part II, page 36.)

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[1] It is well known that a randomness in consumption and production in a matching model causes a heterogeneity in wealth holdings. And the wealth distribution tends to be concentrated around the average wealth level as the degree of divisibility increases, but its limiting function would not be a degenerate one.

To check the above conjecture, we study extensively the connection between the purported expansionary effects of additional money and the indivisibility of money. Our results suggest that, as argued by Hume, some increases in economic activities, in those times, would be ascribed to greater divisibility implied by a larger stock of money.

The second application arises from thinking about the coinage system in the U.S. today, in particular, the elimination of the penny. The existing estimates of seigniorage and the cost of producing pennies suggest that maintaining the penny is a very expensive source of government revenue (expensive relative to estimates of the deadweight loss from other taxes). Given that result, we could ask whether reducing divisibility by eliminating the penny would have welfare cost that we can measure. Our results suggest that it would not.

The rest of this paper is organized as follows. In section 2, we describe the basic model. There are two versions, one with and one without the trading of lotteries. Then, we discuss the analytical results on existence of a steady state. In section 3, we set up the numerical environment and compute the welfare function as a function of degrees of divisibility. More specifically, with a parameterized version of the matching model, we compute a steady state for each degree of divisibility, where a steady state consists of wealth distribution, $\pi$, and value function, $w$. Associated with that steady state is *ex ante* welfare given by expected discounted utility—the inner product, $\pi w$. In section 4, we apply the estimated welfare function to the economy of early modern France and to the current U.S. coinage system. Section 5 summarizes the results and discusses some further issues related to the elimination of the penny. In the appendices, we first describe the numerical algorithm and then present steady states we found.

## II. THE MODEL

The model is the version of the Shi (1995) and Trejos and Wright(1995) random matching models studied by Zhu(2003): indivisible money with a sufficiently large upper bound on individual money holdings and take-it-or-leave-it offers by the buyer.

## 2.1. Basic Environment

Time is discrete. There is a unit measure of each of $N > 2$ types of infinitely lived agents, and there are $N$ distinct produced and perishable types of divisible goods at each date. A type $n$ agent, $n \in \{1, 2, 3,..., N\}$, produces only good $n$ and consumes only good $n+1$ (modulo $N$). Each agent maximizes the expected discounted utility with discount factor $\beta \in (0, 1)$. For a type $n$ person, utility in a period is $u(y_{n+1}) - y_n$, where $y_{n+1} \in \mathbb{R}_+$ is consumption of good $n+1$ and $y_n \in \mathbb{R}_+$ is production of good, $n$. The function, $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing, concave, differentiable, and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, there exists $y' > 0$, such that $u(y') = y'$.

There are three exogenous nominal quantities that describe the stock of money: ($s$, $\bar{a}$, $H$), where $s$ is the smallest unit of money, $\bar{a}$ is the capita stock of money holdings, and $H$ is the upper bound on individual money holdings. One of them can be normalized at unity. We let normalize $s$ be unity so that the feasible set of individual money holdings ($\mathbb{H}$) always consists of integer amounts, namely, $\mathbb{H} = \{0, 1, 2, 3..., H\}$.

In each period, each person meets another person at random. We assume that trading partners see each other's types and money holdings. However, trading histories are private, and agents cannot commit to future actions. In single-coincidence meetings, only relevant meetings in the model, the buyer makes a take-it-or-leave-it offer ($y, p$), where $y$ is the amount of good demanded, and $p$ is the amount of money offered. In a deterministic version, agents are not allowed to trade lotteries, while in a lottery version, buyers can choose lotteries.

## 2.2. Definition of a Steady State and Existence

A symmetric steady state, symmetric across the specialization types, is a pair of a value function, $w : \mathbb{H} \rightarrow \mathbb{R}$, and a measure, $\pi : \mathbb{H} \rightarrow [0, 1]$, where $w(z)$ is the expected discounted value of beginning a period with wealth $z$, and $\pi(z)$ is the fraction of people with wealth $z$ at the start of a period. Here, we formally define a steady state for a version with lotteries in meetings. (For a deterministic version, see Zhu(2003).)

Consider the single coincidence meeting between a buyer with $x \in \mathbb{H}$ amount of money and a seller with $m \in \mathbb{H}$ amount of money. Let $w_t(x)$ be the expected discounted utility of holding $x$ amount of money at the start of period $t$. We first let $\Omega(x, m, w_{t+1})$, a set of probability measures on $\mathbb{R}_+ \times \{\max\{0, x-H+m\}, ..., x\}$, be defined by:

$$\Omega(x, m, w_{t+1}) = \{\sigma : \mathbb{E}_\sigma [-y + \beta w_{t+1}(m + x - z)] \geq \beta w_{t+1}(m)\}, \qquad (1)$$

where $\sigma(y, z)$ is the probability that the trade is y amount of output and a wealth transfer that leaves the buyer with $z \in \mathbb{H}$ units of wealth. Notice that we describe the money transfers in terms of the end-of-trade wealth of the buyer. Hence, the buyer's post-trade wealth $z$ corresponds to the seller's post-trade wealth $m + x - z(= m + p)$. The take-it-or-leave-it offer $(y, p)$ is feasible because $0 \leq p \leq x$ and the seller is willing to produce $y$ if the weak inequality in the right-hand side of (1) holds. Then the buyer's problem is to maximize $\mathbb{E}_\sigma [u(y) + \beta w_{t+1}(z)]$ subject to (1), where $\mathbb{E}_\sigma$ is the expectation with respect to $\sigma$. Let,

$$f(x, m, w_{t+1}) = \max_{\sigma \in \Omega(x, m, w_{t+1})} \mathbb{E}_\sigma [u(y) + \beta w_{t+1}(z)]. \qquad (2)$$

A maximizer in (2) is degenerate in $y$ and is determined by the lottery over the buyer's post-trade wealth, because the constraints in (1) hold with equality. Therefore, we let $\mathbb{S}(x, m, w_{t+1})$, a subset of probability measures on $\mathbb{H}$, be the set of maximizer in (2) described in that way; $\delta \in \mathbb{S}(x, m, w_{t+1})$ is a lottery over the post-trade wealth of the buyer and $\delta(z)$ is the probability for that maximizer that the buyer has wealth $z \in \mathbb{H}$ after trade. Then, we define a set of post-trade distribution on $\mathbb{H}$ by:

$$\Pi(w_{t+1}, \pi_t) = \{\pi_{t+1} : \pi_{t+1}(z) = \frac{1}{N} \sum_{(x, m)} \pi_t(x) \pi_t(m) [\delta(z) + \delta(x - z + m)] +$$

$$\frac{N-2}{N} \pi_t(z) \text{ for } \delta \in \mathbb{S}(x, m, w_{t+1})\}, \qquad (3)$$

where the first probability measure in the right-hand side corresponds to single-coincidence meetings, whereas the second corresponds to all the

other cases. Notice that the buyer's end-of-trade wealth $(x - z + m)$ corresponds to the seller's end-of-trade wealth $z(= x - (x - z + m) + m)$, and hence, $\delta(z)$ is the probability that the buyer ends up with $z$, and $\delta(x - z + m)$ is the probability that the seller ends up with $z$. Finally, the value function, $w_t(x)$, satisfies:

$$w_t(x) = \frac{1}{N} \sum_m \pi_t(m) f(x, m, w_{t+1}) + \frac{N-1}{N} \beta w_{t+1}(x). \tag{4}$$

It consists of the expected payoff from holding $x$ as a buyer in the single-coincidence meeting and the expected payoff from all the other meetings, including as a seller in the single-coincidence meeting. Now, we can define a symmetric steady state for the lottery version.

**Definition 1** *A sequence of $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ with given $\pi_0$ is a symmetric equilibrium, if it satisfies (1)-(4). A symmetric steady state is $(w^*, \pi^*)$, such that $(w_t, \pi_{t+1}) = (w^*, \pi^*)$, for all t, is an equilibrium for $\pi_0 = \pi^*$.*

With a deterministic version of this model, Zhu(2003) proves that if $u'(0)$, $\bar{a}$, $H/\bar{a}$ are sufficiently large, then there exists a *nice* steady state $(w, \pi)$ (*nice* in the sense that $w$ is strictly increasing and strictly concave, and $\pi$ has full support). It is easily confirmed that every step in his argument applies to the version with lotteries in meetings; namely, existence of *nice* steady states in a lottery version is a straightforward extension of his results.

Besides the existence result, Zhu(2003) shows that non-neutrality holds in the following sense. Let $k = (\bar{a}, H)$ and let $\lambda$ be an integer larger than 1. Then the set of all steady states for economy $k$ is a strict subset of those for economy $\lambda k = (\lambda \bar{a}, \lambda H)$. (For both economies, $s$ is normalized at unity.) For any steady state $(w_k, \pi_k)$ for economy $k$, $(w_{\lambda k}, \pi_{\lambda k})$ with $w_k(x) = w_{\lambda k}(\lambda x) = w_{\lambda k}(\lambda x + 1) = \cdots = w_{\lambda k}(\lambda x + \lambda - 1)$ and $\pi_k(x) = \pi_{\lambda k}(\lambda x)$, $\pi_{\lambda k}(\lambda x + 1) = \cdots = \pi_{\lambda k}(\lambda x + \lambda - 1) = 0$ is a steady state for economy $\lambda k$; namely, for any steady state for economy $k$, there is an equivalent (in terms of real allocation) step-function value function steady state for economy $\lambda k$. Besides these kinds of steady states, there is a *nice* steady

state for economy $\lambda k$ that is not duplicated by any steady state of economy *k*. However, nothing is known in general about how nice steady states for economy *k* and economy $\lambda k$ differ.

Despite the somewhat simplicity of the model, further analytical exploration on the properties of a steady state seems not easy, mainly due to an endogenous, non-degenerate distribution of money. Hence, we proceed numerically to explore the real effects of different degrees of divisibility on steady states.

## 2.3. Welfare

Our numerical works largely consist of welfare comparisons across steady states for different degrees of divisibility. The welfare for a given steady state ($w, \pi$) is defined by the inner product $\pi w$, expected utility of a representative agent prior to the assignment of wealth where the assignment is made in accord with the steady-state distribution, $\pi$. In order to compute welfare cost in terms of consumption equivalents, we first rewrite the value function as:

$$w(x) = \frac{1}{N} \sum_{m \in \mathrm{H}} \pi(m)\{(u[y(x,m)] - y(m,x)\} + \beta \sum_{z \in \mathrm{H}} T_{xz} w(z), \qquad (5)$$

where $T_{xz}$ is the probability of a trade that results in transition from having *x* units of money (before trade) to having *z* units of money (after trade) and it can be expressed by:

$$T_{xz} = \begin{cases} \dfrac{1}{N} \sum_{m=0}^{H-x+z} \pi(m)\delta(z;x,m,w) & \text{if } x > z \\[2ex] \dfrac{1}{N} \sum_{m=z-x}^{H} \pi(m)\delta(m-z+x;m,x,w) & \text{if } x < z \\[2ex] 1 - \sum_{x \ne z} T_{xz} & \text{if } x = z. \end{cases} \qquad (6)$$

Here, $\delta(z;x,m,w)$ is a probability that the buyer with *x* units of money offers $x - z$ to the seller having *m* units of money and $\delta(m - z + x; m, x, w)$ is a probability that the buyer with *m* units of money offers $z - x$

to the seller having $x$ units of money. By multiplying $\pi(x)$ both sides and adding over $x \in \mathbb{H}$, we obtain:

$$\sum_{x \in \mathbb{H}} \pi(x)w(x) = \frac{1}{N} \sum_{x \in \mathbb{H}} \sum_{m \in \mathbb{H}} \pi(x)\pi(m)\{u[y(x,m)] - y(m,x)\} +$$
$$\beta \sum_{x \in \mathbb{H}} \sum_{m \in \mathbb{H}} \pi(x)T_{xz}w(z). \tag{7}$$

Since $\sum_{x} \pi(x)T_{xz} = \pi(z)$ by the steady state condition, (7) can be rewritten by:

$$(1-\beta)\pi w = \frac{1}{N}\pi Q\pi'. \tag{8}$$

Here, $q_{xm}$, the row $x$, column $m$ element of $Q$, is given by:

$$q_{xm} = u(y_{xm}) - y_{xm}, \tag{9}$$

where $y_{xm}$ is the buyer's consumption when the buyer enters trade with $x \in \mathbb{H}$ and the seller with $m \in \mathbb{H}$. Notice that an upper bound on $\pi Q \pi'$ is $\max_{y}[u(y) - y]$.

Now our welfare cost measures are computed in terms of consumption equivalents as follows. Suppose Economy 2 has lower welfare than Economy 1. Then, we report $\Delta/[\pi^{(2)}Y^{(2)}\pi^{(2)'}]$, where $y_{xm}^{(2)}$, the Row $x$, Column $m$ element of $Y^{(2)}$, is the buyer's consumption in Economy 2 when the buyer enters trade with $x \in \mathbb{H}$ and the seller with $m \in \mathbb{H}$ and where $\Delta$ satisfies:

$$N(1-\beta)\pi^{(1)}w^{(1)} = \pi^{(2)}Q^{(c)}\pi^{(2)'}, \tag{10}$$

with $q_{xm}^{(c)}$, the Row $x$, Column $m$ element of $Q^{(c)}$, given by:

$$q_{xm}^{(c)} = u(y_{xm}^{(2)} + \Delta) - y_{xm}^{(2)}. \tag{11}$$

We make compensating amount of consumption $(\Delta)$ be additive rather

than multiplicative, as in Lee and Wallace(2006), because an addition to consumption is most valuable for some single-coincidence meetings where consumption is zero. Notice also that the compensating consumption is given only to buyers (consumers in single-coincidence meetings) and that its magnitude is reported relative to average buyer-consumption in Economy 2, not to per capita consumption.

## III. COMPUTING A WELFARE FUNCTION

The standard computational procedure to find a steady state involves iterating on the mapping studied in Section 2, the usual mapping studied in heterogeneous-agent models. (See Appendix 1 for details.)

### 3.1. Parameterization

We first need to specify the utility function ($u$), the discount factor ($\beta$), the number of specialization types ($N$), and the upper bound on money holdings ($H$). Before describing in detail, a brief remark is in order. Because the main objective of this paper is to explore the effects of different degrees of divisibility on welfare in a matching model, we compute steady states for wide-ranging degrees of divisibility in simple numerical environments—that is, among the reasonable parameters satisfying our assumptions, we choose conveniently small numbers and the simplest functional forms to parameterize a background-matching model. Thus, as hardly needs to be mentioned, our numerical results would be indicative.

As for the utility function, we let $u(y) = y^{1/2}$. The square root utility function is the simplest one that satisfies all our assumptions and has the merit of implying a linear first-order condition for the choice of lotteries in meetings. It implies $y^* = 1/4$ and $u(y^*) - y^* = 1/4$. We set the number of specialization types to be $N = 3$. It is the smallest magnitude that eliminates the possibility of a double coincidence of wants in pairwise meetings. We study three different magnitudes of the discount factor, $\beta = 0.999$, $\beta = 0.998$, $\beta = 0.995$, which we regard as arising from an annual discount factor of 0.94 with a weekly meetings, bi-weekly meetings and monthly meetings. Finally, we choose the upper bound on

individual money holdings to be $H = 4\bar{a}$, which is sufficiently large in the sense that numerical results are not sensitive to imposing higher $H$. (See Appendix 2.)

## 3.2. Results

For the above numerical environments, we find a steady state for each $\bar{a}$ and its associated welfare, where our starting point of $\bar{a}$ is 3, sufficiently low degree of divisibility. Notice that we normalize $s$ to be 1, and hence, per-capita stock of money ($\bar{a}$) just represents degree of divisibility ($\bar{a}/s$). Despite the absence of uniqueness results pertaining to nice steady states, our computational procedure converges to a nice steady state, which does not seem sensitive to the initial conditions we choose.

Before presenting computed welfare functions, we first show a detailed picture of some of the underlying outputs. Figure 1 shows behaviors of average consumption over single-coincidence meetings as a function of $\bar{a}$.

**[Figure 1]** Average consumption as a function of $\bar{a}$
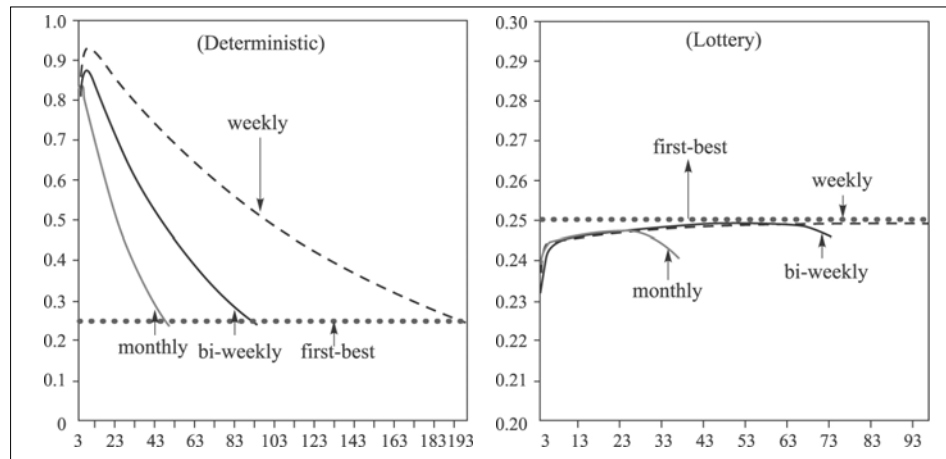


It is well known that, in a deterministic version, the take-it-or-leave-it offer by a buyer generally produces "too much" output (i.e., "too much" relative to the *first-best* level). Unlikely in a deterministic version, average consumption for a lottery version is bounded by the *first-best* level, and it does not vary much relative to that for a deterministic version.

For a deterministic version, there is a substantive difference in the magnitudes of average consumption across different search-frictions for a given $\bar{a}$; the higher search friction, in terms of lower $\beta$, produces lower average consumption. This is because, we conjecture, the less frequent meetings, other things being the same, make the seller more unwilling to incur current disutility in exchange for future consumption.

Figure 2 shows the estimated welfare function as a function of degree of divisibility. In the figure, dotted lines denote the upper bound on welfare corresponding to each search friction, respectively. As expected from Figure 1, welfare increases steadily with $\bar{a}$ but at a decreasing rate. This result is consistent with the findings in Camera and Corbae(1999), and Lee and Wallace(2006). Not surprisingly, in a lottery version, trading benefit is vanished much more quickly than in a deterministic version. This is mainly because small trade is feasible in a lottery version, and hence the distribution of output over single-coincidence meetings is very close to the best output level even in the case of low $\bar{a}$.

**[Figure 2]** Welfare function as a function of $\bar{a}$
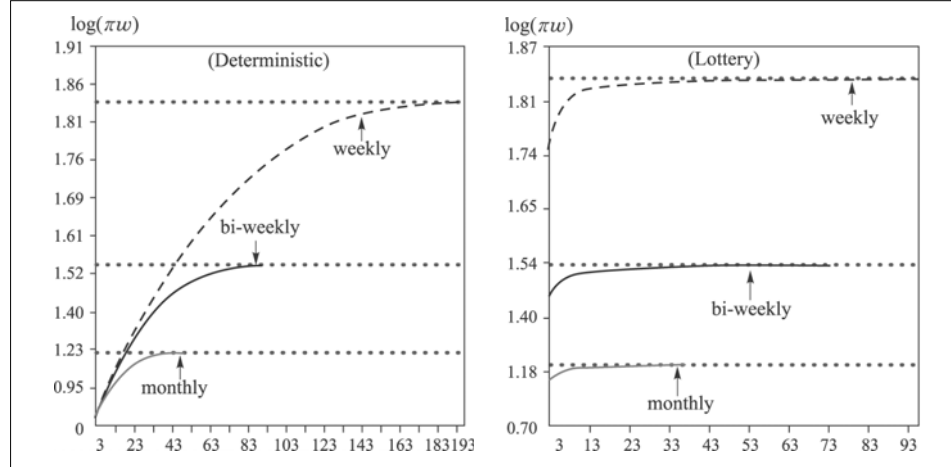


## IV. IMPLICATIONS OF THE RESULTS

To draw some implications from the computed welfare function, we apply it on two different economies; one at a low degree of divisibility and the other at a high degree of divisibility.

### 4.1. An Example of an Economy with a Low Degree of Divisibility

In most of the world before the 19th century, currency consisted of full-bodied coins and conveniently small denominations were not available.[2] In particular, in 16th-century France, the ratio of *per capita* money holdings to the size of smallest pure silver coin was very low.[3] According to Rolnick et al.(1997), the estimates of *per capita* money holdings in terms of grams of pure silver in France from 1300 to 1600 were around 33 to 95 grams. The smallest pure silver coin in the second half of 16th-century France was *quart franc* weighing around 3.5 grams. These imply that $\bar{a}$ at that time ranges from 10 to 30.

For these degrees of divisibility, as can be seen in Figure 2, welfare increases as $\bar{a}$ becomes larger. Table 1 contains consumption equivalent welfare costs relative to the upper bound on $\pi w$ (*first-best*) for each degree of search friction.

**[Table 1]** Welfare cost relative to the *first-best*

| $\bar{a}$ | Deterministic version | | | Lottery version | | |
|---|---|---|---|---|---|---|
| | Monthly | bi-weekly | weekly | Monthly | bi-weekly | weekly |
| 35 | 0.0643 | 0.8470 | 4.4123 | 0.0120 | 0.0221 | 0.0222 |
| 40 | 0.0195 | 0.6002 | 3.4284 | 0.0116 | 0.0189 | 0.0191 |
| 75 | 0.0072 | 0.0292 | 0.9074 | 0.0116 | 0.0086 | 0.0092 |
| 80 | 0.0072 | 0.0129 | 0.7712 | 0.0116 | 0.0082 | 0.0085 |

When $\bar{a}$ is less than 40, a welfare cost decreases monotonically as $\bar{a}$ increases. Even in the case that the estimates of $\bar{a}$ are doubled, say a half of the population held all the money and were engaged in search for monetary trade, a welfare cost still decreases steadily in most of our examples. It suggests that the increase in the stock of money in 16th-century Europe was one of the reasons for the rise in real economic activities.

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[2] See, for example, Redish(2000), and Sargent and Velde(2002).

[3] In those times, an alloy coin of silver and copper, which had a lower value than pure silver coin, was rarely minted and it was not a significant part of the monetary system. See Redish(2000), pages 127-131.

## 4.2. An Example of an Economy with a High Degree of Divisibility

One of the controversial issues in the current coinage system in the U.S. is the elimination or continuation of the penny, which was introduced in 1857. Since 1989, representatives of the United States Congress have tabled a bill four times to stop the production of the penny. From the sense of monetary economics, the penny is one source of government revenue and allows for the divisibility of the currency. Hence, both concerns (seigniorage and trade) benefit from greater divisibility and should be addressed. We first investigate the seigniorage problem. To simplify the argument, we implicitly assume that even if we eliminate the penny, there is no additional demand for other denominations (e.g., nickel[4]). The U.S. Mint reported that total production and distribution cost of 7.13 billion pennies (71.3 million dollars) was 66.7 million dollars in 2004.[5] Therefore, the seigniorage (face value − relevant cost) from the penny at that year was about 4.6 million dollars at a cost, in terms of resources, of 66.7 million dollars. The government could finance the same amount of revenue (4.6 million dollars) through an income tax at a cost (deadweight loss) of 0.9−9.5 million dollars (see Table 2), which is very low relative to the cost of the penny.

Then the remaining rationale for maintaining the penny is its role in ensuring divisibility. One of the survey results shows that non-institutionalized U.S. residents aged 18 or more held about 100 dollars on average. (See Porter and Judson (1996).) Because the smallest unit of money is the penny, $\bar{a}$ is around $1.0 \times 10^4$. Our computed welfare function, however, suggests that such a ratio is high enough compared with the so-called, "upper bound" (upper bound in the sense that there is no or negligible trade benefit of greater divisibility beyond that magnitude). For instance, Figure 3 shows the consumption equivalent welfare cost relative to the upper bound on $\pi w$ for the deterministic version with weekly meetings, when $\bar{a}$ exceeds 100.

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[4] If net amount of nickel mint increases sufficiently resulting from disappearance of the penny, additional revenue from the nickel might exceed the loss from the elimination of the penny.

[5] Recently, the U.S. Mint reported to a Congressional Committee that the cost of a penny might exceed its nominal value (negative seigniorage) in 2006 and 2007, mainly due to rising prices of zinc and copper.

**[Table 2]** Comparison of Deadweight Loss

|  |  | Average DWL[a] | Marginal DWL[b] |
| --- | --- | --- | --- |
| Penny |  | 4.06 | 4.06 |
| Income Tax | Feldstein (1999) | 0.32 | 2.06 |
|  | Fullerton (1998) | − | 0.20 − 0.25 |

a) Total DWL/Revenue

b) DWL per dollar of additional revenue

**[Figure 3]** Welfare cost as a function of $\bar{a}$ : deterministic and $\beta$ =0.999
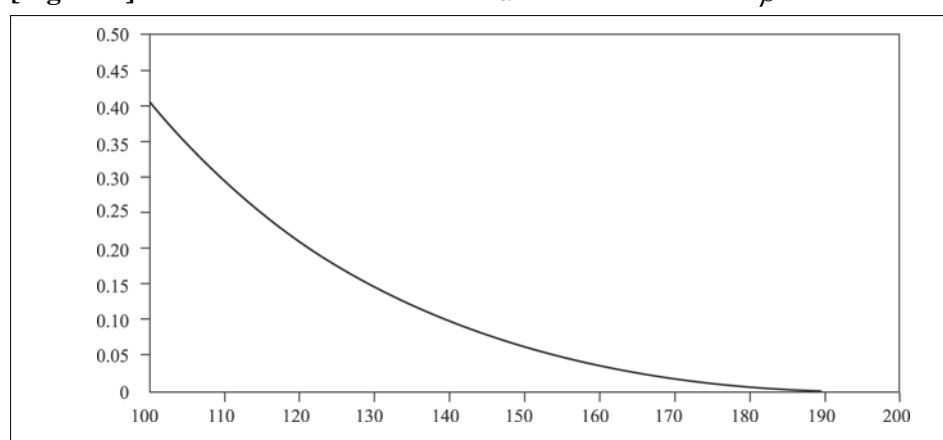


The welfare cost decreases monotonically until $\bar{a}$ reaches around 200. Beyond that degree of divisibility, welfare cost is very inelastic to the change of divisibility. Obviously, 200 is far smaller than 1,000, which is the degree of divisibility when the *per capita* stock of money is as low as 10 dollars. Along with the estimated $\bar{a}$ for the U.S. economy, it suggests that reducing the degree of divisibility by eliminating the penny would have very little or no welfare loss. In contrast, the estimates of $\bar{a}$ from 1900 to 1940 under the assumption of constant ratio of money holdings to *per capita* nominal income over time ranged around 100 to 300.[6] It implies that divisibility would be a compelling reason to circulate the pennies in those periods.

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[6] Suppose people hold a constant fraction of their nominal income in the form of cash, $M_t = \alpha Y_t$ where $M_t$ denotes per-capita cash holdings at time $t$ and $Y_t$ denotes per capita nominal income at time $t$. By using per capita nominal GDP and survey result in Porter and Judson (1996), we calculate $\alpha$ and estimate $\bar{a}$ for other years. The data for nominal GDP in the early 1900s' come from Johnston and Williamson (2006).

# V. SUMMARY AND DISCUSSIONS

This paper explores the real effects of divisibility of money on a steady state of matching models of money. Our results give different messages to an economy having a low degree of divisibility and an economy having a high degree of divisibility. The European economy in the 16th century is a typical example of an economy with low divisible money. Historians generally agree that, in that period, the stock of money increased due to the new-world specie discoveries. Real economic activity was also increased. On this phenomenon, our results, as claimed by David Hume, suggest that the former caused the latter.

One of the issues pertinent to the sufficiently high divisible money is whether to keep the penny or not in the U.S. coinage system. Our results suggest that there would be little or no welfare loss from reducing the degree of divisibility by elimination of the penny.

Note, however, that many impacts of the penny are implied (these are not in our model). For example, the elimination of the penny might have an effect on the overall price level from rounding. We have not taken into account a carrying (or handling) cost of the penny for daily trades. To address these concerns, we need to introduce a multiple-denomination structure, as in the Lee et al.(2005), in place of a single denomination. We conjecture our result would not be eroded even in a multiple-denomination model. Suppose simply, the denomination structure is given by power-of-2 rule [e.g., $d = (1, 2, 4, 8, 16)$], and there is an economy 1 described by $\mathbb{H} = \{0, 1, 2, 3..., H\}$ and $\bar{a} = a_1$. If the penny is removed from the above denomination structure, the new economy (e.g., economy 2), would be characterized by $\mathbb{H}' = \{0, 1, 2, 3..., H/2\}$ and $\bar{a}' = a_1/2$ with $d' = (1, 2, 4, 8)$. Obviously, if $a_1/2$ is sufficiently large, our welfare function implies that there would be no welfare loss in pairwise trades from eliminating the penny. However, the number of coins used in trades, in economy 1, might be larger than that of economy 2. This implies that carrying or handling cost of coins in economy 2 would be smaller than that of economy 1. Meanwhile, Kim and Lee(2007), do a very simple exercise on the deterministic version of Lee et al.(2005). Their results show that the price level in economy 2 is generally lower than that of

economy 1; that is, there would be no inflationary distortion even though we eliminate the penny.

## Appendix 1: Finding A Steady State

We start with an arbitrary initial guess on $g^{(0)} = (w^{(0)}, \pi^{(0)})$, where $w^{(0)}$ is interpreted as an end-of-period value function and $\pi^{(0)}$ is a beginning-of-period distribution. For the given $g^{(0)}$, we can solve the buyer's problem in a single-coincidence meeting in which the buyer has money $x \in \mathbb{H}$ and the seller has $m \in \mathbb{H}$. The solution for a deterministic version can be easily found by way of a global search; choosing $p^*(x, m)$ which maximizes $u[\beta w^{(0)}(m + p) - \beta w^{(0)}(m)] + \beta w^{(0)}(x - p)$ from the set of $\{0, 1, 2, \dots \max\{x, H - m\}\}$.

The process of finding optimal lotteries is little bit tricky. The first step is to find the best meeting-specific integer offer, $p^*(x, m)$. Then we extend the value function, $\widetilde{w}^{(0)} : [0, H] \rightarrow \mathbb{R}_+$ by the linear interpolation of $w^{(0)}$. Then an optimal lottery offer, $p^{R*}(x, m)$, is obtained by finding the optimal deterministic offer from the $\widetilde{w}^{(0)}$. The concavity of value function implies that $p^{R*}(x, m)$ is either in $[p^*(x, m), p^*(x, m) + 1)$ or $(p^*(x, m) - 1, p^*(x, m)]$; if the right derivative of objective function at $p^*(x, m)$ is positive, then it cannot be positive at $p^*(x, m) + 1$; if it is negative at $p^*(x, m)$, then it cannot be negative at $p^*(x, m) - 1$. Hence, positive right derivative at $p^*(x, m)$ implies $p^{R*}(x, m) \in [p^*(x, m), p^*(x, m) + 1)$, where $p^{R*}(x, m)$ is degenerate on $p^*(x, m)$ if $p^*(x, m) + 1$ is not in the feasible set of offers. Similarly, negative right derivative at $p^*(x, m)$ implies $p^{R*}(x, m) \in (p^*(x, m) - 1, p^*(x, m)]$, where $p^{R*}(x, m)$ is degenerate on $p^*(x, m)$ if $p^*(x, m) - 1$ is not in the set of feasible offers or the left derivative of it at $p^*(x, m)$ is positive. Now, suppose right derivative at $p^*(x, m)$ is positive.[7] Then $p^{R*}(x, m) = p^*(x, m) + \delta$, where $\delta \in [0, 1)$ can be obtained by solving the following maximization problem:

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[7] The argument for negative right derivative at $p^*(x, m)$ is symmetric.

$$\max_{\delta \in [0,1]} u[\beta\{w^{(0)}(l) + \Delta w^{(0)}(l)\delta - w^{(0)}(m)\}]$$

$$+ \beta[w^{(0)}(y) - \Delta w^{(0)}(y)\delta] \tag{12}$$

where $l = m + p^*(x,m)$, $\Delta w^{(0)}(l) = w^{(0)}(l+1) - w^{(0)}(l)$, $y = x - p^*(x,m)$ and $\Delta w^{(0)}(y) = w^{(0)}(y) - w^{(0)}(y-1)$. The concavity implies an interior solution such as

$$\delta^* = \frac{1}{\beta \Delta w^{(0)}(l)} u'^{-1}[\frac{\Delta w^{(0)}(y)}{\Delta w^{(0)}(l)}] - [\frac{w^{(0)}(l) - w^{(0)}(m)}{\Delta w^{(0)}(l)}].$$

Now, let $\rho$ satisfy

$$\rho[x - p^*(x,m) - 1] + (1-\rho)[x - p^*(x,m)] = [x - p^{R*}(x,m)].$$

Then the optimal lottery over $[(x - p^*(x,m) - 1), (x - p^*(x,m))]$ is $\sigma^*(x,m) = (\rho, 1 - \rho)$.[8]

Now the function $p^*(x,m)$ and $\sigma^*(x,m)$ can be used to generate an end-of period distribution, $\pi^{(1)}$. In addition, the implied payoff for the buyer and $\pi^{(0)}$ give a beginning-of-period value function, $w^{(1)}$.

The values for the $(i + 1)$-th iteration, denoted $g^{(i+1)} = (w^{(i+1)}, \pi^{(i+1)})$, are formed by a weighted average of the elements $g^{(i)}$ and $g^{(i+1)}$, where different weights are used for the different components. This updating process is repeated until the components of $g$ satisfy the convergence criterion: $\max_j |(w_j^{(i+1)} - w_j^{(i)} / w_j^{(i)}| < 10^{-5}$ and $[\sum_j (\pi_j^{(i+1)} - \pi_j^{(i)})^2]^{1/2} < 10^{-5}$. Finally, to confirm the computational results, we randomly check by hand whether, for a given steady-state value function, the function $p^*(x,m)$ is a solution to the buyer's problem.[9]

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[8] If $p^*(x,m)$ and solution for (12) is unique, then we can always find the unique optimal lottery $\sigma^*(x,m)$ for each single-coincidence meeting.

[9] A copy of the Matlab code is available upon request.

## Appendix 2: Computed Steady States

We first check whether $H = 4\bar{a}$ is sufficiently high. For the conveniently small degree of divisibility, $\bar{a} = 5$, we compute welfare for $\beta = 0.995$ as a function of $H$. As we can see in Figure 4, the variation of welfare is negligible when $H$ exceeds the $3\bar{a}$.

Now we show how a steady state looks like. We illustrate the steady state for $\bar{a} = 15$ and $\beta = 0.995$.[10] Figure 5 shows $w$ and $\pi$. The $\pi$ has full support over $H$, although it looks like almost zero above $3\bar{a}$. An almost inelastic welfare variation when $H$ exceeds $3\bar{a}$ is mainly due to the fact that there is very small mass of agents holding more than three times the average money holdings in a steady state. Not surprisingly, a lottery version has a higher $w$ and a more concentrated $\pi$ around $\bar{a}$.

Recall that welfare for a lottery version is much higher than that for a deterministic version for a given $\bar{a}$. This is primarily because, in a lottery version, as we can see in Figure 6, of more meetings where money holdings of the buyer and the seller are consistent with output near the *first-best* level.

**[Figure 4]** Welfare as a function of $H$; $\bar{a} = 5$ and $\beta = 0.995$.



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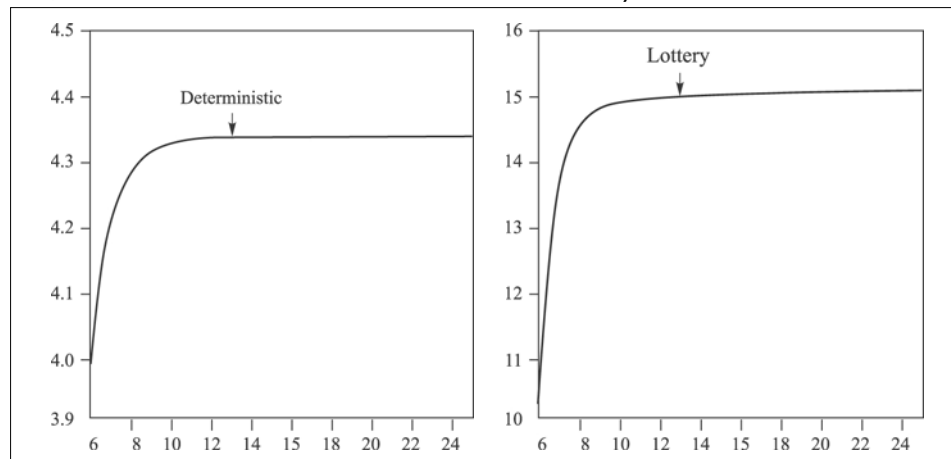
[10] Steady states for other degrees of divisibility and meeting frequencies are virtually similar.

**[Figure 5]** $\pi$ and $w$: $\bar{a} = 15$ and $\beta = 0.995$.
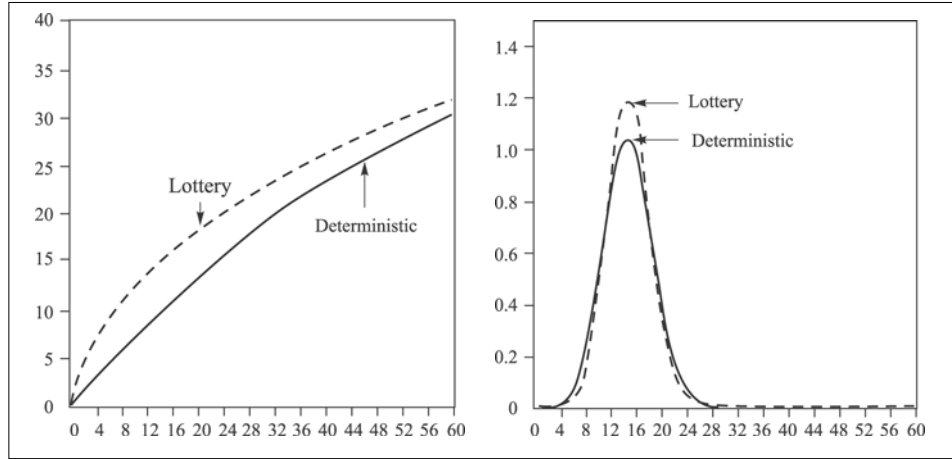


**[Figure 6]** Consumption distributions over single-coincidence meetings: $\bar{a} = 15$ and $\beta = 0.995$.
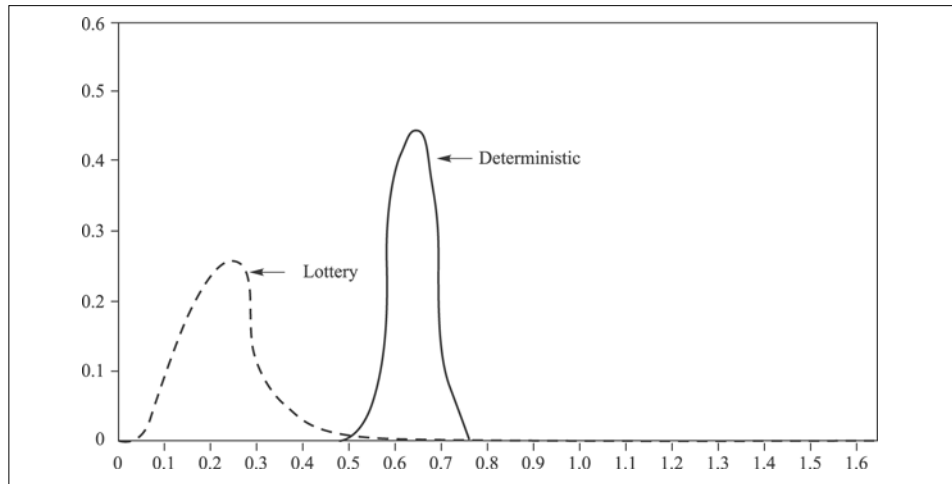
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