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JEVONS'S CURVE FITTING*

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In the Theory of Political Economy Jevons proposed a numerical equation which would conform to the King-Davenant's figures. The equation contained three parameters, of which the values Jevons did not disclose how he had determined. Two kinds of evidence, texts and replications, are presented to contend that Jevons conducted a non-systematic search for what would minimize the sum of squared errors.

I. INTRODUCTION

It is said that the marginal utility theory of price started with the 1871 work of Jevons entitled the *Theory of Political Economy*. But it has rarely been noted that in the book Jevons attempted to ascertain a numerical law from a set of data on price and harvest. Aldrich (1987) pays attention to this neglected attempt of Jevons, suggesting that Jevons would have employed the method of least squares. I shall support Aldrich as to this point. There is, however, a problem. The equation Jevons derived from the data on price and harvest is not linear in parameters, in which case it is complicated to apply the method of least squares. As Jevons did not disclose how he proceeded with the non-linear equation, Aldrich could only speculate on it. It is this speculation of Aldrich that I find questionable. I shall thus present my own speculation on *how* Jevons would have applied the method of least squares. In so doing, I intend to make it more convincing that Jevons employed the method of least squares to fit a curve to data.

The issue at hand is a fairly narrow one as it stands for itself. However, we should take it into consideration that the notion of curve fitting had rarely been put to use in economic investigations until this century. Jevons would have made an exception to that tradition if he had indeed fitted a curve to data. It also

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implies that Jevons had an unusual opinion about the attainability of exactness in economics. It had long been held that no economic law could ever be given an exact numerical expression, partly because no advance from data could be made beyond mere empirical generalizations.[1] In contrast, Jevons considered it possible to erect economics into an exact science by the aid of data. He might have tried to demonstrate the possibility by deriving a numerical law of price from the Gregory King's data. It is thus of historical interest to examine Jevons as one who had an extraordinary view on the nature and method of economics. In particular, we could inquire what and how he tried to learn from statistical data. The present paper is a part of my inquiry in this line, focusing on how he fitted a curve to data.[2]

Four sections follow before they are summarized. Section Ⅱ explicates how Jevons understood the method of least squares. Section Ⅲ concerns another attempt of Jevons to ascertain an exact law from statistical data. We shall then be better positioned to find how Jevons would have derived an exact law of price from the data on price and harvest. Section Ⅳ presents evidence in support of the conjecture that Jevons applied the method of least squares directly to the non-linear equation although the application was not systematic. In Section Ⅴ the conjecture of Aldrich is disputed.

Ⅱ. THE METHOD OF LEAST SQUARES AS DISCUSSED BY JEVONS

Soon after it was proposed in 1805 by a French scientist named Legendre, the method of least squares became a standard tool in astronomy and geodesy.[3] It is, therefore, safe to assume that the method was known to Jevons who had a keen interest in the theory of probabilities and statistical methods.[4] But we shall confirm it by examining his own exposition of the method of least squares in the *Principles of Science*, a book which had occupied his mind since 1866. This examination might help us to gather clues to his procedure with the *non-linear* curve above mentioned.

In the chapter titled "The Law of Error," Jevons explained the method of

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[1] This opinion is explicitly advanced in Cairnes (1875).

[2] A more general discussion about Jevons's use of statistical methods to ascertain exact economic laws is given in Kim (1995). See also Kim (1996) which shows that Jevons (1863, 1869) introduced the notion of probability into statistical inquiry.

[3] For a detailed history of the method of least squares, see S. M. Stigler (1986, Ch. 1-4) and H. L. Harter (1974).

[4] It seems that Jevons was introduced to the theory of probabilities by Augustus De Morgan at the University College. Jevons was also intrigued by Quetelet, as his letter of 30 January 1859 illustrates it. Jevons's trust in the statistical methods was put to work in the "Periodic commercial fluctuations" (1862), followed by the *Value of Gold* (1863) and the *Coal Question* (1865).

least of squares as follows:

> Let a, b, c, &c., be the results of observation, and x the quantity selected as the most probable, that is the most free from unknown errors: then we must determine x so as that $(a-x)^2 + (b-x)^2 + (c-x)^2 + \cdots$ shall be the least possible quantity. We thus arrive at the celebrated Method of Least Squares··· (Jevons 1877, p. 377)

Jevons also explained how to apply the "celebrated method" to obtain the "most probable" values of two or more related constants. An equation with two constants, x and y, was taken as an example (Jevons 1877, pp. 393-4)[5] :

$$ax + by = c$$

The most probable values of x and y, he said, should be determined by solving the "mean equations":

$$(\text{sum of } a^2) \cdot x + (\text{sum of } ab) \cdot y = (\text{sum of } ac)$$
$$(\text{sum of } ab) \cdot x + (\text{sum of } b^2) \cdot y = (\text{sum of } bc)$$

These two equations are such that their solution minimizes the sum of $(c-ax-by)^2$. That is, they are what Legendre (1805) called the "minimum equations," or what Gauss (1809) called the "normal equations."

The mean-equations algorithm, however, cannot be employed unless the equation is linear in parameters. Jevons must have been aware of this limited applicability. Nevertheless, the *Principles of Science* contains no discussion on how to proceed with a non-linear equation. Little more is to be learned from his most likely sources on the method of least squares, namely, De Morgan (1838) and Airy (1861).[6] We should thus close this section without any better idea of how Jevons would have applied the method of least squares to determine the values of constants in a non-linear equation.

Ⅲ. THE WEIGHT-DISTANCE EQUATION

About a year before the *Theory of Political Economy* was first published, Jevons (1870) tried to ascertain a numerical law from a series of experiments. The experiments consisted in measuring the distances to which various weights could be thrown by his arm upon level ground. Noting that a "good average" of the

[5] To make it clear that x and y are constants whereas a, b and c are variables, we could rewrite the equation as $a_i x + b_i y = c_i$.

[6] They both are cited in the *Principles of Science*.

distances was obtained from about fifty-seven experiments with each weight, he reported the averages:

Weight (pounds)	56	28	14	7	4	2	1	1/2
Distance (feet)	1.84	3.70	6.86	10.56	14.61	18.65	23.05	27.15

"A little consideration" then led him to a form of equation with which "these numbers would agree."[7] It was an inverse equation with two constants, p and q:

$$x = \frac{p}{w + q} \tag{1}$$

where x is the distance thrown, and w the weight thrown. He then determined the values of p and q, making a statement which I would take seriously: "by the method of least squares we determine their most probable values to be p = 115.7, q = 3.9" (Jevons 1870, p. 158).[8]

Jevons, however, did not disclose *how* he had applied the method of least squares to the non-linear equation. We can only speculate about the application based on circumstantial evidence. My speculation is that Jevons applied the method of least squares to equation (1) as it is, not to a linear expansion of it or any other transformation.[9] In other words, the sum of squared errors (SSE) which Jevons tried to minimize by selecting the values of p and q is

$$\sum_{i=1}^{8} \left(x_i - \frac{p}{w_i + q} \right)^2 \tag{2}$$

Below I shall explicate and support this conjecture.

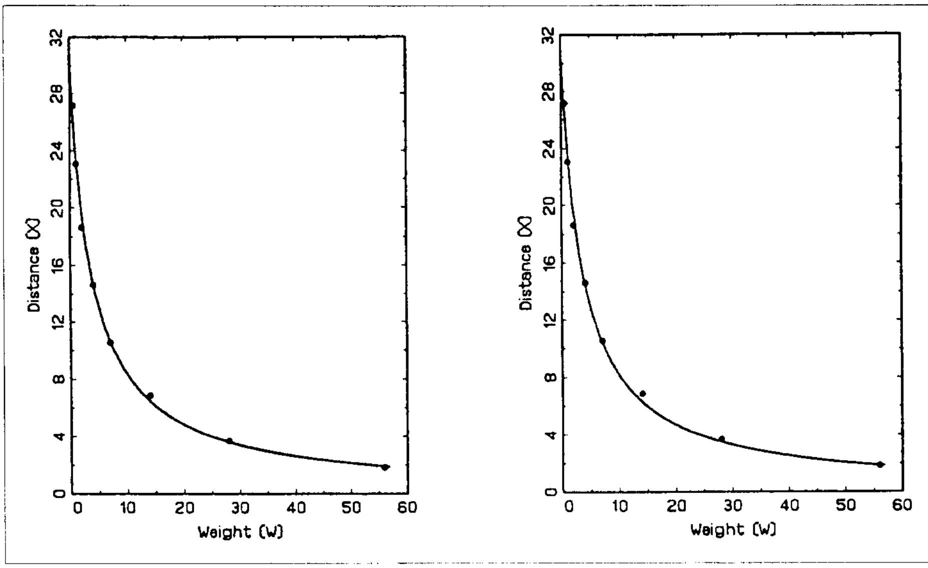
Since the first-order conditions of minimization have no analytic solution, a numerical method has been used to find that the SSE is minimized around p =

[7] Jevos later recalled that this form of equation had been discovered by "a purely haphazard trial" (Jevons 1877, p. 490).

[8] Stigler (1994) suggests that Jevons did not appeal to least squares. In response I would emphasize that Jevons himself referred to the method of least squares in relation to equation (1). Stigler also echoes his 1982 review of Jevons, stating that "to find Jevons using the method with social data in 1871 would have been anomalous". But Jevons *was* an anomalous scientist, who applied "the ordinary methods of the theory of probabilities". This view of mine is advanced in Kim(1995).

[9] Stephen M. Stigler has pointed it out to me that a procedure current at the time was to expand the equation as a linear one before applying the method of least squares. The equation could also be transformed as $wx = p - qx$, as suggested by Aldrich (1987).

[Figure 1a] X = 115.7/[W + 3.9] **[Figure 1b] X = 111.4/[W + 3.7]**



111.4, q = 3.72. Compare this to the Jevons's figures, p = 115.7, q = 3.9. The difference between two pairs is small, and they yield a virtually identical curve so far as the relevant range is concerned. This is illustrated by the two curves in Figure 1, which are respectively drawn for p = 115.7, q = 3.9 and p = 111.4, q = 3.72 against the same eight points of weight and distance. We can also consider the SSE for each curve. Jevons would have got 2.13 from the curve with p = 115.7, q = 3.9. This SSE does not differ much from what the other curve yields, that is, 1.98. I would like to add to it that the two constrained minima of (2), one with q = 3.9 and the other with p = 115.7, are respectively obtained at p = 115.2 and q = 3.91. In other words, no significant reduction of the SSE is to be attained by changing p away from 115.7 while holding q at 3.9, nor by changing q away from 3.9 while holding p at 115.7.

I take these facts as a ground of asserting that Jevons searched for the values of p and q which would minimize the SSE. Of course, a full-scale numerical search would have required too much computation for him to handle. But he could have tried with various curves, visually inspecting how each curve fits the eight points plotted on a w-x space. We know, and Jevons could have concluded, that no significant improvement in the curve fitting is to be attained by changing the values of p from 115.7 or that of q from 3.9. Jevons could also have computed the SSE for a few curves before he selected one of them.

Another kind of facts can be cited in support of my speculation. First, Jevons was very fond of working with graphs.[10] His early statistical work (1862, 1863) illustrates how much he relied upon the graphical method. It might be as much relevant to note that the *Principles of Science* contains a lengthy discussion on the graphical method of curve fitting (Jevons 1877, pp. 492-5). Second, Jevons (1870) drew up another table to include the "calculated distances" and their "differences from experiment," of which the latter amount to the vertical distances between the curve and points in Figure 1(a). It could have been better evidence for my conjecture if he had also reported the squared errors and their sum.

In short, I have contended that Jevons (1870) did what he said he had done. That is, he searched for the values of parameters which would minimize the sum of squared errors although the search was neither systematic nor precise. It has also been suggested that he would have applied the method of least squares directly to a non-linear equation, not to its linear transformation.

Ⅳ. THE QUANTITY-PRICE EQUATION

We are now in a better position to assess the attempt of Jevons to "ascertain the law to which Davenant's figures conform."[11] The figures, which concerned the harvest of corn and its price, were re-stated by Jevons as follows (Jevons 1965, p. 156):

Quantity of Corn	1.0	.9	.8	.7	.6	.5
Price ……	1.0	1.3	1.8	2.6	3.8	5.5

In this table the "normal" harvest and the corresponding price were taken as unity.

Jevons enumerated what an equation should fulfil to represent the relation between the two variables. For instance, he had reasons to believe that "the price… should become infinite before the quantity is zero," and that "the price of corn should never sink to zero" (Jevons 1965, p. 157). These *a priori* considerations led him to an equation of the form,

[10] Keynes (1951, p. 268) may be quoted on this point: "[Jevons] would spend hours arranging his charts, plotting them, sifting them, tinting them neatly with delicate pale colours like the slides of the anatomists, and all the time poring over them and brooding over them to discover their secret." See also the letter of 7 April 1861 printed in Jevons (1973).

[11] There is some uncertainty about the origins of the figures although Davenant, who first put them in print, attributed them to Gregory King. In any case, the figures were widely known since Tooke (1838) reproduced them. Another uncertainty concerns their source; no explanation was given by Davenant on how the figures were obtained. It has often been suspected that they were at best an educated guess, not genuine data (see Creedy 1986). But Jevons took them as an "estimation", a term he earlier used for the average of distances which were measured from experiments.

$$y = \frac{a}{(x - b)^n} \tag{3}$$

where x and y are respectively the quantity and price of corn. Then the three constants, a, b and n, were determined.

> An inspection of the numerical data shows that n is about equal to 2, and, assuming it to be exactly 2, I find that the most probable values of a and b are $a = .824$ and $b = .12$. (Jevons 1965, p. 157)

Note that he called the values as "most probable," a term which he associated with the method of least squares. But we are left only to speculate on how he had applied the method.

The first to note is the similarity between equations (1) and (3). We may thus suppose that Jevons proceeded with equation (3) in the same way as he did with equation (1). That is, he searched for those values of a and b which would minimize the following:

$$\sum_{i=1}^{6} \left(y_i - \frac{a}{(x_i - b)^n} \right)^2 \tag{4}$$

This conjecture shall be put to a test based on my own computations.

The SSE is minimized at $n = 7.921$, $a = 1442$, $b = -1.519$.[12] Given that $n = 2$, the minimum is obtained at $a = .9287$, $b = .09238$. Comparing these two sets of parameter values to what Jevons got, $n = 2$, $a = .824$, $b = .12$, I shall call attention to the following facts. First, the SSE of Jevons's formula is .1226, which constitutes only five per cent of the total variation of price around its mean. This result would have satisfied Jevons. Even if he had looked for a smaller SSE, he could not have reduced it by more than .1168. Once n is set equal to 2, the possible reduction is even smaller, namely, .0346. It is thus not surprising that the three curves in Figure 2, which are drawn against the same eight points of x and y, do not differ much from each other. As Figure 2(a) and 2(b) show it, even the huge difference between the two sets of parameter values ($n = 2$, $a = .824$, $b = .12$ and $n = 7.921$, $a = 1442$, $b = -1.519$) has little significance insofar as the curves are concerned. Second, the SSE-minimizing value of n is closer to 2 than any other integers as long as b is constrained to be positive. Recall the presump-

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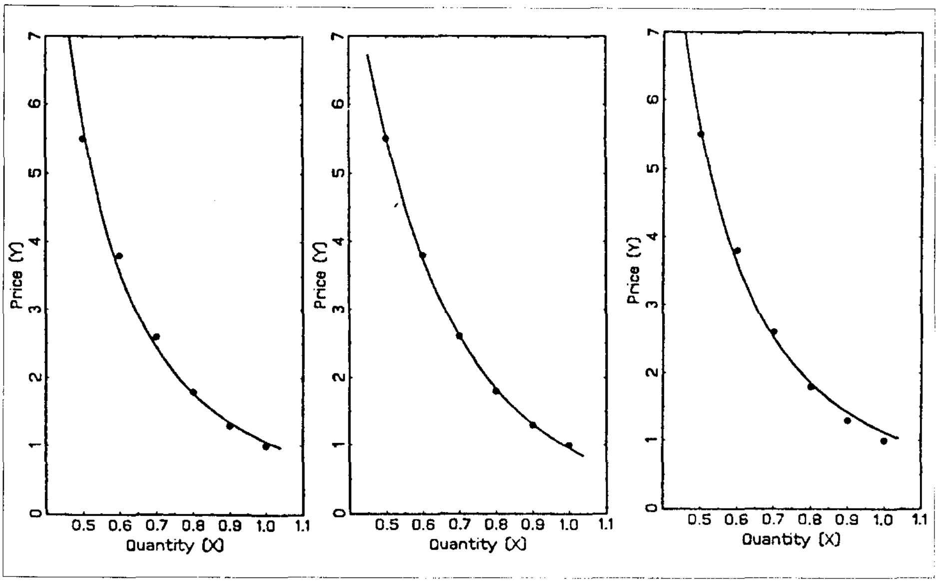
[12] Creedy (1986) reports that an iterative procedure give values for a, b and n respectively of 2.299, −0.631 and 4.736. They are much different from my figures. I was told that Creedy has applied the procedure to the log-transformation of equation (3).

[Figure 2a] $Y = .824/[X - .12]^2$ **[Figure 2b]** $Y = 1442/[X + 1.52]^{7.9}$ **[Figure 2c]** $Y = .929/[X - .09]^2$



tion of Jevons that $a/(x-b)^2$ should approach infinity before x becomes zero. It implies that b should be positive. Therefore, if the constrained search for the minimizing value of n was restricted to integers, $n = 2$ should be the outcome. In addition, the SSE is minimized at $n = 2.01$ if a and b are predetermined to be .824 and .12, respectively.[13] Third and last, a change in a or b alone does not help to reduce the SSE. Such a change only increases the SSE rather rapidly.

Taken together, these facts seem to support my conjecture. To repeat, Jevons would have tried to fit equation (3) to data so that (4) could be minimized. His curve fitting, however, would not have been as systematic as ours. He would have relied on visual inspection, limiting the computation of errors to a few curves. (But I should also point out that he did not report the differences of "calculated prices" from Davenant's figures.) It seems to be this crude application of the method of least squares that led Jevons to $n = 2$, $a = .824$, $b = .12$.

[13] As for the value which Jevons chose of n, White (1989) contends that it could have been drawn from Whewell. Jevons indeed quoted Whewell as stating that "the price varies inversely as the square of the supply" (Whewell 1862, pp. 51-2; Jevons 1965, p. 158). But the quotation was preceded by Jevons's emphasis on "the close approximation" of his empirical formula to Davenant's figures. I would thus say that Whewell might have led Jevons to consider $n = 2$, but not to choose it.

V. ALDRICH'S CONJECTURE

Attention shall be turned to Aldrich (1987) which has been cited in the beginning. According to him, Jevons would have transformed equations (1) and (3) as follows before he applied the method of least squares[14]:

$$wx = p - qx \tag{5}$$

$$y^{1/2} = a^{1/2} x^{-1} + bx^{-1} y^{1/2} \tag{6}$$

This conjecture is supported by the results of the method of least squares applied to the above equations; the parameter values so determined are very close to Jevons's. The argument of Aldrich, however, belies itself. There is discrepancy between equations (5) and (6). Given the similarity between equations (1) and (4), it is unlikely that Jevons would have proceeded in different ways. If Jevons had transformed his weight-distance equation to (5), he would rather have transformed his price-quantity equation to

$$xy^{1/2} = a^{1/2} + by^{1/2} \tag{7}$$

Further, this transformation is more obvious and makes the computation much simpler. However, it yields what Aldrich would call a "poor" result, that is, a = .7770, b = .1365 as compared to a = .824, b = .12. Hence I find it hard to accept the speculation of Aldrich.

VI. SUMMARY

In a study of muscular exertion Jevons fitted a non-linear curve to his data, stating that he had used "the method of least squares" to find the "most probable" values for parameters. He also fitted a similar curve to Davenant's figures and called the result as "most probable" values. We thus have a reason to believe that Jevons used the method of least squares to fit *both* curves. It should also be noted that the SSE is nearly minimized by his fitted curves. Further, the SSE is increased rather rapidly if other values are tried for one and only one parameter. These facts have been taken as evidence supporting my speculation: Jevons searched for the SSE-minimizing values of parameters, by visually inspecting the fit of alternative curves and computing the SSE for a few of them.

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[14] Equation (6) is reconstructed based on Aldrich's statement that his own estimation of a and b was obtained through a regression of $y^{1/2}$ on x^{-1} and $y^{1/2}/x$.

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