

TRADER'S OPTIMAL ORDER PLACEMENT STRATEGIES WITH LIMIT AND MARKET ORDERS

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This paper analyzes traders' strategic order placement behaviors with limit and market orders in an organized securities exchange. We analyze both traders' and the specialist's optimal behavior allowing a trader to choose between limit and market orders. It has been shown that sellers (buyers) with low (high) valuation on the securities will trade via market orders and sellers (buyers) with high (low) valuation will trade via limit orders. We show that limit order prices is a non-decreasing function of a reservation price and it has "price jumps"(which are termed as "gravitational pull" by Cohen, Maier, Schwartz and Whitcomb (1979)); one from a change in an optimal order type from a market order to a limit order; another in the limit price as a response the other trader's switch in order type. We also show that the specialist (broker-dealer) trading system is more efficient than the broker system and than the dealer system under certain conditions.

I. INTRODUCTION

Most speculative markets allow investors to choose from a number of different kinds of orders in trading securities. The two of the most common type of orders frequently used in the stock market such as on the New York Stock Exchange (NYSE), are the market and the limit orders. The primary characteristic of a limit order is that the trader specifies an upper (lower) limit price at which he is willing to buy (sell), so that the trader is an active price setter. No such price accompanies a market order; a trader merely instructs the broker to buy or sell at the current market price so that a trader is a price taker. The price for a market order could be set by the specialist or by the limit order from other traders. The main concern of this paper is to analyze how an investor would choose between market and limit orders.

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Little attention has been paid to the strategic aspects of choice between limit and market orders in the literature on the microstructure of security markets. Most papers focus on the specialist's bid-and ask-price determination behavior: order processing costs (Demsetz (1968), Tinic (1972)), inventory holding costs (Garman (1976), Stoll (1978), Amihud and Mendelson (1980), Ho and Stoll (1981), Mildenstein and Schleef (1983), O'Hara and Oldfield (1986)) and adverse selection costs (Copeland and Galai (1983), Milgrom and Glosten (1986), Glosten (1987)) have been analyzed as factors of the *specialist's* bid-and ask-price determination. In these papers, all orders are *market orders* and *limit orders* are not included in the analysis so that the specialist functions only as a dealer. Furthermore, *traders'* behavior has either not been considered or treated as an exogenous probability distribution of market order arrival. O'Hara and Oldfield (1986) and Conroy and Winkler (1986) include limit orders in their analysis of the specialist's price-determination, but their limit order is exogenously given rather than endogenously derived from trader's optimal behavior.

In reality, the specialist functions more as a broken than as a dealer¹ which indicates that limit orders would play just as an important role as market orders in determining various market outcomes such as the specialist's quote. As Glosten and Milgrom (1986, p.76) point out, a model that includes limit orders "should include investors optimally choosing the type of order to submit" with "detailed description of individual behavior."

There are a few studies that include traders' behavior in the analysis (Cohen, Maier, Schwartz, and Whitcomb (CMSW) (1981), Mendelson (1982)², Rock (1987)). Rock (1987) analyzes the informed trader's strategic choice between the limit and the market orders. He shows that an informed trader always submits a market order if the distribution of uninformed buy and sell market orders is symmetric.³ His analysis, however, is for a broker market where the market order is crossed with the limit order, and the content of limit order book is completely known to the public before the market order is placed. The specialist (intermediary), therefore, does not play any role in his analysis. CMSW (1981) analyze trader's strategic

¹Stoll (1985, pp. 11-16) estimates that only 25 percent of reported volume on NYSE in 1979 involves the specialist as a dealer (market orders) and 48 percent involves the specialist as a broker (limit orders). The remaining 28 percent are traded either by blocks or by an opening auction which are orders which specify its transaction prices as limit orders. In 1985, the specialist participated in only 10.6 percent of shares purchased and sold on the NYSE (Schwartz (1988) p. 52).

²Mendelson (1985) analyzes a competitive exchange model with the Walrasian auctioneer, not a dealer market. However, in his concluding remarks (p.275), he shows that traders' truth-revealing limit prices constitute a Nash equilibrium in an exchange mechanism in which each trader submit a limit order and orders are executed at market bid-and ask-price.

³Theorem 1 in p. 17 of Rock (1987). In his analysis, traders are exogenously categorized into the "time sensitive" and "time insensitive." The "time sensitive" uninformed and the informed traders place market orders and the "time insensitive" uninformed traders place limit orders.

behavior in the order placement. They show, by means of a dynamic price model with transaction costs, that the probability of a limit order execution does not rise to unity as the limit price at which the order is placed gets arbitrarily close to a counterpart market order. This gap in the probability induces a "gravitational pull" to change from a limit order to a market order. Their results are based on the assumption that the bid-and ask-prices are generated by a Poisson stochastic process, so that no explicit considerations are given to the specialist's role or the trader's strategic behavior in determining the limit price. However, their model explicitly considers the choice between limit and market order and it is very closely related to our analysis.

Our analysis is motivated by the desire to obtain a better understanding of why and how traders choose between limit and market orders in security exchanges where the specialist functions as both dealer and broker.⁴ Especially in our analysis, unlike in CMSW, a trader's strategies - in choosing between limit and market orders and in determining a limit price - are conditioned not only on the specialist's bid-and ask-price determining behavior but also on the other trader's strategies. The following are some of the questions to be addressed. (1) when would a trader choose to trade via a market order and when would he trade via a limit order? (2) how should a trader determine the limit price when a limit order is chosen? (3) how should the specialist determine the bid-and ask-price for market orders? In addition, we also discuss the relative efficiency-in terms of ex-ante expected profit from trade-of a trading system that includes the specialist, where both limit and market orders are allowed, to other trading systems; one where only limit orders are allowed, one where only market orders are allowed, and one where direct bargaining is used.

We construct an equilibrium in which a trader's optimal choice can be either type of orders. In this equilibrium, traders' order placement strategies are characterized as follows: for a seller (buyer), those who place a relatively low (high) value on the security will trade via a market order and those who place a relatively high (low) value on the security will trade via a limit order when the traders' information on the security value is symmetric. The limit price is a non-decreasing function of trader's true valuation on the security but limit price to sell (buy) will be systematically higher (lower) than the true valuation.

Such strategies can be explained by the characteristic differences between limit and market orders: probability of trading is unity in the market order, but there is uncertainty with regard to the transaction price; the transaction price is specified in the limit order, but there is uncertainty with regard to the transaction probability. The trade-off between the gain (or loss) in the transaction price and the loss (or gain) in the transaction probability is the key factor which determines the proper

⁴The specialist as dealer executes market order against his own account. As a broker, he crosses market order against limit order or crosses limit order against limit order.

order type. For a seller with a low reservation price, the opportunity of sure trading guaranteed by a market order outweighs the expected profit that could be obtained by choosing high price in a limit order. A seller with a high reservation price has a very small chance that the market bid price would be higher than his reservation price. Thus he gains more from quoting a high limit price than from the sure trading opportunity of a market order.

In Section 1, we present the assumptions, the trading rules used in the model, and the information structure among the agents involved in a trade. In Section 2, we define the equilibrium in which both market and limit orders can be chosen by traders as optimal strategies. And in Sections 3-5, we show how such an equilibrium is constructed. Section 3 describes the specialist's optimal pricing behavior. Section 4 contains trader's strategic behavior with limit and market orders. Section 5 deals with the special case of uniform distribution. In Section 6, we compare the efficiency of the specialist trading system analyzed in this paper to other trading systems analyzed in earlier literature in terms of ex-ante expected profit from trade. And finally, Section 7 contains a summary and conclusion.

II. TRADER'S OPTIMAL ORDER PLACEMENT STRATEGIES WITH LIMIT AND MARKET ORDERS.

1. The Model and Assumptions

There are three types of agents involved in a trade; sellers who own shares of stocks and intend to sell them whenever it is worthwhile to do so; buyers who intend to buy some shares whenever this is worthwhile; and a specialist who acts as a broker-dealer. Traders can place either a market order or a limit order. A market order will be executed at the market price which is determined (as we will see later) either by the specialist or by traders placing limit orders. When a trader places a limit order⁵, he specifies the transaction price-the limit price-to the specialist. We normalize the quantity traded to one unit so that the decisions which trader is facing are what type of order to place and what price to submit when placing a limit. In reality, every order specifies the quantity to be traded. Thus normalizing the quantity is restrictive, but this is necessary for the sake of tractability.⁶ This is not so restrictive as it seems, because the probability of transaction can be interpreted as trading volume times probability. In reality, the specialist's quote on NYSE are valid only for a specified number of shares. Once we normalize the trading volume, we can also normalize the number of traders

⁵In the organized stock exchanges, there are two types of limit order, those which specify a price limit and those which specify a time limit. The price-limit orders are more frequently used in reality, and by limit order we mean price-limit order.

⁶Normalizing a trading volume to one share is also employed in Copeland and Galai (1983) and Glosten and Milgrom (1985),

without much loss of generality. So we will proceed as if there are one buyer and one seller. We further assume the following.

(A. 1) The trade will be executed only through the specialist, and traders are not allowed to communicate with one another.

(A. 2) The specialist can short-sell without incurring any cost.⁷

(A. 3) Each agent, including the specialist,⁸ has his own reservation price⁹ on the security before the trade, but is uncertain about the other's reservation prices. This uncertainty is described by probability assessments on the others' reservation prices; these probability assessments are common knowledge.

(A. 4) The traders and the specialist are assumed to be risk-neutral. Thus, traders place orders to maximize their expected profit from the trade, and the specialist determines the bid-and ask-price to maximize the expected profit.

The sequence of the trade is as follows. First, both the buyer and the seller decide whether to place a market order or a limit order. If a trader decided to place a limit order, then he chooses a limit price and submits it to the specialist. If a trader decided to place a market order, then he waits to observe the market price. The specialist first waits for limit orders before he posts bid-and ask-price. There are four possible cases depending on whether or not the specialist receives a limit order from traders. If the specialist receives a limit order from both side; he matches orders as a broker; if prices are compatible, trade occurs; otherwise, no trade occurs. If the specialist receives a limit order from only side, he posts the limit price as a market price. If he receives market orders from the both sides, he determines both bid-and ask-prices (hence, he acts as a dealer). The trader who decided to place a market order gets a chance to observe the market price and decides whether to place an order or not.¹⁰ If one trader placed a limit order and the other trader decided not to place an order, then no trade occurs in this period. If one trader places a market order and the other trader does not place an order, then the specialist becomes the trading partner of the market order.

⁷This assumption is necessary for the specialist to act as a dealer (market maker) since his bid-and ask-prices for the market orders are his commitments to buy and sell for his account at these prices.

⁸The specialist's reservation price should be interpreted somewhat differently from the traders'. For the specialist, the stock is valued as an inventory for the future trading while it is valued as an investment for the traders.

⁹We do not elaborate what determines reservation prices. This is the price at which the trader is indifferent between making a trade and not making a trade. For the cases of asymmetric information among traders, especially when there is a more informed trader about the "true" underlying security's value than other traders, we simply assert that the information is already reflected in the reservation price.

¹⁰We can consider this trading sequence as if order types (and a limit price in a limit order) are submitted to the specialist before the specialist posts quotes and market order placing traders make trading decisions after they observe the market prices.

Three things need to be explained. First, we do not consider the inventory problem. Implicitly, we assume that the inventory will not accumulate in either direction in the long run. Secondly, when only one trader placed a limit order, the specialist will not make any profit. Hence, requiring the specialist to post the limit price as a market price constrains the specialist from acting as both a broker and a dealer in one trade. At a first glance, this may look too restrictive, but it is in accordance with the NYSE Rule 104.¹¹ Thirdly, traders have no chance to observe the market price before they place a limit order. In reality, traders may observe the market price before they place limit orders. However, the market price to be posted at the time when the trade occurs could be different from the market price they observed. What is relevant to the trading probability is the former, not the latter. In this sense, traders do not observe the market price before they place limit orders. However, we do admit the fact that trader may obtain some information by observing (previous) market prices. Incorporating this aspect will lead to a rather complicated multi-period problem. We want to avoid this complication by adopting a single period model.

The traders and the specialist have a certain reservation price on the security when they participate in the trade, as assumed in (A. 3). We denote the seller's and the buyer's reservation price by v_s and v_b respectively, and the specialist's by v . Each participant is uncertain, however, about the other's reservation price; each has only a probability assessment on the other's possible reservation price. These assessments are common knowledge. We denote the probability density functions of the seller's and the buyer's reservation price by f_s and f_b , respectively, and the specialist's by f_d . Corresponding distribution functions will be represented by F with appropriate subscripts. For simplicity, we assume that these are independent and distributed over the identical domain of $[0,1]$.

2. The Equilibrium

Traders's decision about the order type and the limit price (if he chooses the limit order) should be optimal with respect to the other trader's strategy and to the specialist's pricing behavior and depends on the trader's reservation price. Since trader's reservation price is unknown, the relevant equilibrium concept is

¹¹The specialist's function in the Exchange is defined as maintaining "a fair and orderly market", The Rule 104 established by the Board of Governors of the Exchange states (Teweles and Bradley (1982) p. 154):

The function of a member acting as a regular specialist on the floor of the Exchange, includes, in addition to the effective execution of commission orders entrusted to him, the maintenance, in so far as reasonably practicable, of a fair and orderly market on the Exchange in the stocks in which he is so acting.

If the specialist is allowed to compete with limit orders, the specialist is at a greatly advantageous position in a trading relative to the limit order-placing traders since he has a complete information about the limit orders while traders have no information about the specialist.

the Bayesian Nash equilibrium. In this Section, we define a Bayesian Nash equilibrium with additional restrictions on the trader's strategy. We characterize this equilibrium in section 2.1. In section 2.2, we explain why this is the most plausible choice of equilibrium.

2.1 Dichotomized Strategy.

The trader's decision is made in two stages; first, he decides whether to place a limit order or a market order, and then choose a limit price if he choose limit order. $b(\cdot)$ and $s(\cdot)$ denote the buyer's and seller's limit pricing schedule respectively. The specialist needs to make a non-trivial decision when there are no limit orders. In that case, he must choose both the bid-and ask-prices. In general, the bid-and ask-prices depend on the sepecialist's evaluation of the stock value. Let $p(v) = (A, B)$ represent the ask-and bid-price when the specialist's own reservation price is v .

Given others' strategies, his own reservation price, and the probability assessments about other's reservation prcie, each trader can calculate the expected payoff resulting from his own strategy. A Bayesian Nash equilibrium is a strategy profile such that each player's strategy is a best response to the others' in the sense that it maximizes the (conditional) expected payoff. In the following, we will restrict out attention to the "dichotomized strategies" according to which traders switch from a market order to a limit order (or from a limit order to a market order) only once in the whole domain of his reservation price. A formal definition is given below.¹² other types of strategies in which traders change the order types more than once fro some given distribution of reservation price might be possible, but those are not considered here on the ground of simplicity and reality.

Definition: A "dichotomized strategy" is a strategy such that there exists a critical value v_i^* for $i = b, s$ at which trader i is indifferent between a market order and a limit order, and switches from one order type to the other.

The following is a typical example: the seller chooses a market order when his reservation price is lower than v_s^* , $v_s \leq v_s^*$, and he chooses a limit order with a limit price of $s(\cdot)$ when his reservation price is higher than v_s^* , $v_s > v_s^*$; the buyer chooses a market order when his reservation price is higher than v_b^* , $v_b \geq v_b^*$, and he chooses a limit order with a limit price of $b(\cdot)$ when his reservation price is lower than v_b^* , $v_b < v_b^*$. Although, there are three other possible configuration, this one looks most natural to us, and hence we will always refer to this type when we mention a dichotomized strategy.¹³

¹²It is also possible to have a strategy in which there are more than one critical level of reseason price at which trader is indifferent between two order types and change from one order type to another. However, this kind of strategy is very unlikely.

¹³Equilibria with a reversed strategy could be also possible: a seller choose a market order with a high reservation price, $v_s^* \leq v_s$, and a limit order with a low reservation price, $v_s < v_s^*$; a buyer choose

Now, we characterize an equilibrium. Let $EII_b(b, v_b)$ be the buyer's expected profit when his reservation price is v_b and he places a limit order with a limit price of b . Let $EII_b^m(v_b)$ be the buyer's expected profit when his reservation price is v_b and he places a market order. Similarly define $EII_s(s, v_s)$ and $EII_s^m(v_s)$ for the seller.

Then the equilibrium requires followings.

- (1) $\max_b EII_b(b, v_b) \geq EII_b^m(v_b)$ for all $v_b \leq v_b^*$
- (2) $EII_b^m(v_b) \geq \max_b EII_b(b, v_b)$ for all $v_b \geq v_b^*$
- (3) $\max_s EII_s(s, v_s) \geq EII_s^m(v_s)$ for all $v_s \geq v_s^*$
- (4) $EII_s^m(v_s) \geq \max_s EII_s(s, v_s)$ for all $v_s \leq v_s^*$

Let b and s be functions from $[0, 1]$ to $[0, 1]$ which satisfy

- (5) $b(v_b) \in \operatorname{argmax}_b EII_b(b, v_b)$ for all $v_b \in [0, 1]$
- (6) $s(v_s) \in \operatorname{argmax}_s EII_s(s, v_s)$ for all $v_s \in [0, 1]$

Then, the following proposition is a well known result.

Proposition 1: b and s are non-decreasing.

The proposition can be proved by the same technique used in Theorem 1 of Chatterjee and Samuelson (1983), hence we skip the proof.

If we define $II_b(v_b; b) = EII_b[b(v_b), v_b] - EII_b^m(v_b)$ and $II_s(v_s; s) = EII_s[s(v_s), v_s] - EII_s^m(v_s)$, then inequalities (1) to (4) can be rewritten as

- (1') $II_b(v_b; b) \geq 0$ for all $v_b \leq v_b^*$
- (2') $II_b(v_b; b) \leq 0$ for all $v_b \geq v_b^*$
- (3') $II_s(v_s; s) \geq 0$ for all $v_s \geq v_s^*$
- (4') $II_s(v_s; s) \leq 0$ for all $v_s \leq v_s^*$

In an equilibrium, when the specialist receives both market buy and sell orders, he can infer that $v_b > v_b^*$ and $v_s < v_s^*$. Accordingly he changes his probability assessments about traders' type. His choice of bid-and ask-price should be optimal in the sense that it maximizes his expected profit with respect to the posteriors. Let us denote the expected profit of the specialist by $EII_d(A, B; v)$. Then, his strategy P must satisfy the following.

a limit order with a high reservation price, $v_b^* \leq v_b$, and a market order with a low reservation price, $v_b < v_b^*$. This is, however, very unlikely for the following reason. The benefit of a market order is that it could be executed either at specialist's price or at the other trader's limit price. However, if a seller with a high reservation price place a market order, it has no chance of getting executed at the specialist's bid price which is lower than the ask price, and it also has very small chance of getting executed at the buyer's limit price since buyer with high reservation price will place a limit order and its price, $b(v_b)$ will be lower than his true reservation price v_b .

(7) $p(v) \in \operatorname{argmax}_{A,B} E\Pi_d(A,B; v)$ for all $v \in [0, 1]$.

The optimization problem will be analyzed in detail in the next two Sections. Specialist's problem will be considered in Section 3 traders' problem in Section 4.

2.2 Economic Intuition

The intuition behind the equilibrium is as follows. A market order is equivalent to making a decision as to whether or not he will participate in a trade after the market price is observed. The seller(buyer) agrees to trade at the market price only when the market price is higher(lower) than his reservation price. In other words, the probability of a market order being executed is unity if a trader decides to participate in a trade. However, a limit order will be executed only at its limit price and the probability of trade is not unity. When a seller has a relatively low reservation price, the gains from trade is already high, and hence the value of trading probability is high. This provides incentive for market order. In order for a limit to be a superior to a market order, the quoted limit price must be higher than the average buyer's quote. However, high price further reduces the possibility of transaction, which is very costly when gains from transaction is high. Hence, a seller with low reservation price would prefer market order. As seller's reservation price increases, the gains from trade decreases and the probability of trading becomes less important. At a certain level of reservation price, v_s^* , this tradeoff between the trading probability and the gains from high quoted price will balance. At a very high level of reservation price, the only possible trade occurs when the buyer places market order. Seller has no chance of trading via a market order since the specialist's bid price is lower than the ask price. Hence, limit order is better in this case. For similar reasons, the buyer will choose a market order when his reservation price is high, and will choose a limit order when his reservation price is low¹⁴.

In the rest of this paper, we will analyze the equilibrium in more detail. We will derive the specialist's optimal bid-and ask-price for market orders in Section 3, and the traders' optimal strategies in Section 4. In section 5, we examine the special case of uniform distribution.

¹⁴The reverse case is also possible: a seller with a high reservation price places a market order, but places a limit order with a low reservation price; a buyer with a low reservation price place a market order, but places a limit order with a high reservation price. It is however, very unlikely for the following reason. First, since the specialist's bid price is lower than the ask price, a seller's market order with a high reservation price is very unlikely to be executed against the specialist's bid price; and since the buyer places a limit order at the high level, the buyer's limit price will be lower than his reservation price and in turn the seller's market order with high reservation price is unlikely to be executed against the buyer's limit price. When a seller places a limit order with a low reservation price, he has no reason to call a limit price lower than the specialist's bid price.

3. Specialist's Pricing Behavior

In this Section, we derive the specialist's optimal bid-and ask-price for market orders. The specialist posts bid-and ask-prices when neither buyer nor seller placed limit orders. The security will be transferred from a seller to a buyer through the specialist when both buy-and sell-orders are executed at the bid-and ask-price, and the specialist does not take any position. The specialist should, however, take a position when one of the buy-and sell-orders is not executed. Since the bid-and ask-price are his commitment to traders such that he is willing to buy or sell at these prices, the specialist will have an inventory of the security when only the sell trade is successful, and he should short sell the security when only the buy trade is successful. In cases that the specialist takes a position, the specialist should have his own assessment on the underlying future value of the security. This assessment is his reservation price in the trade which will be denoted as v .

The specialist's expected profit from the successful trade comes from two sources. First, the difference between the buy-and sell-limit price. When both traders place limit orders, the trade is successful only if the buy limit price is not lower than the sell limit price ($b \geq s$), and the specialist will execute the trade at each order's limit price and will realize the difference in limit prices as a profit. There is no profit available for the specialist when one of the buy and sell orders is a limit order since the trade will be made at the limit price. Whenever a limit order is executed, the specialist does not determine the price and acts only as a broker.

The second source of profit for the specialist is the bid-ask spread. This occurs when the specialist acts as a dealer for market orders. The specialist determines bid-and ask-prices, B and A , to maximize the expected profit from the trade, which is given as follows.¹⁵

$$\begin{aligned}
 (8) \quad E\Pi_d(A, B; v) = & (A-B) \int_A^1 \int_0^B g_s(v_s) g_b(v_b) dv_s dv_b \\
 & + (A-v) \int_A^1 \int_B^{v_s^*} g_s(v_s) g_b(v_b) dv_s dv_b \\
 & + (v-B) \int_{v_b^*}^A \int_0^B g_s(v_s) g_b(v_b) dv_s dv_b
 \end{aligned}$$

The g_s and g_b represent posterior probability density functions; i.e., $g_s = f_s/F_s(v_s^*)$ on $[0, v_s^*)$ and $g_b = f_b/[1-F_b(v_b^*)]$ on $(v_b^*, 1]$. The first part of the right

¹⁵The expected profit given by equation (11) is case when both buy and sell orders are market orders. The expected profit from limit orders is given below. The specialist, however, does not make decision.

$$E\Pi_d = \int_{b^{-1}(s)}^{v_b^*} \int_{v_s^*}^{s^{-1}(b)} (b(v_b) - s(v_s)) f_s(v_s) f_b(v_b) dv_s dv_b$$

hand side of equation (8) is the expected profit obtained when both the buy-and sell-trade are successful, and the second part is the expected profit obtained when only the buy trade is successful. The third part is the expected profit obtained when only the sell trade is successful. The first order conditions for the above maximization problem with respect to A and B are given below.

$$(9) \quad [1-G_b(A)] - (A-v) g_b(A) = 0$$

$$(10) \quad (v-B) g_s(B) - G_s(B) = 0$$

where G_b and G_s are distribution functions for g_b and g_s . (9) shows that, by increasing A, the specialist can extract surplus from buyers who are willing to trade. The first term in (9), the posterior probability that the buyer will accept the offer, represents this gain. However, there is a counterbalancing loss. As the ask price increases, less buyers will accept the offer. The reduction in buyer's acceptance is represented by $g_b(A)$. In both cases, the loss is the unrealized gain, $A - v$. Thus, the second term in (9) represents the expected loss due to reduction in the probability of trade. The condition requires these counterbalancing effects to be equal at the margin. (10) has a similar interpretation. We suppose that these first order conditions have a unique solution. This is true if, say, f_b is not too concave for (9). (More precisely, $-F_b'' < 2F_b'$ is sufficient.) However, there might not be an interior solution to the first order conditions, in which case there will be a corner solution. Let us denote the (unique) solution to the optimization problem by $A(v)$ and $B(v)$.

4. Traders' Strategic Behavior

In this Section, we consider first how traders choose a limit price if they were to place a limit order. The decision as to whether to place a limit or market order will be made after traders compare the maximum expected profit obtainable from limit order with the expected profit from the market order. This decision problem will not be considered in this Section. Since the argument is symmetric, we will primarily focus on seller's pricing behavior.

4.1 Seller's Choice of Limit Price.

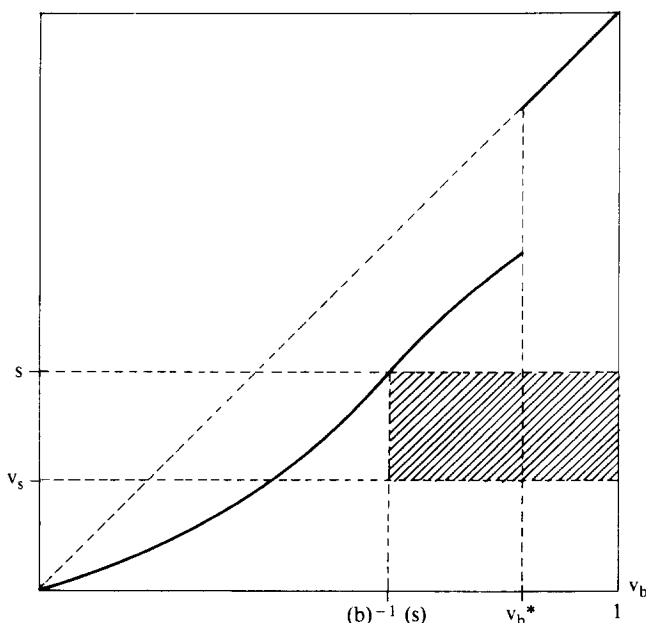
Consider a seller who decided to place a limit order. The price that maximize his expected gain depends on his reservation price and on buyer's strategy. As explained earlier, the specialist has no strategic role when a limit order is placed. Another useful observation is that buyer's market order is equivalent to the truth-revealing limit order from limit seller's point of view. Thus, we can treat the buyer's market order as a limit order with bid function $b(v_b) = v_b$. This simplifies the analysis, because the problem is now reduced to a 2-person bargaining game with a well-specified trading rule. Following Chatterjee and Samuelson (1983), we will

focus on differentiable pricing strategies.¹⁶ However, certain “irregularities” such as discontinuity and nondifferentiability are unavoidable due to the change in order types in trading partner’s strategy.

Figure 1 depicts a buyer’s strategy viewed by a seller placing a limit order. Buyer’s strategy is represented by the limit price (or the equivalent when a market order is chosen by buyer), denoted by $b(v_b)$, as a function of buyer’s reservation price, v_b . On the interval of $[v_b^*, 1]$, buyer chooses the market order, which is equivalent to quoting the reservation price, hence the strategy is represented by the 45 degree line. The way a seller chooses an optimal limit price given the buyer’s strategy is analogous to the way a monopolist chooses an optimal price, given a market demand curve. If a seller with reservation price v_s chooses a limit price s , then the expected gain is

$$(s - v_s) \text{ Prob } [v_b \geq b^{-1}(s)]$$

The probability that trade will occur is equal to the probability that $v_b \geq b^{-1}(s)$, because the buyer will quote a price higher than s according to proposition 1. There is a tradeoff between the size of the gain, $(s - v_s)$, and the probability of the gain being realized, $\text{Prob } [v_b \geq b^{-1}(s)]$. The higher the seller’s limit price



[Figure 1] Seller's Expected Profit

¹⁶The discontinuous pricing strategies are also possible. See Leiniger, Linhart, and Radner (1986),

s, the larger the potential gain but the smaller the probability of that gain being realized. Notice that when there is discontinuity in the buyer's strategy (as at v_b^* in Figure 1) it is never optimal for the seller to quote a price lower than the upper limit of the buyer's bid price (which is v_b^* when $v_b = v_b^*$). This is because asking higher price within the bound does not reduce the probability of trading. If we denote the upper limit and lower limit at the discontinuity point by b^* and b_* respectively, the seller's optimal limit price will be either (weakly) higher than b^* or (weakly) lower than b_* . Hence, we expect to have a discontinuity in seller's strategy as well. With this property in mind, let us now consider seller's optimal strategy.

First, suppose the seller's reservation price is sufficiently high, say close to 1. Only buyers with high reservation prices would be relevant in this case, since trade is possible only when the buyer's reservation price is higher than the seller's ($v_b > v_s$). Since buyers with high reservation prices, say $v_b > v_b^*$, choose market order, the seller's expected profit can be written as follows:

$$\begin{aligned} E\Pi_s(s; v_s) &= (s - v_s) \Pr \{v_b \geq s\} \\ &= (s - v_s) [1 - F_b(s)] \end{aligned}$$

The first order condition is:

$$(11) \quad [1 - F_b(s)] - (s - v_s) f_b(s) = 0$$

It has the same interpretation as (9) or (10). The seller should balance the gains from the higher price with the losses from the lower trading probability. We suppose that (11) has a unique solution and denote it by $F_b^m(v_s)$. $F_b^m(\cdot)$ is the seller's limit pricing strategy against the buyer's market order. We also assume that F_b^m is increasing and differentiable. This is true if F_b is not too concave ($-F_b'' < 2F_b'$) and twice differentiable. $F_b^m(\cdot)$ is the optimal pricing strategy as long as seller expects that the potential buyer places a market order, i.e., as long as $v_s \geq (F_b^m)^{-1}(v_b^*)$. For v_s lower than $(F_b^m)^{-1}(v_b^*)$, $F_b^m(v_s)$ will be lower than v_b^* . However, the optimal price is still v_b^* for v_s slightly lower than $(F_b^m)^{-1}(v_b^*)$ for the following reason. A buyer who has a reservation value of v_b^* is indifferent between placing a market order and a limit order. The optimal limit price for this buyer must be lower than his reservation price, because otherwise he can get at most zero profit, whereas a market order will generate positive expected profit. The seller's optimal price should remain constant for a certain range of v_s below $(F_b^m)^{-1}(v_b^*)$, since further decrease in price does not increase trading probability. Let v_s^2 denote the lowest value of v_s for which the optimal price is v_b^* .

As v_s falls below v_s^2 , the optimal limit price falls below v_b^* , and there must be a jump at v_s^2 , because the trading probability does not increase otherwise.¹⁷ Hence,

¹⁷The size of the jump in s should be greater than the size of jump in the buyer's strategy, $v_b^* - b(v_b^*)$, in order to induce an additional trading probability.

at v_s^2 , there are two prices which give rise to the same expected profit; one with high s ($s = v_b^*$) and low trading probability, the other with low s ($s < v_b^*$) and high trading probability. Formally, $EII_s(s = v_b^*; v_s^2) = EII_s(s = s^2; v_s^2)$ where s^2 denotes the lower limit price at v_s^2 . Then the following equation must hold:

$$(v_b^* - v_s^2) \Pr \{v_b \geq v_b^*\} = (s^2 - v_s^2) [\Pr \{(v_b \geq v_b^*)\} + \Pr \{b(v_b) \geq s^2\}]$$

or equivalently,

$$(12) (v_b^* - s^2) [1 - F_b(v_b^*)] = (s^2 - v_s^2) \Pr \{b(v_b) \geq s^2\}$$

Since $v_b^* > s^2$, it must be true that $\Pr \{b(v_b) \geq s^2\} > 0$. Notice, however, that limit buyers will never choose a limit price higher than s^2 , because s^2 is the highest price below v_b^* that a limit seller will ever ask for in equilibrium. Hence, we have $\Pr \{b(v_b) = s^2\} > 0$, i.e., the buyer's pricing function must be constant at the top portion of $[0, v_b^*]$.

Proposition 2: As the seller's reservation price falls, his limit price stays constant at the point where the buyer switches from market to limit order, and then falls to the buyer's highest limit price. This requires the buyer's limit price to be constant in the top part of his limit order region.

Generally speaking, where there is a discontinuity in the opponent's strategy, there is a flat part in the optimal response (constant response) and consequently a discontinuity. Hence, we expect to have at least two points of discontinuity in each trader's strategy: one due to the buyer's critical value v_b^* at which he changes from a limit order to a market, and the other due to the seller's critical value v_s^* . These price discontinuities are the analogy of "gravitational pull" effect of market order analyzed by CMSW (1981), "Intuitively stated, as a trader contemplates placing a bid closer and closer to an ask already established on the market, he is increasingly attracted by this counterpart offer; at some point, the "gravitational pull" exerted by the established ask will dominate. The trader will "jump" his price and execute with certainty via a market order" (quoted from CMSW (1981) p.289). However, in our model, one jump is due to the switching a trader's own order type and the other is in response to the other trader's switch in order type. The following proposition summarizes these price jumps.

Proposition 3: There are two price jumps in trader's strategy: At v_s^* where a seller changes his choice of order type from a market order to a limit order; at v_s^2 where is a shift in trading probability due to a change in buyer's order type from a market order to a limit order.

For $v_s < v_s^2$, the seller's expected profit is;

$$EII_s(s; v_s) = (s - v_s) \{1 - F_b[b^{-1}(s)]\}.$$

The first order condition for maximization is;

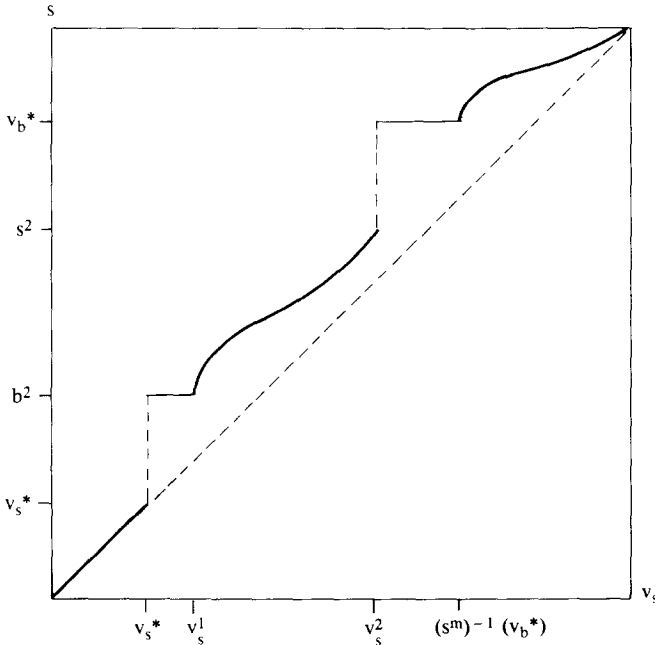
$$(13) [1-F_b(x)]-(s-v_s) f_b(x)/b'(x)=0$$

where $x = b^{-1}(s)$ and b' is the derivative of b . (13) has the same interpretation as (11). The appearance of $b(\cdot)$ reflects the fact that trading probability now depends on the potential buyer's strategy as well. We suppose that (13) has a unique solution and denote it by $s^\ell(v_s)$, $s^\ell(\cdot)$ is the seller's limit pricing strategy against the buyer's limit order. We also assume that s^ℓ is (strictly) increasing in v_s . As v_s decreases further, $s^\ell(v_s)$ reaches the point of discontinuity in the buyer's limit price schedule. Then, for the same reason that the buyer's limit price function is constant at the top portion of $[0, v_b^*]$, the seller's optimal limit price will be constant at the bottom portion of $[v_s^*, 1]$. More precisely, let $b^2 = \inf \{b(v_b): b(v_b) > v_s^*\}$ and $v_s^1 = (s^\ell)^{-1}(b^2)$, then the optimal price will be b^2 for v_s below v_s^1 .

The seller's optimal pricing strategy with a limit order is summarized in the following proposition. (cf. Figure 2.)

Proposition 4: As the seller's reservation price, v_s , decreases from 1, his optimal limit price are:

$$(14) \begin{aligned} s &= s^m(v_s), & (s^m)^{-1}(v_b^*) \leq v_s \leq 1 \\ v_b^*, & & v_s^2 \leq v_s \leq (s^m)^{-1}(v_b^*) \\ s^\ell(v_s), & & v_s^1 \leq v_s \leq v_s^2 \\ b^2, & & v_s^* \leq v_s \leq v_s^1 \\ & & \text{where } v_s^1 = (s^\ell)^{-1}(b^2) \end{aligned}$$



[Figure 2] Seller's Optimal Strategy

4.2 Buyer's Strategic Behavior

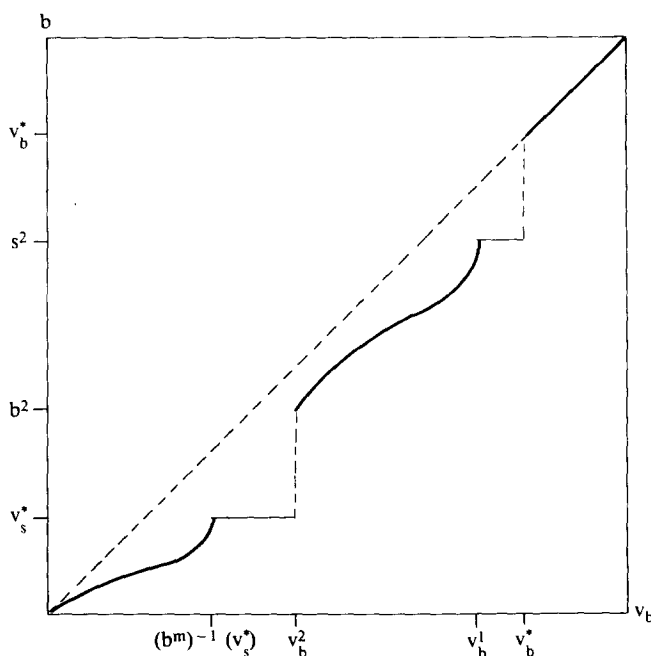
In this Section, we analyze the buyer's strategic choice of limit price. We will be very brief since the arguments are virtually symmetric to the ones for the seller. The Figure 3 would supplement any deficiency in explanation. We start with the behavior of buyers with low reservation prices whose potential trading partners (sellers) have low reservation value and place market orders. Hence, the buyer's expected profit can be written as;

$$E\Pi_b(b; v_b) = (v_b - b) F_s(b)$$

and its first order condition is;

$$(15) (v_b - b) f_s(b) - F_s(b) = 0$$

We suppose that (4-5) has a unique solution and denote it by $b^m(v_b)$, $b^m(\cdot)$ is the buyer's limit pricing strategy against the seller's market order. b^m is assumed to be differentiable and strictly increasing. For $v_b \leq (b^m)^{-1}(v_s^*)$, $b^m(v_b)$ will be an optimal strategy. The buyer's optimal limit price will stay constant for a range of v_b above $(b^m)^{-1}(v_s^*)$ since there is a jump in the seller's strategy at v_s^* . Let v_b^2 be the highest value of v_b for which the optimal limit price is v_s^* . As v_b rises



[Figure 3] Buyer's Optimal Strategy

above v_b^2 , the optimal limit price rises above v_s^* , and there is a jump at v_b^2 . Hence, at v_b^2 there are two prices which give rise to the same expected profit; one with low b and high trading probability and one with high b and low trading probability. Denote the higher limit price by b^2 and note that this is the same b^2 defined at the end of last subsection. Since both prices generate the same expected profit, i.e. $E\Pi_b(b = v_s^*; v_b^2) = E\Pi_b(b = b^2; v_b^2)$, we have the following equation.

$$(v_b^2 - v_s^*) F_s(v_s^*) = (v_b^2 - b^2) [F_s(v_s^*) + \Pr\{s(v_s) = b^2\}]$$

or equivalently,

$$(16) (b^2 - v_s^*) F_s(v_s^*) = (v_b^2 - b^2) \Pr\{s(v_s) = b^2\}$$

For $v_b > v_b^2$, the buyer's expected profit is;

$$E\Pi_b(b; v_b) = (v_b - b) F_s[s^{-1}(b)].$$

and its first order condition is;

$$(17) (v_b - b) f_s(y) / s'(y) - F_s(y) = 0$$

where $y = s^{-1}(b)$ and s' is the derivative of s . We suppose that (17) has a unique solution and denote it by $b^\ell(v_b)$. $b^\ell(\cdot)$ is the buyer's limit pricing strategy against the seller's limit order. b^ℓ is assumed to be differentiable and strictly increasing. $b^\ell(v_b)$ will be the optimal limit price until its value reaches s^2 , where it remains constant. The buyer's optimal limit pricing strategy is summarized in the following proposition.

Proposition 5: As the buyer's reservation price, v_b , increases from 1, his optimal limit prices are:

$$(18) \begin{aligned} b &= b^m(v_b), \quad 0 \leq v_b \leq (b^m)^{-1}(v_s^*) \\ v_s^*, \quad &(b^m)^{-1}(v_s^*) \leq v_b \leq v_b^2 \\ b^\ell(v_b), \quad &v_b^2 \leq v_b \leq v_b^1 \\ s^2, \quad &v_b^1 \leq v_b \leq v_b^* \\ &\text{where } v_b^1 = (b^\ell)^{-1}(s^2). \end{aligned}$$

4.3 Traders' Equilibrium Strategy Profile

In this Section, we describe how traders' equilibrium strategies are obtained for general case. A special case of uniform distribution for reservation prices is considered in the next Section. Limit pricing strategy is given by (14) and (18), s^m and b^m can be separately calculated from (11) and (15) which are given below

again. s^ℓ and b^ℓ are obtained simultaneously as a solution to differential equation system (19) and (20) given below, which are obtained from (13) and (17) after substituting $v_b = b^{-1}(b)$ $v_s = s^{-1}(s)$, $s = b(x)$, and $b = s(y)$.

$$(11) [1-F_b(s)] - (s-v_s) f_b(s) = 0$$

$$(15) (v_b-b) f_s(b) - F_s(b) = 0$$

$$(19) [1-F_b(x)] b'(x) - [b(x)-s^{-1}(b(x))] f_b(x) = 0$$

$$(20) [b^{-1}(s(y)) - s(y)] f_s(y) - F_s(y) s'(y) = 0$$

To completely describe traders' strategy profile, we still need to specify what the values of v_b^* , v_s^* , v_b^1 , v_s^1 , v_b^2 , v_s^2 , b^2 and s^2 are. We can obtain these after solving following 8 equations simultaneously.

$$(21) (v_b^*-v_s^2) [1-F_b(v_b^*)] = (s^2-v_s^2) [1-F_b[(b^\ell)^{-1}(s^2)]]$$

$$(22) (v_b^2-v_s^*) F_s(v_s^*) = (v_b^2-b^2) F_s[(s^\ell)^{-1}(b^2)]$$

$$(23) (b^2-v_s^*) [1-F_b(v_b^2)] \\ = \int_{v_b^2}^{v_b^1} [b^\ell(v_b)-v_s^*] dv_b + \int_{v_b^1}^{v_b^*} (s^2-v_s^*) dv_b$$

$$(24) (v_b^*-s^2) F_s(v_s^2) \\ = \int_{v_s^*}^{v_s^1} (v_b^*-b^2) dv_s + \int_{v_s^1}^{v_s^2} [v_b^*-s^\ell(v_s)] dv_s$$

$$(25) v_s^1 = (s^\ell)^{-1}(b^2)$$

$$(26) v_b^1 = (b^\ell)^{-1}(s^2)$$

$$(27) s^2 = s^\ell(v_s^2)$$

$$(28) b^2 = b^\ell(v_b^2)$$

Equations (21) and (22) are from (12) and (16) which equates the same expected profit from two different strategies at v_b^2 and v_b^2 respectively (cf. Figure 2 and 3) Notice that

$$\begin{aligned} \Pr \{b(v_b) \geq s^2\} &= \Pr \{b(v_b) = s^2\} \\ &= F_b(v_b^*) - F_b(v_b^1) && \text{from (18)} \\ &= F_b(v_b^*) - F_b[(b^\ell)^{-1}(s^2)] && \text{by definition of } v_b^1 \end{aligned}$$

$$\begin{aligned} \Pr \{s(v_s) = b^2\} &= F_s(v_s^1) - F_s(v_s^*) && \text{from (14)} \\ &= F_s[(s^\ell)^{-1}(b^2)] - F_s(v_s^*) && \text{by definition of } v_s^1 \end{aligned}$$

The other two equations, (23) and (24), are derived from the expressions which equate the same expected profit obtained from a market order to the maximum expected profit obtained from a limit order for v_s^* and v_b^* respectively. (25), (26), (27) and (28) describe the relationships among v_s^1 , v_b^1 , v_s^2 , v_b^2 , b^2 and s^2 (cf. Figure 2 and 3).

Proposition 6: The equilibria are obtained from limit order strategies described in proposition 4, 5 and points, v_b^* , v_s^* , v_b^1 , v_s^1 , v_b^2 , v_s^2 , b^2 , and s^2 , which solve equations from (21) to (28),

In the next Section, we provide a specific example in order to give some insight on the above description of the equilibrium strategy profile with the special case where reservation values are uniformly distributed.

5. An Example: Unifrom Distribution Case.

In this section, we will derive an equilibrium strategy profile for the unifrom distribution case. Even for this simple case the equation system involves solving a fourth order polynomial equation, and we have to use numerical method. However, except fro the last step, the expressions are so simple that we can easily see how traders' strategic behaviors are interrelated.

Specialist's Strategy

The specialist's objective function (3-1) reduces to

$$(29) \quad E\Pi_d(A, B; v) = (A-v) (1-A)v_s^* + (v-B) (1-v_b^*)B$$

and the optimal strategy is

$$(30) \quad A(v) = \max \{ (v+1)/2, v_b^* \}$$

$$(31) \quad B(v) = \min \{ v/2, v_s^* \}$$

Seller's Strategy with High Reservation Price.

The seller's objective function reduces to

$$(32) \quad E\Pi_s(s, v_s) = (s-v_s) (1-s)$$

and the optimal limit pricing strategy against a buyer's market order is

$$(33) \quad s^m(v_s) = (v_s + 1)/2$$

Buyer's Strategy with Low Reservation Price

The buyer's objective function reduces to

$$(34) \quad E\Pi_b(b, v_b) = (v_b - b)b$$

and the optimal limit pricing strategy against a seller's market order is

$$(35) \quad b^m(v_b) = v_b/2$$

Seller's and Buyer's Strategy with Intermediate Values

The differential equation system (19) and (20) reduces to

$$(36) (1-x)b'(x) - [b(x) - s^{-1}(b(x))] = 0$$

$$(37) [b^{-1}(s(y)) - s(y)] - ys'(y) = 0$$

Assuming that s and b are linear, we get the seller's and buyer's optimal limit pricing strategy against each other's limit order as

$$(38) s^l(v_s) = v_s/2 + 1/3$$

$$(39) b^l(v_b) = v_b/2 + 1/6$$

Other Variables

We need to specify s^2 , and b^2 . (21) and (22) reduce to

$$(40) (v_b^* - s^2)(1 - v_b^*) = (s^2 - v_s^2)(v_b^* - v_b^l)$$

$$(41) (b^2 - v_s^*)v_s^* = (v_b^2 - b^2)(v_s^l - v_s^*)$$

Two more equations can be obtained from the definition of v_s^* and v_b^* . Seller's expected profit from limit order at v_s^* is

$$(42) E\Pi_2(b^2, v_s^*) = (b^2 - v_s^*)(1 - v_b^2)$$

His expected profit from market order at v_s^* is

$$\begin{aligned} (43) E\Pi_s^m(v_s^*) &= E\{b(v_b) \geq v_s^*\} [b(v_b) - v_s^*] \\ &= \int_{v_b^2}^{v_b^l} [b^l(v_b) - v_s^*] dv_b + \int_{v_b^l}^{v_b^*} (s^2 - v_s^*) dv_b \\ &= \int_{v_b^2}^{v_b^l} (v_b/2 + 1/6 - v_s^*) dv_b + (s^2 - v_s^*)(v_b^* - v_b^l) \end{aligned}$$

Notice that the seller can expect positive profit only when buyer places limit order, because $B(v) \leq v_s^*$ for all v . By equating these two expressions, we get (44) from (23),

$$(44) (s^2 - b^2)(2v_b^* - v_b^l - v_b^2) = 2(b^2 - v_s^*)(1 - v_b^2)$$

Similarly by equating the buyer's expected profit from a limit order with that obtained from market order v_b^* , we get (45) from (24),

$$(45) (s^2 - b^2)(v_s^2 + v_s^l - 2v_s^*) = 2(v_b^* - s^2)v_s^*$$

From the definitions of v_s^1 , v_b^1 , s^2 and b^2 , and equations (25), (26), (27) and (28), we have;

$$(46) \quad b^2 = v_s^1/2 + 1/3$$

$$(47) \quad s^2 = v_b^1/2 + 1/6$$

$$(48) \quad s^2 = v_s^2/2 + 1/3$$

$$(49) \quad b^2 = v_b^2/2 + 1/6$$

Equations, (40), (41) and equations (45) through (50) can be solved simultaneously for v_s^* , v_s^2 , v_s^1 , v_b^* , v_b^2 , v_b^1 , s^2 and b^2 . The unique solution which satisfies all the required conditions described is.¹⁸

$$v_s^* = 0.05883, \quad v_s^2 = 0.41873, \quad v_s^1 = 0.24793, \quad s^2 = 0.54270$$

$$v_b^* = 0.94117, \quad v_b^2 = 0.58127, \quad v_b^1 = 0.75207, \quad b^2 = 0.45730$$

One can verify that the equilibrium conditions (3-1') through (3-4') are satisfied. Consequently, an equilibrium exists in which traders' optimal strategies are as given below. The seller's optimal strategy is: a market order for $0 \leq v_s < v_s^*$, and a limit order for $v_s^* \leq v_s \leq 1$ with the following limit price s .¹⁹

$$(50) \quad s = b^2, \text{ for } v_s^* \leq v_s < v_s^1 \\
v_s/2 + 1/3, \text{ for } v_s^1 \leq v_s < v_s^2 \\
v_b^*, \text{ for } v_s^2 \leq v_s < 2v_b^* - 1 \\
v_s/2 + 1/2, \text{ for } 2v_b^* - 1 \leq v_s \leq 1$$

The buyer's optimal strategy is: a limit order for $0 \leq v_b \leq v_b^*$ with the following limit price b , and a market order for $v_b^* \leq v_b \leq 1$.²⁰

¹⁸These numbers are rounded ones from the exact solutions which are irrational numbers.

¹⁹For each seller's strategy, the expected profits are as follows.

for $0 \leq v_s < v_s^*$,

$$\pi_2(s(v_s)) = (v_s^* - v_s)2 + (v_s^* - v_s)(v_b^1 - 2v_s^*) \\
+ (1/4)(v_b^1)2 - (1/4)(v_b^2)2 + (1/6 - v_s)(v_b^1 - v_b^2) + (s^2 - v_s)(v_b^* - v_b^1) \\
+ (1 - v_b^*)(v_s^* - v_s)(1 - v_s^* - v_s)$$

for $v_s^* \leq v_s < v_s^1$, $\Pi_s(s(v_s)) = (b^2 - v_s)(1 - v_b^2)$

for $v_s^1 \leq v_s < v_s^2$, $\Pi_s(s(v_s)) = 2(1/3 - v_s/2)^2$

for $v_s^2 \leq v_s < 2v_b^* - 1$, $\Pi_s(s(v_s)) = (v_b^* - v_s)(1 - v_b^*)$

for $2v_b^* - 1 \leq v_s \leq 1$, $\Pi_s(s(v_s)) = (1/4)(1 - v_s)^2$

²⁰For each buyer's strategy, the expected profits are as follows.

for $0 \leq v_b \leq 2v_s^*$, $\Pi_b(v_b) = (1/4)v_b^2$

for $2v_s^* \leq v_b < v_b^2$, $\Pi_b(b(v_b)) = (v_b - v_s^*)v_s^*$

for $v_b^2 \leq v_b < v_b^1$, $\Pi_b(b(v_b)) = 2(v_b/2 - 1/3)^2$

for $v_b^1 \leq v_b < v_b^*$, $\Pi_b(b(v_b)) = (v_b - s^2)v_s^2$

for $0 \leq v_b \leq v_b^*$, $\Pi_b(b(v_b)) = (v_b - b^2)(v_s^1 - v_s^*) + (v_b - 1/3)(v_s^2 - v_s^1) \\
- (1/4)(v_s^2)^2 + (1/4)(v_s^1)^2 + 2(v_b - 1/2)(v_b - v_b^*) \\
+ (v_b - v_b^*)(2v_b^* - 1 - v_s^2) - (1/4)(2v_b - 1)^2 \\
+ (1/4)(2v_b^* - 1)^2 + v_s^*(v_b - v_b^*)(v_b + v_b^* - 1)$

(51)

$b = v_b/2,$

$v_s^*,$

$v_b/2 + 1/6,$

$s^2,$

for $0 \leq v_b < 2v_s^*$

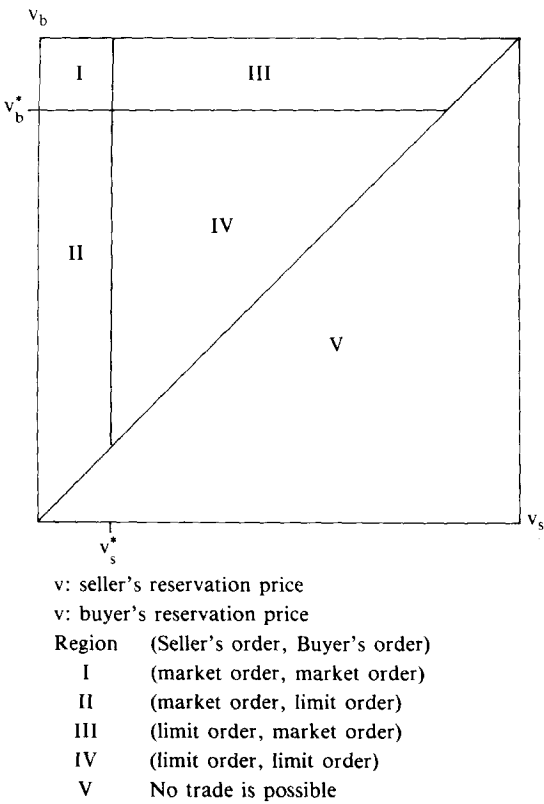
for $2v_s^* \leq v_b < v_b^2$

for $v_b^2 \leq v_b \leq v_b^1$

for $v_b^1 \leq v_b \leq v_b^*$

When a seller has a low reservation price, a market order is optimal, and when he has a high reservation price, a limit order is optimal. For a buyer, a market order is optimal when he has a high reservation price, and a limit order is optimal when he has a low reservation price. Even if a trader has no control over the transaction price with a market order, a market order constitutes an optimal strategy, since it provides higher probability of successful trade.

Given the uniform distribution of trader’s reservation price, the probability for each type of order is illustrated in Figure 4: in region I, both buyer and seller place market orders; in region II, a seller places a market order and a buyer places a limit order; in region III, a seller places a limit order and a buyer places a market order; in region IV, both place limit orders; in region V, no trade is possible. Among



[Figure 4] Market and Limit Order Placing Regions

the regions where a trade is possible (region I, II, III and IV), the probability of a market order being involved in at least of side of trade (region I, II, and III) is only about 21.7%. According to Stoll's (1985) estimate, the proportion of market order out of total trading volume is about 24%. By no means, our market order probability of 21.7% is intended to justify Stoll's estimate of 24%. However, our analysis certainly shows limit orders are dominant part of possible orders under certain condition.

6. Relative Efficiency of the Specialist System

In this Section, we compare the efficiency of the trading system with a specialist analyzed above to the one analyzed in Jang (1987), as well as to the case of the direct bargaining analyzed by Chatterjee and Samuelson (1983). Jang (1987) has shown that the limit order is the dominant strategy for both the buyer and the seller in a two stage trading model; the traders choose order types in the first stage and report their offer to the specialist knowing the other's other type.²¹ In other words, a trading system in which the traders place only limit orders, and the specialist acts only as a broker yields more ex-ante expected profit to the traders and to the society than dealer market systems.²² We will refer this system as the broker market system.

In this paper, we have analyzed a trading system in which traders place either a limit order or a market order before the other trader's order type is known, and the specialist acts as a broker as well as a dealer. We refer this trading system as the broker-dealer market system. For this broker-dealer market system, we can derive the ex-ante expected profit²³ by integrating the interim expected profit²⁴ over the corresponding reservation price ranges. The total ex-ante expected profit is .115752, which is the sum of three participant's profit; the seller gets .053227; the buyer gets .053227; and the specialist gets .009297.²⁵ The total ex-ante expected profit in the broker market system is .12346 of which the seller gets .04938, the buyer gets .04938, and the specialist gets .02469.

In the direct bargaining with the split-the-difference rule²⁶ analyzed by Chat-

²¹Refer to section 5 in Jang (1987).

²²The other two trading systems considered in Jang (1987) are; one is which traders place either a market order or a limit order, and the specialist acts both as a broker and as a dealer in the same trading; one in which traders place only market orders, and the specialist acts only as a dealer.

²³The ex-ante expected profit is the expected profit before each agent in a trade draws his own reservation price. The expected profit we derived in the footnotes 19 and 20 is the interim expected profit that is after each agent know his own reservation price.

²⁴Refer to footnotes 19, 20, and 23.

²⁵These numbers for the ex-ante expected profit, including the ones for other cases presented in the table 1, are for the identical uniform distribution of the reservation prices over the support of $[0, 1]$.

²⁶They showed that the split-the-difference rule maximizes the total ex-ante expected profit in the direct bargaining between the buyer and the seller.

[Table 1] The Ex-Ante Expected Profit

System	1	2	3
Seller	.053227	.04938	.07031
Buyer	.053227	.04938	.07031
Specialist	.009297	.02469	
Total	.115752	.12346	.14063

*System 1: Broker-Dealer Market

System 2: Broker Market

System 3: Direct Bargaining

terjee and Samuelson (1983), the total ex-ante expected profit for the identical uniform distribution on $[0, 1]$ is .14063 which is the maximum attainable given the information structure. Table 1 summarizes the ex-ante expected profit of the different trading system.

Proposition 7: The broker-dealer market system is less efficient than the broker market system in terms of the total ex-ante expected profit yielded to the society. Traders, however, are better off with the broker-dealer market system than with the broker market system.

Trader get a higher ex-ante expected profit in the broker-dealer market system than in the broker market system, while the specialist gets less in the broker-dealer market system. In other words, the broker-dealer system is more desirable for traders than the other systems considered, even if it is not necessarily true for the society. This result implies that the system which provides two different order types - a limit order and a market order - to traders, and in which the specialist plays the dual role of a dealer and a broker is better, at least, for traders than the order trading systems such as: one in which traders place only limit orders, and the specialist plays only the broker role; and one in which traders place only market orders, and the specialist plays only the dealer role. The broker-dealer market system is, in fact, close to the specialist system used in the New York Stock Exchange.

VII. SUMMARY AND CONCLUSION

We have shown that there exists an equilibrium in which either of a market or limit order can be the traders' optimal strategy. In such an equilibrium, a seller with a low reservation price trades via a market order since the gains in the high trading probability outweigh the gains in the surplus due to a high limit price. A seller with a high reservation price will place a limit order since the gains in the surplus through a high limit price outweigh the loss in the trading probability. Similarly, a buyer with a high reservation price will choose a market order, and a buyer with a low reservation price will choose a limit order. Given that a limit order is chosen, the limit price is determined in such a way that it balances the

gains in the surplus and the loss in the trading probability. The limit price is a non-decreasing function of trader's reservation price with two price jumps due to the changes in the order types. The price jumps are particularly interesting given that more than 37% of number of trades²⁷ are executed at a price in between of bid and ask price (i.e. within the spread). Assuming many of these trades are new limit orders arrived after the outstanding quote was posted (either as a limit order from the public or as an order that floor trader trying to get better price than the quoted price), our result implies that a trader should consider a shift in trading probability not only due to his switch in order type but also due to the possibility of the other trader's switch in order type on the opposite side of a trade. We also have shown that the system which provides traders with two different types of order - a limit order and a market order, and in which the specialist plays the dual role of a dealer and a broker is better, at least, for traders than other trading systems, even if this is not the case for the society as a whole.

The model, in essence, addresses the issue of the trader's strategic behavior in order placement. Further researches are, however, necessary in several directions. The quantity decision needs to be included in the model and a dynamic model will more adequately capture the effect of the specialist's inventory position on the prices. The effects of asymmetric information among agents would also be a very intriguing research project. The question of whether an informed trader will trade via a market or limit order is of particular importance in analyzing traders' strategic order placement behavior.

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²⁷This figure is based on author's separate research project which analyzes all trades and quotes and quotes made for 855 actively traded stocks (top 50% in number of trades) on NYSE during 253 trading days in 1988. A rough draft will be provided upon a request.

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