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MULTIPERIOD MULTITRADE BARGAINING

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Recently, Bae ("Multitrade Bargaining," Journal of Economic Theory, 1991) developed a model of multitrade bargaining games in which the object of bargaining can be traded repeatedly, and distinguished the contract bargaining game (CBG) in which one contract is made at most, and the repeated bargaining game (RBG) in which contracts can be made in each period. He showed that the two-period CBG and RBG are equivalent in the sense that each player's payoff and the sequence of trades made are identical in the equilibria of the two games. We derive conditions under which the n-period CBG and RBG are equivalent.

I. INTRODUCTION

Recently, Bae (1991) developed a model of multitrade bargaining games in which the object of bargining can be traded repeatedly, and distinguished the contract bargaining game (CBG) in which one contract is made at most, and the repeated bargaining game (RBG) in which contracts can be made in each period. He showed that the two-period CBG and RBG are equivalent in the sense that each player's payoff and the sequence of trades made are identical in the equilibrial of the two games. However, because he did not say anything for the n-period CBG, and RBG for $n \geqslant 3$, it is not clear whether the two-period result can extend to the general n-period bargaining games. In this paper, we derive conditions under which the n-period CBG and RBG are equivalent.

In Bae (1991), the buyer has a reservation price which is his private information. In each period, the seller offers a price and the buyer either accepts or rejects it. Multitrade bargaining games differ from single-trade sequential bargaining games (SBG) (Rubinstein (1982) and Sobel and Takahashi (1983) et. al.). Bargaining over services or the rental of durable goods are natural candidates for a multitrade bargaining game. If the contract made in a multitrade bargaining game is long-term so that the contract is effective throughout the time horizon, then at most one contract is made, as in the SBG. In this case, the multitrade bargaining game is referred to as the contract bargaining game (CBG). Bargaining over a long-term contract of services or a sale of a durable good is a CBG. The structure of CBG

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is similar to that of SBG, which has been analyzed extensively. When the contract term is only one period, the multitrade bargaining game is referred to as the repeated bargaining game (RBG), which is an instance of repeated incomplete information games.[1]

The leading solution concept for games with incomplete information is the sequential equilibrium (Kreps and Wilson, 1982). A sequential equilibrium of a bargaining game, in which the uninformed party (the seller) makes all the offers, is said to satisfy the simple partitioning property (SPP) if after any history and for any price offer, there is a cut-off value such that the buyer accepts the offer if and only if his reservation price exceeds or equals that cut-off value. We know that every sequential equilibrium of CBG has SPP (see Fudenberg, Levine and Tirole (1985)).

We compare the sequential equilibria with SPP in the CBG and RBG. When the range of the buyer's reservation price is narrow so that it is optimal for the seller to charge the lowest reservation price in the range in the static set-up, the n-period CBG and RBG have the same unique equilibrium in which the seller charges the lowest reservation price in each period. If the range is wide and there is little discounting, the n-period CBG and RBG are not equivalent. Actually the n-period RBG may not have a sequential equilibrium with SPP. But, if there is large discounting, and the (n-1)-period subgames of CBG and RBG are equivalent, and the n-period CBG and RBG have unique equilibria, then either i) the n-period CBG and RBG are equivalent, or ii) the seller's expected profit is greater in the n-period RBG than in the n-period CBG.

Hart and Tirole's (1988) research is closely related to our study. They consider the n-period CBG and RBG with two types of buyers. One of their findings is that the seller's expected payoff is not greater in the n-period RBG than in the n-period CBG. This finding of Hart and Tirole contrasts to our result ii) in the n-period bargaining games, though we do not show the existence of a case in which ii) holds. This contrast may stem from the difference in the number of the buyer types in their model and our model. If there does not exist a case in which ii) holds, then our equivalence result becomes more general.

The remainder of this paper is organized as follows. Section II describes the model and presents the definition of equivalence. In Section III, we derive conditions under which the n-period CBG and RBG are equivalent. Section IV includes a summary and suggested directions for further research.

II. THE MODEL

The seller wishes to sell a service to a buyer for finite periods. The seller and

[1] Bergin (1989) characterizes sequential equilibria in infinitely repeated incomplete information games when types of players are finite.

buyer are bargaining over the price of the service. Each agent has a time-invariant per-period reservation price for the service. The seller's reservation price is common knowledge and is assumed to be 0. The buyer's reservation price is v which is known only to him. All the rest is common knowledge. Both agents have a common discount factor q, where $0<q<1$. A probability distribution F(v) represents the seller's belief that the buyer has a reservation price less than or equal to v. We assume that F( ) is supported on the closed interval $[\underline{m}, 1]$, where $0\leqslant\underline{m}<1$.

In the n-period CBG, the seller sets a price and gives the buyer an opoortunity to purchase one unit of the service in each period from that period to period n at the offered price. If the buyer accepts the offer, then the game ends. When the buyer rejects the offer, the process is repeated, with the seller making a new offer if current period is before period n. At the end of period n, the game ends whether or not the buyer accepts the offer. The n-period RBG, in contrast, has the following structure. The seller sets a price and gives the buyer an opportunity to purchase one unit of the object that period. The buyer then either accepts or rejects the offer. The process repeats itself each period for n periods.

We assume that the function $G(v)\equiv vF(v)$ is continuously differentiable and that $G''(v)>0$. This is a decreasing expected marginal revenue assumption, and guarantees that the problem, $\max_{\underline{m}\leqslant p\leqslant x} p(F(x) - F(p))$, has a unique solution $p^*(x)$ for all $x\in[\underline{m}, 1]$. For the analysis of the n-period bargaining games, we need to consider their subgames.[2] We will focus on the sequential equilibria with the SPP for the bargaining subgames, and simply call them equilibria hereafter. Thus, it is sufficient to consider the subgames in which the seller's posterior belief is a conditional probability distribution of F(v) on a subinterval of the interval $[\underline{m}, 1]$.

Let $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ denote the n-period subgames of CBG and RBG, respectively, where there are n periods remaining and the seller's Bayes-consistent belief is the conditional probability distribution of F(v) on the subinterval, $[m, m']$, of the interval, $[\underline{m}, 1]$.[3] Also, let $R^i(m, m'; n)$, $p^i(m, m'; n)$ and $x^i(m, m'; n)$ be the seller's expected payoff and the first-period price and cut-off value, respectively, in the equilibrium of $BG^i(m, m'; n)$, for $i=c$ and r. For simple exposition, we let $n^i(m, m'; n)=R^i(m, m'; n)(F(m')-F(m))$, for $i=c$ and r. Then, $n^i(m, m'; n)$ is the seller's expected payoff under unconditional probabilities in the equilibrium of $BG^i(m, m'; n)$, for $i=c$ and r. Now, we define equivalence between $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$.

[2]When subforms and information sets are defined in terms of nodes (see Kreps and Wilson (1982)), the n-period bargaining games have no proper subgames. But, if we define subforms and information sets in terms of paths (see Bergin (1989)), then we have (n − 1)-period bargaining subgames in the beginning of the second period.

[3]The seller's belief may be a half open interval $[m, m')$, but it does not matter because F( ) is continuous and hence m' cannot be an equilibrium price when we replace $[m, m')$ by $[m, m']$.

Definition 1 An equilibrium of $BG^c(m, m'; n)$ and an equilibrium of $BG^r(m, m'; n)$ are *equivalent* if for every reservation price of the buyer, payoffs of the seller and buyer and the sequence of trades made are the same in the two equilibria.

Definition 2 $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are *equivalent* if for each equilibrium of $BG^c(m, m'; n)$ ($BG^r(m, m'; n)$), there exists an equivalent equilibrium of $BG^r(m, m'; n)$ ($BG^c(m, m'; n)$).

III. N-PERIOD BARGAINING GAMES

In this section, we derive some results for the comparision of $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$. We assume that if the buyer is indifferent between accepting and rejecting an offer, then he accepts it. The results are summarized in the following theorems and corollary.

Theorem 1 If $m \geqslant p^*(m')$, then $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ have the same unique equilibrium in which the seller charges m and the buyer accepts the offer in each period.

Proof See APPENDIX 1.

Theorem 1 shows that when the range of the buyer's reservation price is narrow so that it is optimal for the seller to charge m in the static set-up, the n-period (subgames of) CBG and RBG have the same unique equilibrium in which the seller charges m in each period, and hence they are equivalent.

Lemma 2 $x^c(m, m'; n) = m$ if $m \geqslant p^*(m')$, and $x^c(m, m'; n) > m$ otherwise.

Proof See APPENDIX 2.

Theorem 3 If $m < p^*(m')$ and $q + \dots + q^{n-1} \geqslant 1$, then $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are not equivalent.

Proof See APPENDIX 3.

Theorem 3 shows that when the range of the buyer's reservation price is wide and there is not much discounting, $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are not equivalent. Actually, $BG^r(m, m'; n)$ may not have a sequential equilibria with SPP.

The intuition behind the proof of Theorem 3 is as follows. Suppose that $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent. Then, a v-buyer accepts the first offer if and only if $v \geqslant x^r(m, m'; n)$. We prove that $m < x^r(m, x^r(m, m'; n); n-1)$

$<x^r(m, m'; n)$. A v-buyer with $v \in [x^r(m, x^r(m, m'; n); n-1), x^r(m, m'; n)]$ rejects the first offer and accept an offer from the second period on in the equilibrium. Call the v-buyer with $v = x^r(m, m'; n)$ the indifferent buyer, because the indifferent buyer cannot earn positive surplus in the equilibria of $BG^r(m, x^r(m, m'; n); n-1)$, his first-period surplus from accepting the first offer should not be less than future surplus from rejecting the first offer and accept the offer from the second period on. Then, a v-buyer with $v \in [x^r(m, x^r(m, m'; n); n-1), x^r(m, m'; n)]$ prefers to accept the first offer because, his payoff is less than the indifferent buyer's payoff by $x^r(m, m'; n) - v$ when both of them accept the first offer and by $(q + \ldots + q^{n-1})(x^r(m, m'; n) - v)$ when they reject the first offer and accept an offer from the second period on. This is a contradiction to the definition of $x^r(m, m'; n)$. This problem does not occur in the n-period CBG because a v-buyer's surplus form accepting the first offer p is $(1 + q + \ldots + q^{n-1})(v - p)$ rather than $v - p$.

Theorem 4 Suppose than $m < p^*(m')$ and $q + \ldots + q^{n-1} < 1$. Further, suppose that $BG^c(z, z'; n-1)$ and $BG^r(z, z'; n-1)$ are equivalent for all z and z' such that $m \leqslant z \leqslant z' \leqslant m'$. $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent if and only if there does not exist an $x^r(m, m'; n)$ such that $x^r(m, m'; n) < p^*(m')$.

Proof See APPENDIX 4.

If a trade is made in $BG^c(m, m'; n)$, it continues to period n. Therefore, for $BG^r(m, m'; n)$ to be equivalent to $BG^c(m, m'; n)$, trade continues to period n after once made, in the equilibria of $BG^r(m, m'; n)$. If $x^r(m, m'; n) \geqslant p^*(m')$, then by Theorem 1, the buyer accepts an offer from the first period to the last period in the equilibrium of $BG^r(m, m'; n)$. From the proof of Theorem 4, we can see that when the n-period subgames of CBG and RBG are equivalent, equilibrium outcomes of the n-period subgame of RBG (CBG) can be easily calculated from equilibrium outcomes of the equivalent n-period subgame of CBG (RBG).

Corollary 5 Suppose that $m < p^*(m')$ and $q + \ldots + q^{n-1} < 1$. Further, suppose that $BG^c(z, z'; n-1)$ and $BG^r(z, z'; n-1)$ are equivalent for all z and z' such that $m \leqslant z \leqslant z' \leqslant m'$, and that $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ have unique equilibria. Then,

i) $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent if $x^r(m, m'; n) \geqslant p^*(m')$,
ii) the seller's expected payoff is greater in $BG^r(m, m'; n)$ than in $BG^c(m, m'; n)$ if $x^r(m, m'; n) < p^*(m')$.

Proof See APPENDIX 5.

Hart and Tirole (1988) consider than n-period CBG and RBG when there are two types of buyers. One of their findings is that the seller's expected payoff is not greater in the n-period RBG than in the n-period CBG. This finding of Hart and Tirole contrasts to our result ii) in Corollary 10, though we do not show the existence of a case in which ii) holds. This contrast may stem from the difference in the number of the buyer types in their model and our model. If there does not exist a case in which ii) holds, then our equivalence result in Corollary 10 becomes more general.

Theorem 6 $BG^c(m, m'; 2)$ and $BG^r(m, m'; 2)$ are equivalent for all m and m′ such that $\underline{m} \leqslant m \leqslant m' \leqslant 1$.

Proof See APPENDIX 6.

Theorem 6 shows that the two-period subgames of CBG and RBG are equivalent.

Theorem 7 Suppose that $(\sqrt{5}-1)/2 \leqslant q \leqslant 1$. Then, $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent if and only if $n=2$ or $m \geqslant p^*(m')$.

Proof Theorem 7 follows from Theorems 1, 3 and 6. Q.E.D.

If we use the interest rate as the discount rate, then it is likely to have $q \geqslant (\sqrt{5}-1)/2 = 0.6 \ldots$ with any reasonable length of one service period. Theorem 7 shows that in this case, the n-period CBG and RBG are equivalent if and only if $n=2$ or $\underline{m} \geqslant p^*(1)$.

IV. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

We have considered n-period multitrade bargaining games, distinguishing between contract bargaining games and repeated bargaining games, and have compared their sequential equilibria with the simple partitioning property. We find that when the range of the buyer's reservation price is narrow, the n-period CBG and RBG have the same unique equilibrium and hence they are equivalent. When the range is wide and there is little discounting, the n-period contract bargaining game and repeated bargaining game are not equivalnet. If there is large discounting, and the $(n-1)$-period subgames of contract bargaining game and repeated bargaining game are equivalent, and the n-period bargaining games have unique equilibria, then either i) the n-period contract bargaining game and repeated bargaining game are equivalent, or ii) the seller's expected profit is greater in the n-period repeated bargaining game than in the n-period contract bargaining game.

Though we do not show the existence of a case in which the result ii) holds, it contrasts to one of the Hart and Tirole's (1988) results. This contrast may stem from the difference in the number of the buyer types in their model and our model. If there does not exist a case in which ii) holds, then our equivalence result becomes more general. Further investigation of the contrast is necessary. Our anlaysis of the n-period bargaining games is restricted to the case with large discounting. Extending the analysis to the general n-period bargaining games is thus a high-priority topic for further research. In our model, the uninformed player (the seller) makes all the offers. Testing equivalence results in the alternating-offers model (Rubinstein (1985) and Grossman and Perry (1986) et al.) may also be an important area of future research.

APPENDIX 1

Theorem 1 If $m \geqslant p^*(m')$, then $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ have the same unique equilibrium in which the seller charges m and the buyer accepts the offer in each period.

Proof We will prove that if $m \geqslant p^*(m')$, then $p^i(m, m'; n) = x^i(m, m'; n) = m$ for $i = c$ and r, by induction on n. Because $p^i(m, m'; 1) = x^i(m, m'; 1) = \max\{m, p^*(m')\}$ for $i = c$ and r, $p^i(m, m'; 1) = x^i(m, m'; 1) = m$ for $i = c$ and r if $m \geqslant p^*(m')$. Suppose that $p^i(z, z'; n-1) = x^i(z, z'; n-1) = z$ for $i = c$ and r if $z \geqslant p^*(z')$, and that $m \geqslant p^*(m')$.

Now, we prove the lemma for $BG^c(m, m'; n)$. $p^c(m, m'; n)$, is the value of p that solves:

$$\text{(A1)}\quad \max_{m \leqslant p \leqslant m'} (1 + q + \dots + q^{n-1})p(F(m') - F(x^c(p))) + (q + \dots + q^{n-1})m(F(x^c(p)) - F(m)),$$

where the cut-off value $x^c(p)$ satisfies

$$\text{(A2)}\quad x^c(p) = \begin{cases} x & \text{where } (1 + q + \dots + q^{n-1})(x - p) = (q + \dots + q^{n-1})(x - m) \\ & \text{if such an } x \in [m, m'] \text{ exists,} \\ m' & \text{otherwise.} \end{cases}$$

Using (A2) to eliminate p in (A1), $x^c(m, m'; n)$, is the value of x which solves:

$$\text{(A3)}\quad \max_{m \leqslant x \leqslant m'} x(F(m') - F(x)) + (q + \dots + q^{n-1})m(F(m') - F(m)).$$

Thus, $x^c(m, m'; n) = m$. Finally, by (A2), $p^c(m, m'; n) = m$.

Two cases have to be distinguished for $BG^r(m, m'; n)$. First, suppose that $q + \dots + q^{n-1} < 1$. $p^r(m, m'; n)$, is the value of p that solves:

$$\text{(A4)}\quad \max_{m\leq p\leq m'} p(F(m')-F(x^r(p))) + (q+\ldots+q^{n-1})x^r(p)(F(m')-F(x^r(p))) + (q+\ldots+q^{n-1})m(F(x^r(p))-F(m)),$$

where the cut-off value $x^r(p)$ satisfies

$$\text{(A5)}\quad x^r(p) = \begin{cases} x & \text{where } x-p=(q+\ldots+q^{n-1})(x-m) \\ & \text{if such an } x\in[m, m'] \text{ exists,} \\ m' & \text{otherwise.} \end{cases}$$

Because $q+\ldots+q^{n-1}<1$, the cut-off value, $x^r(p)$, is well defined in (A5). Using (A5) to eliminate p in (A4), $x^r(m, m'; n)$, is the value of x which solves;

$$\text{(A6)}\quad \max_{m\leq x\leq m'} x(F(m')-F(x)) + (q+\ldots+q^{n-1})m(F(m')-F(m)).$$

Thus, $x^c(m, m'; n)=m$. Finally, by (A5), $p^c(m, m'; n)=m$.

Lastly, suppose that $q+\ldots+q^{n-1}\geq 1$. Let x(p) be the cut-off value when the seller charges p in the first period of $BG^r(m, m'; n)$. Then, $x(m)=m$. Now, we consider the case with $p>m$. Then, $x(p)>m$. The v-buyer with $v=x(p)$ cannot earn positive surplus from the second period on when he accepts the first offer. But, if he rejects the first offer, then he obtains $(q+\ldots+q^{n-1})(v-m)$ in the future. Because $(q+\ldots+q^{n-1})(v-m)>(v-p)$ for all v and p such that $v>m$ and $p>m$, $x(p)=m'$ for all $p>m$, that is, if the seller charges $p>m$ in the first period of $BG^r(m, m'; n)$, the buyer rejects the offer. Note than $p^*(m')>0$ and hence $m>0$. Finally, because the first offer m yields greater expected profit to the seller and leads to the same subgame as the first offer $p>m$, $p^r(m, m'; n)=m$ and hence $x^r(m, m'; n)=m$. Q.E.D.

APPENDIX 2

Lemma 2 $x^c(m, m'; n)=m$ if $m\geq p^*(m')$, and $x^c(m, m'; n)>m$ otherwise.

Proof The first part of the lemma follows from Theorem 1. The second part holds when $n=1$ because $x^c(m, m'; 1)=\max\{m, p^*(m')\}$. Now, we prove the second part when $n\geq 2$. If $m=0<m'$, then $x^c(m, m'; n)>m$ because the seller's payoff is 0 when he induces all v-buyers to accept the first offer while he earns positive expected profit when he induces no v-buyer to accept an offer until the last period and charges $p^*(m')$ in the last period. Suppose that $0<m<p^*(m')$ and let

$$\text{(B1)}\quad \bar{x}=\max\{x|p^*(x)=m\},$$

$$\text{(B2)}\quad \bar{p}=(1+q+\ldots+q^{k-1})^{-1}(\bar{x}+(q+\ldots+q^{n-1})m).$$

Then, $m<\bar{p}<\bar{x}<m'$. If the seller charges $p \in [m, \bar{p}]$ in the first period of $BG^c(m, m'; n)$ and charges m in the second period, then the first-period cut-off value is in the interval, $[m, \bar{x}]$. Moreover, if $x \in [m, \bar{x}]$, then the seller charges m and the buyer accepts the offer in the first period of $BG^c(m, x; n-1)$, by Theorem 1.

We consider the case in which the seller charges $p \in [m, \bar{p}]$, in the first period of $BG^c(m, m'; n)$ while he is free to charge any price from the second period on. Then, the seller solves:

$$\text{(B3)}\ \max_{m \leqslant p \leqslant \bar{p}}\ (1+q+\ \dots\ +q^{n-1})p(F(m')-F(x^c(p))) + (q+\ \dots\ +q^{n-1})m(F(x^c(p))-F(m)),$$

where the cut-off value $x^c(p)$ satisfies

$$\text{(B4)}\ x^c(p) = \begin{cases} x \quad \text{where } (1+q+\ \dots\ +q^{n-1})(x-p) = (q+\ \dots\ +q^{n-1})(x-m) \\ \quad \text{if such an } x \quad [m, m'] \text{ exixsts,} \\ m' \text{ otherwise.} \end{cases}$$

Using (B4) to eliminate p in (B3), the seller's problem becomes:

$$\text{(B5)}\ \max_{m \leqslant x \leqslant \bar{x}}\ x(F(m')-F(x)) + (q+\ \dots\ +q^{n-1})m(F(m') - F(m)).$$

The solution of the problem (B5) is

$$\text{(B6)}\ x^c = \min\{p^*(m'), \bar{x}\}.$$

We showed that the seller obtains higher profit by inducing the cut-off value, x^c, than by inducing the cut-off value, m. This proves the second part of the lemma when $n \geqslant 2$, because $x^c>m$. Q.E.D.

APPENDIX 3

Theorem 3 If $m<p^*(m')$ and $q+\ \dots\ +q^{n-1} \geqslant 1$, then $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are not equivalent.

Proof Suppose that $m<p^*(m')$ and $q+\ \dots\ +q^{n-1} \geqslant 1$, and that $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent. Then,

(C1) $x^r(m, m'; n) = x^c(m, m'; n)$,

(C2) $x^r(m, x^r(m, m'; n); n-1) = x^c(m, x^r(m, m'; n); n-1)$, and $BG^r(m, x^r(m, m'; n); n-1)$ and $BG^c(m, x^r(m, m'; n); n-1)$ are equivalent.

Claim 1 $m<x^r(m, m'; n)<m'$.

Proof of Claim 1 By (C1) and Lemma 6, $x^r(m, m'; n)>m$. If $x^r(m, m'; n)=m'$, then the seller obtains zero expected profit in the first period of $BG^r(m, m'; n)$ and faces $BG^r(m, m'; n-1)$ in the second period. But, if the seller charges m in the first period of $BG^r(m, m'; n)$, then he earns $m(F(m')-F(m))$ and faces $BG^r(m, m'; n-1)$ in the second period. Because $x^r(m, m'; n)\neq m$, $x^r(m, m'; n)\neq m'$.

Q.E.D.

Claim 2 $x^r(m, x^r(m, m'; n); n-1)<x^r(m, m'; n)$.

Proof of Claim 2 If $m\geqslant p^*(x^r(m, m'; n))$, then Claim 2 holds by (C2) and Lemma 6. if $m<p^*(x^r(m, m'; n))$, then Claim 2 holds by the same reason as Claim 1.

Q.E.D.

Because $x^r(m, m'; n)<m'$ and $BG^r(m, x^r(m, m'; n); n-1)$ and $BG^c(m, x^r(m, m'; n); n-1)$ are equivalent,

$$\text{(C3)}\quad \begin{aligned} &x^r(m, m'; n)-p^r(m, m'; n)\\ &\geqslant (q+\ \dots\ +q^{n-1})(x^r(m, m'; n)-p^c(m, x_r(m, m';n); n-1)). \end{aligned}$$

But, if $v\in[x^r(m, x^r(m, m'; n); n-1), x^r(m, m'; n))$, the buyer prefers to accept $p^r(m, m'; n)$ in the first period of $BG^r(m, m'; n)$ because by (C3),

$$\text{(C4)}\quad v-p^r(m, m'; n)\geqslant(q+\ \dots\ +q^{n-1})(v-p^c(m, x^r(m, m'; n); n-1)).$$

This is a contradiction to the definition of $x^r(m, m'; n)$. Q.E.D.

APPENDIX 4

Theorem 4 Suppose that $m<p^*(m')$ and $q+\ \dots\ +q^{n-1}<1$. Further, suppose that $BG^c(z, z'; n-1)$ and $BG^r(z, z'; n-1)$ are equivalent for all z and z′ such that $m\leqslant z\leqslant z'\leqslant m'$. $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent if and only if there does not exist an $x^r(m, m'; n)$ such that $x^r(m, m'; n)<p^*(m')$.

Proof Choose an arbitrary pair of equivalent equilibria of $BG^c(z, z'; n-1)$ and $BG^r(z, z'; n-1)$, for each pair of z and z′ such that $m\leqslant z\leqslant z'\leqslant m'$.

$BG^c(m, m'; n)$

In $BG^c(m, m'; n)$, the equilibrium first-period price, $p^c_1=p^c(m, m'; n)$, is the value of p that solves:

$$\text{(D1)}\quad \max_{m \leqslant p \leqslant m'} (1+q+\ \dots\ +q^{n-1})p(F(m')-F(x^c(p))) + q\,\pi^c(m,\ x^c(p);\ n-1),$$

where the cut-off value $x^c(p)$ satisfies

$$\text{(D2)}\quad x^c(p) = \begin{cases} x & \text{where } (1+q+\ \dots\ +q^{n-1})(x-p) \\ & \quad = (q+\ \dots\ +q^{n-1})(x-p^c(m,\ x;\ n-1)) \\ & \text{if such an } x \in [m,\ m'] \text{ exists,} \\ m' & \text{otherwise.} \end{cases}$$

In (D1), we work with unconditional probabilities. This renormalization simplifies exposition without affecting the seller's optimal behavior.

Using (D2) to eliminate p in (D1), the equilibrium cut-off value, $x^c = x^c(m,\ m';\ n)$, is the value of x which solves:

$$\text{(D3)}\quad \max_{m \leqslant x \leqslant m'} \varphi^c(x;\ n),$$

where

$$\text{(D4)}\quad \varphi^c(x;\ n) = (x+(q+\ \dots\ +q^{n-1})p^c(m,\ x;\ n-1))(F(m')-F(x)) + q\,\pi^c(m,\ x;\ n-1).$$

Finally, by (D2),

$$\text{(D5)}\quad p^c{}_1 = (1+q+\ \dots\ +q^{n-1})^{-1}(x^c+(q+\ \dots\ +q^{n-1})p^c(m,\ x^c;\ n-1)).$$

$BG^r(m,\ m';\ n)$

In $BG^r(m,\ m';\ n)$, the equilibrium first-period price, $p^r{}_1 = p^r(m,\ m';\ n)$, is the value of p that solves:

$$\text{(D6)}\quad \max_{m \leqslant pf \leqslant m'} p(F(m')-F(x^r(p)))+q\,\pi^c(x^r(p),\ m';\ n-1) + q\,\pi^c(m,\ x^r(p);\ n-1),$$

where the cut-off value $x^r(p)$ satisfies

$$\text{(D7)}\quad x^r(p) = \begin{cases} x & \text{where } x-p=(q+\ \dots\ +q^{n-1})(x-p^c(m,\ x;\ n-1)) \\ & \text{if such an } x \in [m,\ m'] \text{ exists,} \\ m' & \text{otherwise.} \end{cases}$$

In (D6) and (D7), we replace $BG^r(x,\ m';\ n-1)$ and $BG^r(m,\ x;\ n-1)$ by their equivalents $BG^c(x,\ m';\ n-1)$ and $BG^c(m,\ x;\ n-1)$. Because $q+\ \dots\ +q^{n-1}<1$, the cut-off value, $x^r(p)$, is well defined in (D7).

Using (D7) to eliminate p in (D6), the equilibrium cut-off value, $x^r = x^r(m, m'; n)$, is the value of x which solves:

(D8) $\max_{m \leqslant x \leqslant m'} \varphi^r(x; n)$,

where

(D9) $\varphi^r(x; n) = (x - (q + \ldots + q^{n-1})(x - p^c(m, x; n-1)))(F(m') - F(x)) + q\, \pi^c(x, m'; n-1) + q\, \pi^c(m, x; n-1)$.

Finally, by (D7),

(D10) $p^r_1 = x^r - (q + \ldots + q^{n-1})(x^r - p^c(m, x^r; n-1))$.

Equivalence

If the seller charges x in the first period of the $BG^c(x, m'; n-1)$, then the buyer accepts the offer. Thus, by Lemma 6,

(D11) $\pi^c(x, m'; n-1) = (1 + q + \ldots + q^{n-2})x(F(m') - F(x))$ if $x \geqslant p^*(m')$,
(D12) $\pi^c(x, m'; n-1) > (1 + q + \ldots + q^{n-2})x(F(m') - F(x))$ if $x < p^*(m')$.

Then, by (D4) and (D9),

(D13) $\varphi^r(x; n) = \varphi^c(x; n)$ if $x \geqslant p^*(m')$,
(D14) $\varphi^r(x; n) > \varphi^c(n; n)$ if $x < p^*(m')$.

By (D13) and (D14), for each $x^c(x^r)$, there exists an $x^r(x^c)$ such that $x^c = x^r$, if and only if there does not exist x^r such that $x^r < p^*(m')$. Suppose that there does not exist x^r such that $x^r < p^*(m')$, and choose an equilibrium of the $BG^c(m, m'; n)$ and an equilibrium of the $BG^r(m, m'; n)$ such that $x^c = x^r$. Then,

$$\begin{aligned}(\text{D15})\quad (1 + q + \ldots + q^{n-1})p^c_1 &= x^c + (q + \ldots + q^{n-1})p^c(m, x^c; n-1)\\ &= x^r + (q + \ldots + q^{n-1})p^c(m, x^r; n-1)\\ &= p^r_1 + (q + \ldots + q^{n-1})x^r.\end{aligned}$$

The first equality of (D15) comes from (D5), the second from the fact that $x^c = x^r$ and the last from (D10).

Consider an arbitrary reservation price of the buyer. In the equilibria of $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$, the sequence of trades made is the same because $x^c = x^r$ and the (n − 1)-period subgames of $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent. If $v<x^c$, payoffs of the seller and buyer are the same in the two equilibria because the $(n-1)$-period subgames of $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent. If $v \geqslant x^c$, they are the same because the buyer pays $(1+q+ \ldots +q^{n-1})p^c_1$ in $BG^c(m, m'; n)$ and $p^r_1 + (q+ \ldots +q^{n-1})x^r$ in $BG^r(m, m';n)$, by Theorem 5. Q.E.D.

APPENDIX 5

Corollary 5 Suppose that $m<p^*(m')$ and $q+ \ldots +q^{n-1}<1$. Further, suppose that $BG^c(z, z'; n-1)$ and $BG^r(z, z'; n-1)$ are equivalent for all z and z′ such that $m \leqslant z \leqslant z' \leqslant m'$, and the $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ have unique equilibria. Then,

i) $BG^c(m, m'; n)$ and $BG^r(m, m'; n)$ are equivalent if $x^r(m, m'; n) \geqslant p^*(m')$,
ii) the seller's expected payoff is greater in $BG^r(m, m'; n)$ than in $BG^c(m, m'; n)$ if $x^r(m, m'; n)<p^*(m')$.

Proof i) follows from theorem 4. suppose that $x^r=x^r(m, m'; n)<p^*(m')$. Then, by (D14).

(E1) $\varphi^r(x^r; n)>\varphi^c(x^r; n)$.

Also, by the definition of x^r and (D13) and (D14),

(E2) $\varphi^r(x^r; n)>\varphi^r(x; n) \geqslant \varphi^c(x; n)$ for all $x \neq x^r$.

Finally, (E1) and (E2) prove ii). Q.E.D.

APPENDIX 6

Theorem 6 $BG^c(m, m'; 2)$ and $BG^r(m, m'; 2)$ are equivalent for all m and m′ such that $\underline{m} \leqslant m \leqslant m' \leqslant 1$.

Proof Theorem 6 follows from Theorem 1, if $m \geqslant p^*(m')$. Suppose that $m<p^*(m')$. Because $q<1$ and $BG^c(z, z'; 1)$ and $BG^r(z, z'; 1)$ are equivalent for all z and z′ such that $m \leqslant z \leqslant z' \leqslant m'$, it is sufficient to show that $x^r(m, m'; n) \geqslant p^*(m')$. Because $p^r(z, z'; 1) = \max(z, p^*(z'))$, the seller charges p^r_1 in the first period of $BG^r(m, m'; 2)$, which is the value of p that solves:

$$\text{(F1)}\quad \max_{m \leqslant p \leqslant m'} p(F(m')-F(x^r(p))) + q\max\{x^r(p), p^*(m')\}(F(m')-F(\max\{{}^r(p), p^*(m')\})) + q\max\{m, p^*(x^r(p)\}(F(x^r(p))-F(\{m, p^*(p))\})),$$

where the cut-off value $x^r(p)$ satisfies

$$\text{(F2)}\quad x^r(p) = \begin{cases} x & \text{where } x - p = q(x - \max\{m,\ p^*(x)\}) \\ & \quad \text{if such an } x \in [m,\ m'] \text{ exists,} \\ m' & \text{otherwise.} \end{cases}$$

Using (F2) to eliminate p in (F1), $x^r = x^r(m, m'; 2)$, is the value of x which solves:

$$\begin{aligned} \text{(F3)}\quad \max_{m \le x \le m'} \ & ((1-q)x + q \max\{m,\ p^*(x)\})(F(m') - F(x)) \\ & + q \max\{x,\ p^*(m')\}(F(m') - F(\max\{x,\ p^*(m')\})) \\ & + q \max\{m,\ p^*(x)\}(F(x) - F(\max\{m,\ p^*(x)\})). \end{aligned}$$

The partial derivative with respect to x of the objective function in (F3) for $x < p^*(m')$, is

$$\text{(F4)}\quad H^r(x) \equiv \begin{cases} (1-q)(F(m') - G'(x)) + q\, p^{*\prime}(x)(F(m') - G'(p^*(x))) & \text{if } p^*(x) > m, \\ (1-q)(F(m') - G'(x)) & \text{if } p^*(x) < m. \end{cases}$$

Note that $F(m') - G'(x) > 0$ if $x < p^*(m')$ and that $p^{*\prime}(x)(F(m') - G'(p^*(x))) > 0$ if $p^*(x) > m$. The latter follow from $p^{*\prime}(x) > 0$ and $G'(p^*(x)) = F(x)$ if $p^*(x) > m$. Then, $H^r(x) > 0$ for $x \in [m,\ p^*(m'))$. Therefore, $x^r \in [p^*(m'),\ m']$. Q.E.D.

REFERENCES

BAE, H. (1991), "Multitrade Bargaining," *Journal of Economic Theory,* forthcoming.

BERGIN, J. (1989), "A characterization of Sequential Equilibrium Strategies in Infinitely Repeated Incomplete Information Games," *Journal of Economic Theory* 47, pp. 51-65.

FUDENBERG, D., LEVINE, D. and TIROLE, J. (1985), "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in Roth, A. (ed.), *Game-Theoretic Models of Bargaining,* Cambridge: Cambridge University Press, pp. 73-98.

GROSSMAN, S. and PERRY, M. (1986), "Sequential Bargaining under Asymmetric Information," *Journal of Economic Theory* 39, pp. 120-154.

HART, O. and TIROLE, J. (1988), "Contract Renegotiation and Coasian Dynamics," *Review of Economic Studies* 55, pp. 509-540.

KREPS, D. and WILSON, R. (1982), "Sequential Equilibria," *Econometrica* 50, pp. 863-894.

RUBINSTEIN, A. (1982), "Perfect Equilibrium in a Bargaining Model," *Econometrica* 50, pp. 97-109.

RUBINSTEIN, A. (1985), "A Bargaining Model with Incomplete Information," *Econometrica* 53, pp. 1151-1172.

SOBEL, J. and TAKAHASHI, I. (1983), "A Multistage Model of Bargaining," *Review of Economic Studies* 50, pp. 411-426.