Global Dynamics in a Simple Macro Model

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I. INTRODUCTION

The usual dynamic analysis of macroeconomic models is based on analyzing the "local" stability of systems of differential equations describing the models. Such analysis can only answer questions about whether or not the system converges to equilibrium after "sufficiently" small shocks or "sufficiently small" policy changes. "Local Stability" cannot even ensure that if one moves to a position "close" to the equilibrium, then the trajectory will *stay* close to the equilibrium. The analysis of large shocks and discrete (noninfinitesimal) changes and of the behavior of trajectories near the equilibrium are clearly of considerable importance for macroeconomic policy making. The analysis of these issues necessarily rely on tools of global stability analysis. In spite of the importance of these issues, the inherent intractability of the techniques of global stability analysis has resulted in such analysis being largely ignored, even in the simplest macroeconomic models. In this paper we take a step towards filling this gap.

Unlike a couple of decades ago when the IS-LM curve model ruled supreme, there is currently no single model which is universally accepted. Hence, in selecting a model we have tried to pick one which would accomodate a broad spectrum of opinions. Indeed, we have taken special care not to prejudge either of the two questions which have in recent years been the focus of a continuing and lively debate: Firstly, which of two

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types of policies-fiscal and monetary-has the greater impact on aggregate demand. Secondly, is it possible by judiciously controlling aggregate demand to influence the unemployment rate in the "long run", in particular is a trade-off between inflation and unemployment possible in the long run? The simple mobel we describe based, roughly, on a monetarist model in Vanderkamp [11][1] is general enough to be consistent with the Keynesian and Monetarist, the activistic and passivistic positions on these issues. More often than not, the positions in these debates are based on "global" compartive statics. For instance, Friedman's celebrated x precent rule (based on convergence to the natural rate of unemployment *no matter what the value of x is*) clearly depends on global stability of the system. Similarly, the Keynesian argument for a longrun trade-off between inflation and unemployment does not specify that such a trade-off is infinitesimal or that such a trade-off is achievable only through several "small" changes and hence is based on a presumption of global stability. Our analysis is not intended to resolve any of the above controversies but rather by proving global stability to provide a background for and a meaningfulness to the debates (in the same sense that "local" stability analysis provides a background for and makes "marginal" (calculus based) comparative static analysis meaningful).

Thus, in this paper we provide a proof of global asymptotic stability of a conventional macro model by means of Lyapounov's second method. While a "Lyapounov function" is provided for the model, it has the difficulty that its time derivative may be zero outside the steady state, though it is everywhere nonpositive. However, we argue, even in this case one can use a refined version of the Lyapounov method to prove the global stability. To our knowledge, since even the standard method of Lyapounov function is rarely used in the context of macroeconomic models, macroeconomists might find both the metnod and the content of this paper to be interest.

II. THE MODEL

Our model will be described by four basic equations[2]:

$$z=p+y \tag{1}$$

$$p=h(u)+kp^*,\ u(0)>0,\ h'<0,\ h(0)=\infty,\quad 0<k\leq 1 \tag{2}$$

$$\dot{u}=(\bar{y}-y),\ g'>0,\ g(0)=0 \tag{3}$$

$$\dot{p}^*=f(p-p^*),\ f'>0,\ f(0)=0 \tag{4}$$

Equation (1) is an identity describing the relationship between the rate of growth of nominal income (z), the rate of inflation (p) and the rate of growth of real income (y). We will assume that the variable z can be controlled by the government through the appropriate use of fiscal and/or monetary policy.

Equation (2) is the standard equation for Phillips curve, where u is the rate of unemployment, and p^* the expected rete of inflation. $h'<0$ and $h(0)=\infty$ ensure that the short-run (i.e., $p^*=$constant) Phillips curve has a negative slope and friction prevents the unemployment rate from being equal to zero at any finite inflation rate. The constant k is a measure of "money illusion." In particular, $k=1$ represents the case of "no money illusion" and yields a vertical long-run ($p=p^*$) Phillips curve with natural rate of unemployment $h^{-1}(0)$, while $k<1$ gives us a negatively sloped long-run Phillips curve with a slope $h'(u)/(1-k)$. $u(0)>0$ reflects the assumption that the unemployment rate is positive at the initial point, $t=0$.

Equation (3), a variant of "Okun's Law,"[3] describes the relation between the real rate of growth of the economy (y) and changes in the unemployment rate($\dot{u}$). $\bar{y}$ is a constant representing the rate of growth needed to keep the unemployment rate constant ($g(0)=0$). It is taken to be determined by long-run demographic factors and corresponds to Harrod's "natural rate of growth." If the actual rate (y) exceeds (resp. is less than) the rate $\bar{y}$, then unemployment declines (resp. increases). (This follows from $g(0)=0$, $g'>0$.)

Equation (4) describes an adaptive expectation formation mechanism. It reflects the highly plausible behavioural assumption that the expected rate of inflation (p^*) adjusts upwards (resp. downwards) if and only if the actual rate exceeds (resp. is less than) the expected rate. The *linear* adaptive process $\dot{p}^*=a\cdot(p-p^*)$, $a>0$, postulated by Cagan ([13]) (and justified by the work of Burmeister and Turnovsky([2]) as a continuous approximation of a discrete error learning mechanism, and by Friedman ([7]) in relation to information constraints on "rational expectations") is a special case of equation (4).

The equations (1), (2), (3), and (4) fully described the model discussed in this paper.

III. THE PROBLEM

The model described in the previous section is consistent with the traditional Keynesian and Monetarist positions. If we assume that z, the rate of growth of nominal income, is controlled primarily by using fiscal policy and if the constant k (in equation (2)) is less than one, we get an "interventionist" Keynesian model with fiscal policy as an instrument for trade-off between inflation and unemployment in the short and long runs. On the other hand, with $k=1$ together with the assumption that the growth of nominal income (z) is primarily influenced by the rate of growth of money supply, equation (1) would become a version of the quantity theory and we would get a Friedmanian model of the Phillips curve with no trade-off possible between u and p in the long run (see Vanderkamp ([11]). The relevance of either position (Keynesian or Monetarist) given an appropriate choice of z depends crucially on the stability of the model, in particular, on whether following some random shock or a discrete (possibly "large") policy change, the model would converge to the long run equilibrium. In other worde, is the system of equations (1) to (4) globally stable for and arbitrarily fixed constant value of the policy parameter z ?

In what follows, setting $z=\bar{z}$(a constant) we reduce (1) to (4) to a pair of differential equations and sketch the phase diagram for the system as a preliminary step towards the investigation of its stability.

Defining $\bar{p}=\bar{z}-\bar{y}$, from equations (1) and (3) we get :

$$\dot{u}=g(p-\bar{p}) \tag{5}$$

Differentiating (2) with respect to time t and using (2), (4), and (5), we obtain :

$$\dot{p}=h'(u)g(p-\bar{p})+kf[(1-k^{-1})+k^{-1}h(u)] \tag{6}$$

To obtain the equilibrium of the system consisting of (5) and (6), we set $\dot{u}$

$=\dot{p}=0$. From (5) we get $p=\bar{p}(\equiv\bar{z}-\bar{y})$. From (6) and $\dot{u}=0$ and $\dot{p}=0$, noticing that by equation (4) $f^{-1}(0)=0$, $u=\bar{u}\equiv h^{-1}[(1-k)\bar{p}]$. Thus, the system has a unique equilibrium$(\bar{u},\ \bar{p})$.

The phase diagriam for the differential equation system (5) and (6) is given in Figure 1. The horizontal arrows in Figure 1 represent the fact that $\dot{u}\equiv0$ depending on $p\equiv\bar{p}$. The vertical arrows are derived from (6) and these reflect the fact that $\partial\dot{p}/\partial p=h'g'+(k-1)f'<0$ (i.e., for any givenvalue of u, at any point above (resp. below) the line $\dot{p}=0$, $\dot{p}$ is negative (resp. positive)). To complete the explanation of the phase diagram, we only need to justify the location of the line $\dot{p}=0$.

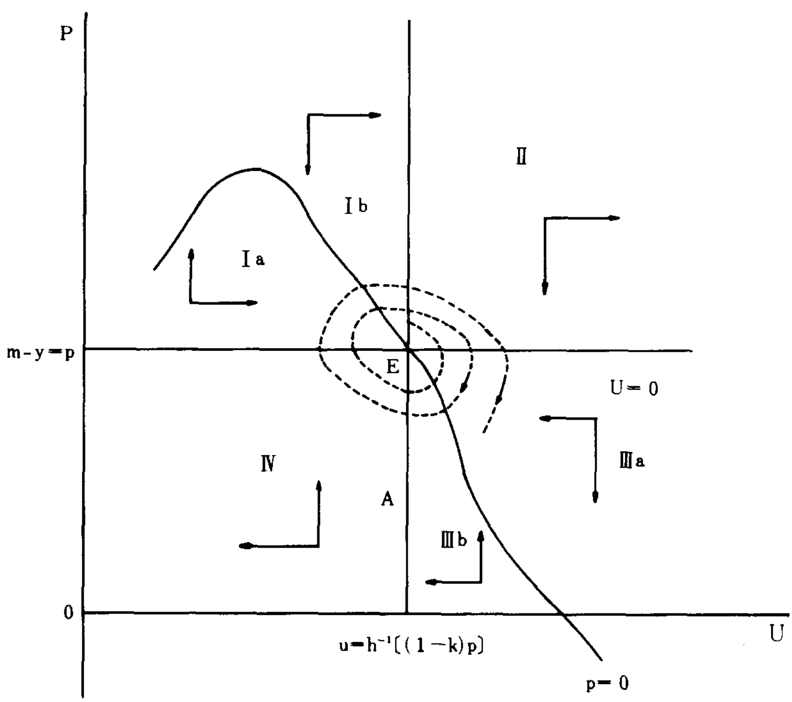
Differentiating (6) with respect to u and p we get:

$$\frac{\partial\dot{p}}{\partial u}=h''g+f'h' \tag{7a}$$

$$\frac{\partial\dot{p}}{\partial p}=h'f'+(k-1)f'<0 \tag{7b}$$

Noticing that $\dot{p}=0$ defines a function[4] $p=p(u)$ and using (7a) and (7b), we

Figure 1.



have[5] :

$$[\partial \dot{p} / \partial u]_{\substack{p=\bar{p} \\ u=\bar{u}}} = -\frac{f'(0)h'(\bar{u})}{h'(u)g'(0)+(k-1)f'(0)} < 0 \tag{8}$$

Ineqality (8) together with the fact that the equilibrium $(\bar{u}, \bar{p})$is unique ensures us that the line p = p(u)given by $\dot{p}=0$ lies in quadrants I and III as shown in Figure 1.

Noting that all the arrows in the phase diagram do not point " inwards" towards the equilibrium and that a trajectory such as the one indicated by the dotted line may be possible, one may draw the following (false!) conclusions : Firstly, the equilibrium may draw not be "locally" stable and that trajectories, even if they started "close" to $(\bar{u},\bar{p})$, may not converge. Secondly, even if the system is locally stable, it may not be stable in the sense of " Lyapounov" [6] (i.e., trajectories starting near the equilibrium may not remain near the equilibrum, moving away before converging). Finally, the system may not be " globally stable ; in other words, for some initial condition the trajectory may never approach $(\bar{u}, \bar{p})$.

As pointed out earier, instability of equilibrium seriously jeopardizes any meaningful comparative static analysis and reduces the significance of policy conclusions derived from such analysis. Below, we define formally the three concepts of stability described above and prove a theorem to show that the system of differential equations (5) and (6) is stable in all three sense.

Definition. Let (u(t),p(t))be a solution path of the equation system (5) and (6) for the initial values of u and p given by u = u (0)>0 and p =p(0), and let $(\bar{u},\bar{p})$ be the equilibrium.

(1.1) $(\bar{u},\bar{p})$ is *locally stable,* if and only if there exists $\delta>0$ such that for all (u(0), p(0)), $|(\bar{u}, \bar{p})-(u(0),p((0))| < \delta$ implies that as $t \to \infty$, $(u(t),p(t)) \to (\bar{u}, \bar{p})$.

(1.2) $(\bar{u},\bar{p})$ is *Lyapounov stable* , if and only if for all $\varepsilon > 0$ there exists $\delta>0$ such that $|(\bar{u},\bar{p})-(u(0),p((0))| < \delta$ implies that $|(\bar{u},\bar{p})-(u(t))| < \varepsilon$ for all t.

(1.3) $(\bar{u},\bar{p})$ is *globally stable,* if and only if for all (u(0),p(0)),as $t \to \infty$,$(u(t)),p(t)) \to (\bar{u},\bar{p})$.

(1.4) $(\bar{u},\bar{p})$ is *asymptotically locally stable,* if and only if it is (a) locally stable and (b) Lyapounov stable.

(1.5) $(\bar{u},\bar{p})$ is *asymptotically globally stable,* if and only if it is (a) globally stable and (b) Lyapounov stable.

Remark. Clearly, asymptotic global stability implies all the other types of stability.

Theorem. $(\bar{u},\bar{p})$, the equilibrium of our model, is asymptotically globally stable.

Ⅳ PROOF OF THEOREM

The Jacobian matrix, J, for the system of equations (5) and (6) is given by

$$J=\begin{bmatrix} 0 & g' \\ h''g+f'h' & h'g'+(k-1)f' \end{bmatrix}$$

It has a negative trace. Furthermore, the determinant of J evaluated at the equilibrium $(\bar{u}, \bar{p})$ is positive.[8] This establishes the following lemma.

Lemma 1. $(\bar{u}, \bar{p})$ is locally stable.

Next, we show

Lemma 2. $(\bar{u}, \bar{p})$ is asymptotically locally stable.

Proof: By Lemma 1 and definition (1.4), it is sufficient to prove Lyapounov stability of $(\bar{u}, \bar{p})$. This can be done by constructing a real valued "Lyapounov" function $V(u(t), p(t))$ satisfying the following conitions (see Hahn ([7]):

(i) $V(u(t), p(t)) \geqq 0$.

(ii) $V(u(t), p(t))=0$ if and only if $u(t)=\bar{u}$ and $p(t)=\bar{p}$.

(iii) $\dot{V}(u(t), p(t)) \leqq 0$.

Consider the following function V:

$$V \equiv \int_{\bar{p}}^{p(t)} g(x-\bar{p})dx - k\int_{\bar{u}}^{u(t)} f[\bar{p}(1-k^{-1})+k^{-1}h(x)]\,dx \tag{9}$$

By the fundamental theorem of integral calculus and the differentiability (and hence the continuity) of f, g, and h, the integrarls in (9) exist. Hence, V is well defined.[9] Furthermore, V has the following properties:

(a) $p(t)>$(resp.$<$) $\bar{p}$ if and only if $x>$(resp.$<$) $\bar{p}$ and hence, if and only if , $g(x-\bar{p})>$(resp.$<$) 0. Therefore, for all $p(t)\neq\bar{p}$,

$\int_{\bar{p}}^{p(t)} g(x-\bar{p})dx > 0$, and for $p(t)=\bar{p}$, $\int_{\bar{p}}^{p(t)} g(x-p)dx = 0$.

Similarly, remembering that $f[\bar{p}(1-k^{-1})+k^{-1}h(\bar{u})]0$ and $f'>0$, $h'<0$, it follows that for all $u(t) \neq u$, the integral

$-\int_{\bar{u}}^{u(t)} f[\bar{p}(1-k^{-1})+k^{-1}h(x)]\,dx$

is postitive, and 0 if $u(t)=\bar{u}$.

(b) Differentiating (9) and using (5) and (6), we have:

$$\begin{aligned}\dot{V}(u(t),\ p(t)) &= g(p-\bar{p})\dot{p} - kf[\bar{p}(1-k^{-1})+k^{-1}h(u)\dot{u} \\ &= g(p-\bar{p})[h'(u)g(p-\bar{p})+kf[\bar{p}(1-k^{-1})+k^{-1}h(u)]] \\ &\quad - kf[\bar{p}(1-k^{-1})+k^{-1}h(u)]\,g(p-\bar{p}) \\ &= g(p-\bar{p})k[f[p(1-k^{-1})+k^{-1}h(u)] - f[\bar{p}(1-k^{-1})+k^{-1}h(u)]] \\ &\quad + h'(u)[g(p-\bar{p})]^2 \leqq 0.\end{aligned} \tag{10}$$

[10]

(a) and (b) imply that V satisfies the conditions (i), (ii), and (iii), hence $(\bar{u}, \bar{p})$ is Lyapounov stable.[11]

Proof of Theorem: Let E be the locus of points (u, p) such that $\dot{V}=0$ and M the largest invariant set[12] contained in E. Note that the Lyapounov function introduced in the proof of Lemma 2 satisfies the following properties:

(i) $V(u(t),\ p(t)) > 0$, for $(u(t),\ p(t)) = (\bar{u},\ \bar{p})$,

(ii) $\dot{V}(u(t),\ p(t)) \leqq 0$,

(iii) $V(u(t),\ p(t) \rightarrow \infty$ as $u(t),\ p(t) \rightarrow \infty$.[13]

Hence, by Theorem VI in La Salle and Lefschetz ([9], p.66), all solutions approach M as $t \rightarrow \infty$. Thus, to complete the proof, it is sufficient for us to show that $M = \{(\bar{u}, \bar{p})\}$. By equation (10), E consists of all the points (u, p) such that $p=\bar{p}$. Clearly, $(\bar{u}, \bar{p}) \in M \subseteq E$. Assume to the contrary that the largest invariant set contained in E is not $\{(\bar{u}, \bar{p})\}$. Then choose a point $(\bar{\bar{u}}, \bar{\bar{p}}) \leftarrow M$ such that $(\bar{\bar{u}}, \bar{\bar{p}}) = (\bar{u}, \bar{p})$. Consider the solutin path $(u(t_0),\ p(t_0))$ such that $(u(t_0),\ p(t_0)) = (\bar{\bar{u}}, \bar{\bar{p}}) = (\bar{\bar{u}}, \bar{p})$. By the definitions of M and E, $p(t)=\bar{p}$. for any $t \geqq t_0$. Thus, $p(t)=\bar{p}$ for all $t>t_0$. By (5), moreover, $p(t)=\bar{p}$ implies $\dot{u}=0$. Thus, for all $t>t_0$, $\dot{u}=\dot{p}=0$. But this implies that for all $t>t_0$ $u(t)=\bar{u}$ and $p(t)=\bar{p}$. By continuity this implies $u(t_0)=\bar{u}$ and $p(t_0)=\bar{p}$, contradicting $(\bar{\bar{u}}, \bar{\bar{p}}) = (\bar{u}, \bar{p})$.

Footnotes

1) See Footnote 2 for the differences.
2) Vanderkamp's ([11]) model can be obtained by (a) omitting equstion (4), (b) assuming g in equation (3) to be linear, (c) $h(u)=u^{-1}$ and $k=1$ in equation (2), and (d) assuming that z is the rate of growth of money supply.
3) For further details, see for example, Dornbusch and Fischer ([6]).
4) From (7b), $\dot{p}$ is a decreasing function in p for any given value of u. Hence, for any u, there exists *at most* one p such that $p=0$. For sufficienctly large (resp. small) p, $f[p(1-k^{-1})+k^{-1}h(u)]$ is less (resp. greater) than $f(0)=$ (given u). Hence, given u, $\dot{p}$ is less (resp. greater) than 0 for sufficiently large (resp. small) p. By continuity, there *exists* a unique p such that for the given u, $p=0$. Hence, $\dot{p}=0$ defines a function $p=p(u)$.
5) Note that $u=\bar{u}$ and $p=\bar{p}$ imply $g=0$.
6) Local stability and Lyapounov stability are independent concepts: Neither implies the other (see Cesari([4], p. 96)).
7) See Footnote 6.
8) At points other than ones where $p=\bar{p}$, we have $g\neq 0$. Hence (under our assumptions) the sign of the determinant cannot be determined. This prevents us from applying "Olech's Theorem." (See Olech ([10]).)
9) For the function V to be well defined, it is necessary that $u(t)>0$ (since for $u(t)\leqq 0$, h(u) would not be defined). Given the initial value $u(0)>0$ of u, our assumptions imply that $u(t)>0$. Assume to the contrary that there exists $t=\bar{t}$ such that $u(\bar{t})=0$. (If $u(t)<0$ for some $t>0$, then by continuity such $\bar{t}$ exists.) Then, by (2), as $t\rightarrow\bar{t}$, $h(u(t))\rightarrow\infty$ and $p^*(t)\rightarrow-\infty$ (since $p(t)$ is finite). Pick an increasing sequnce $\{t_n\}$, $t_n\rightarrow\bar{t}$ (as $n\rightarrow\infty$) such that $\dot{p}^*(t_n)<0$ for all t_n. By (4), $\dot{p}^*(t_n)>0$ for sufficiently large n, since $[p(t_n)-p^*(t_n)]\rightarrow[p(t)-p^*(t)]=\infty$. This contradicts our choice of t_n.
10) The inequality follows from the fact that (a) $h'<0$ and (b) $g(0)=f(0)=0$ and $g'>0$ and $f'>0$ imply that p(resp.$<$)p^-, if and only if, $g\geqq$ (resp.$<$) 0. Also, $[f(p(1-k^{-1})+k^{-1}h(u))-f(\bar{p}(1-k^{-1}0+k^{-1}h(u))]\leqq$ (resp.$>$) 0, if and only if, $p(1-k^{-1})$ (resp.$>$) $\bar{p}(1-k^{-1})$, if and only if $p\geqq$ (resp.$<$) $\bar{p}$.
11) V in Lemma 2 just falls short of being the "standard" type of Lyapounov function needed to prove global stability, since $\dot{V}=0$ at $u\neq\bar{u}$ and $p=\bar{p}$.
12) An invariant set M is defined to be a set such that if a point x_0 belongs to the set, then its whole path (forward and backward) lies in M.
13) Assume that the sequence $\{t_n\}$, $t_n>t_{n-1}$, is such that either $|p(t_n)|$ or $|u(t_n)|$ becomes unbounded. If $p(t_n)\rightarrow\infty$, or $p(t_n)\rightarrow\infty$, then the first integral on R.H.S. of (9) remains negative. If $u(t_n)\rightarrow\infty$, then while the first integral remains positive, the secon one goes to $-\infty$. By (2) and footnote 9, $u(t)>0$ for all t. Hence, in any case we have: For the increasing sequence $\{t_n\}$, $V(t_n)\rightarrow\infty$, as $n\rightarrow\infty$.

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