The Marginal Efficiency of Capital and the Aggregate Investment Demand Schedule

Kwan Chi Oh*

Since Lord J.M. Keynes introduced the concept of the marginal efficiency of capital (MEC) in the economists' tool box, many writers clarified and showed various difficulties associated with the concept.[1] Thanks to Professor A.P. Lerner it is now generally recognized that the concet of MEC must be replaced by or relabeled as the marginal efficiency of investment (MEI) in deriving an investment demand schedule.[2] Both of MEC and MEI, however, hinge upon the concept of the internal rate of return which is defined as that rate of discount that makes the discounted income stream of an investment option equal to the value of the costs associated with the option. What if, then, there exist more than one internal rate of return, or there does not exist a positive real rate of return that equates the discounted value of the income stream to that of the costs for a specific investment option? In both cases the derivation of investment demand schedule from either MEC or MEI will encounter conceptual as well as practical difficulties.

Professors Lorie and Savage have shown that multiple internal rates of return for an investment option can exist. It is, however, Professor Jack Hirshleifer who has shaken the concept of the internal rate of return even more fundamentally by showing a possibility of nonexistence of it for a respectable investment option.[1] In this paper first, Keynes' MEC and Fisher's rate of return over cost will be briefly reviewed; second, non-existence of the internal rate of return for some particular cases of income and cost streams will be proved; and finally, a correct derivation of the aggregate

* Assistant Professor of Economics, Korea Military Academy

1. J. M. Keynes, *The General Theory of Employment, Interest and Money*, New York, 1936 pp.135-46.
2. A. P. Lerner, "On the Marginal Productivity of Capital and Marginal Efficiency of Investment," *Journal of Political Economy*, 61 (1953), pp. 1-14.
1. J.H. Lorie and L.J. Savage, "Three Problems in Rationing Capital," *Journal of Business* (October, 1955), pp. 236-9. Jack Hirshleifer, "On the Theory Optimal Investment Decision," *Journal of Political Economy* (August, 1958), pp. 329-52.

investment demand schedule will be discussed.

1. Keynes' MEC and Fisher's Rate of Return over Cost.

Keynes postulates that when a man buys a capital-asset, he obtains the right to the series of annuities R_1, R_2,......, R_n called the prospective yield of the investment. The cost of the capital-asset is not the market price at which an asset of the type in question can actually be purchased in the market, but the price which would just induce a manufacturer newly to produce an additional unit of such assets, i.e., what is sometimes called its replacement cost. He then defines MEC as being equal to that rate of discount which would make the present value the series of annuities given by the returns expected from the capital-asset during its life just equal to its supply price. Let the replacement cost of the capital-asset under consideration be denoted by C. Suppose that the amount C is paid in the beginning of the first period and that the annuities are received in the end of each period. Then the MEC is the discount rate denoted by r which equates the present value of the annuities R_1, R_2,......, R_n with the replacement cost C:

$$C=\frac{R_1}{1+r}+\frac{R_2}{(1+r)^2}+\cdots+\frac{R_n}{(1+r)^n}\cdots\cdots\cdots\cdots(1)$$

If the annuity is a continuous function of time denoted by $R(t)$, then the MEC is the discount rate satisfying the following equation:

$$C=\int_{t_0}^{t_n}R(t)e^{-rt}dt\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Equation (1) or (2) gives us MEC's of particular types of capital-assets The greatest of these MEC's can then be regarded as the MEC in general.

Keynes argues that the marginal efficiency of any type of capital will diminish as the investment in it is increased during any period of time partly because the prospective yield will fall as the supply of that type of capital increases, and partly because pressure on the facilities for producing that type of capital good will cause its marginal cost to increase. Thus for each type of capital a schedul which shows by how much investment in it will have to increase within the period in order that its marginal efficiency should fall to any given figure can be constructed. The investment demand schedule, or the schedule of the MEC in general can be derived by aggregating these schedules for all different types of capital-assets relating the rate of aggregate investment to the corresponding MEC in general which that rate of investment will establish. Thus it is clear that the rate of invesrment will be pushed to the point on the investment demand schedule where the MEC in general is equal to the market rate of interest.

In deriving the MEC Keynes acknowledges that his MEC is identical with Fisher's rate of return over cost.[1] In a strict sense, however, the two concepts are not identical as pointed out by Alchian.[2] Fisher argues that when we compare two investment options either may be preferable to the other according to the discount rates used to compute the present value of them; and thus the two options would stand on a par if the right intermediate rate were employed as a discount rate. He refers to this intermediate rate of discount which if used in calculating the present worth of the two options compared would equalize them or their differences, positive and negative differences between the income streams in each period over the horizon, the rate of return over cost.[3] Let $\underset{1}{\overset{1}{R}}, \underset{2}{\overset{1}{R}}, \ldots, \underset{n}{\overset{1}{R}}$ and $\underset{2}{\overset{1}{R}}, \underset{2}{\overset{2}{R}}, \ldots \ldots, \underset{n}{\overset{2}{R}}$ be the prospective yields for the n time periods on investment option 1 and 2 respectively. Then the rate of return over cost r is that rate of discount which satisfies the following equation:

$$\frac{R_1^1-R_1^2}{1+r}+\frac{R_2^1-R_2^2}{(1+r)^2}+\cdots\cdots+\frac{R_n^1-R_n^2}{(1+r)^n}=0\cdots\cdots(3)$$

If the annuities are assumed to be continuous functions of time, the rate of return over cost can be found by solving the equation:

$$\int_{t_0}^{t_n}[R^1(t)-R^2(t)]e^{-rt}dt=0 \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

Thus Keynes' MEC is not identical with Fisher's rate of return over cost in a strict sense. In comparing investment option 1 with option 2 Fisher looks at the differences between the two annuities $\underset{t}{\overset{1}{R}}-\underset{t}{\overset{2}{R}}$; the present value of the positive differences referred to as returns should be greater than that of the negative differences called cost when discounted at the going market rate of interest for investment option 1 to be preferable to option 2. If negative differences appear first followed by positive differences, that is, the investor receives positive returns after incurring costs, the present value of these differences will increase as the rate of discount falls; the present value will fall as the rate rises. The rational investor, therefore, chooses the investment option 1 in preference to option 2 if the present value of the differences is greater than zero at the going market rate of interest employed as the discount rate. Fisher, however, goes one step further where

1. J.M. Keynes, op. cit., pp. 140-1.
2. A.A. Alchian, "The Rate of Interest, Fisher's Rate of Return over Cost, and Keynes' Internal Rate of Return," *American Economic Review* (December, 1955), pp. 938-43.
3 Irving Fisher, *The Theory of Interest*, New York, 1936, pp. 158-77.

he errs. Suppose that the present value of the differences is greater than zero at the discount rate equal to the going market rate of interest. Since the present value falls as the rate of discount rises, it seems to be reasonable to assume that we can find a rate of return over cost greater than the market rate of interest. If this is so, ranking investment options by comparing the rate of return over cost with the market rate will be the same as ranking them by the present value.[1] Though Fisher painstakingly explains the existence of rate of return over cost greater than the market rate in the case under discussion by numerical example, he never actually proves it.[2]

Suppose that we have two investment options shown in Table 1. The computed rate of return over cost is one of the two values $r=1\pm\sqrt{i}$ and neither of which can be thought of as a rate of return over cost comparable with the market rate. This triggered economists to question the existence of internal rate of return upon which Keynes' aggregate investment demand schedule is founded.

TABLE 1 INVESTMENT OPTIONS COMPARED

	Option 1 $(\overset{1}{R}_t)$	Option 2 $(\overset{2}{R}_t)$	Difference $(\overset{1}{R}_t-\overset{2}{R}_t)$
1st year	2	3	−1
2nd year	9	5	4
3rd year	3	8	−5

2. Economically Meaningful Internal Rate of Return May not Always Exist.

Consider a discrete finite income stream denoted by $R_1, R_2, R_3, \ldots\ldots, R_{k-1}, R_k, \ldots\ldots, R_n$. It is not possible to state the general condition for the existence of the internal rate of return for any discrete finite income stream. In a simpler case, however, a condition for the existence of the internal rate of return can be derived.

Suppose that

$$R_i=\begin{cases}-a, & i\neq k,\ a\text{: a positive real number}\\ A, & i=k,\ A\text{: a positive real number.}\end{cases}$$

1. Irving Fisher, op. cit., p. 159.
2. Ibid., pp. 161-2.

Then the income stream becomes

$$-a,\ -a,\ -a,\ldots\ldots -a,\ A,\ -a,\ldots\ldots,\ -a \cdots\cdots\cdots\cdots\cdots (5)$$

Now we want to find a rate of discount at which the present value of the stream becomes zero, that is, we seek a discount rate satisfying the equation

$$-a\ -a(1+r)^{-1}-a(1+r)^{-2}-\ldots\ldots-a(1+r)^{-(k-2)}+A(1+r)^{-(k-1)}-(1+r)^{-k}$$
$$-a(1+r)^{-k+1}-\ldots\ldots-a(1+r)^{-(n-1)}=0.0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

By rearranging terms we get

$$L(r)\equiv a(1+r)^{k}-\frac{a}{(1+r)^{n-k}}=(A+a)r\equiv R(r)\cdots\cdots\cdots\cdots(7)$$

The existence of the internal rate of return for the income stream (1), therefore, implies that the left-hand function $L(r)$ crosses the right-hand $R(r)$, a linear function of r at least once. Since the slope of $R(r)$ is constant, the necessary condition for the $L(r)$ to cross $R(r)$ is as follows:

(i) $L''(r)>0$ if $L'(0)<R'(0)$ $0\leq r<\infty$

(ii) $L(r)$ should have an inflection point if $L'(0)>R'(0)$, $L''(0)>0$ $0\leq r<\infty$

(iii) $L''(r)<0$ if $L'(0)>R'(0)$, $0\leq r<\infty$

By differentiating $L(r)$ with respect to r we get

$$L'(r)=ka(1+r)^{k-1}+{}^{(n-k)}a(1+r)^{-(n-k+1)}\cdots\cdots\cdots\cdots\cdots\cdots(8)$$

Since $k>0$, $a>0$, and $k\leq n$, $L'(r)>0$ for any $r>0$, that is, $L(r)$ is monotonically increasing with r. Next, consider the second derivative $L''(r)$:

$$L''(r)=ak(k-1)(1+r)^{k-2}-a(n-k)(n-k+1)(1+r)^{-(n-k+2)}$$
$$=a(n-k)(n-k+1)(1+r)^{k-2}\left[\frac{k(k-1)}{(n-k)(n-k+1)}-{}^{(1+r)-n}\right]\cdots\cdots(9)$$

Since the maximum value of $(1+r)^{-n}=1.0$ for $0\leq r<\infty$, $L''(r)>0$ for $n>k$ if $k(k-1)>(n-k)(n-k+1)$. This implies that if

$$k>\frac{n+1}{2}\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(10)$$

then $L(r)$ is a strictly convex function in the domain $0\leq r<\infty$.

Since the the value of $L'(r)$ at $r=0.0$ is

$$L'(0)=ka+(n-k)$$
$$=na$$

$L(r)$ never cut through $R(r)$, that is, no positive real internal rate of return exists if

$$L'(0)=na>A+a\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

$$\text{and } k>\frac{n+1}{2}$$

In some cases, even though $L'(0)=na>Aa$, $L''(0)<0$, $L(r)$ has an infle-

TABLE 2 MONTHLY PAYMENT AND RECEIPT SCHEME OF 50,000 WON-20 POSITION SEQUENCE KYE

Month	Sequence position 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	**50000**	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-5000	-1850	-1650	-1550
2	-3350	**50000**	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
3	-3350	-3350	**50050**	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
4	-3350	-3350	-3300	**50100**	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
5	-3350	-3350	-3300	-3250	**50150**	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
6	-3350	-3500	-3300	-3250	-3200	**50200**	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
7	-3350	-3350	-3300	-3250	-3200	-3150	**50250**	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
8	-3350	-3350	-3300	-3250	-3200	-3150	-3100	**50350**	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
9	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	**50450**	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
10	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	**50500**	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
11	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	**50600**	-2650	-5250	-2450	-2300	-2150	-2000	-1850	-1650	-1550
12	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	**50700**	-2550	-2450	-2300	-2150	-2000	-1850	-1650	-1550
13	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	**50800**	-2450	-2300	-2150	-2000	-1850	-1650	-1550
14	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	**50900**	-2300	-2150	-2000	-1850	-1650	-1550
15	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	**51050**	-2150	-2000	-1850	-1650	-1550
16	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	**51200**	-2000	-1850	-1650	-1550
17	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	**51350**	-1850	-1650	-1550
18	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2510	-2000	**51500**	-1650	-1550
19	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2750	-2650	-2550	-2450	-2300	-2150	-2000	-1850	**51700**	-1550
20	-3350	-3350	-3300	-3250	-3200	-3150	-3100	-3000	-2900	-2850	-2570	-2650	-2550	-2450	-2300	-2150	-2000	-1850	-1650	**51800**
Total payments[a]	63650	63650	62700	61940	60800	59850	58900	57000	55100	54150	52250	50350	48450	46550	43700	40850	38000	35150	31350	29450

Source: The author's sample survey in Seoul, 1971.

[a]The sum of undiscounted monthly payments in current wons.

ction point above $R(r) \equiv (A+a)r$, and thereby preempting any possibility of the existence of the internal rate of return.

The above analysis has a direct bearing upon the structure of the rates of return in kyes.[1] As is well known the sequence structure of rates of return is embedded in the predetermined monthly-payment-receipt scheme of them. Table 2 shows one of the most typical sequence kyes prevailing in Seoul. The negative income stream stands for monthly payments and the positive for receipts. Simple arithmetic shows that sequence positions the 8th, 9th, 10th and 11th do not meet the conditions (10) and (11) simultaneously as summarized in Table 3 and Figure 1.

TABLE 3 CALCULATION OF THE INTERNAL RATE OF RETURN

(n=20)

Sequence position (k)	Monthly payments (a)	Receipt (A)	na	A+a
8	3,000wons	50,350wonss	60,000	53,350
9	2,900	50,450	58,000	53,350
10	2,850	50,500	57,000	53,350
11	2,750	50,600	55,000	53,350

Source: The author's sample survey in Seoul, 1971.

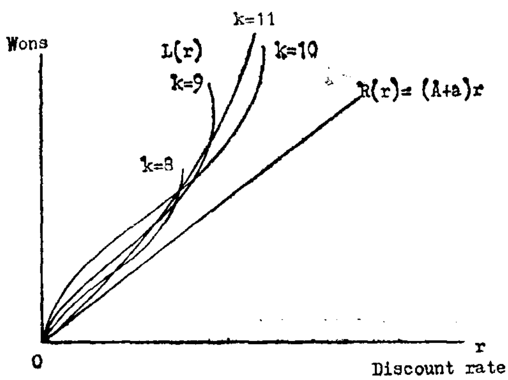

Figure 1

Consequently there do not exist the relevant internal rates of return for these sequence positions.

The problem under discussion is an extremely simple case of discrete

1. Kwan Chi Oh, "Two Essays on the Economics of Kye," *The Journal of the Korean Statistical Society* (June, 1974), pp. 31-57.

finite income and cost streams. Except this simple case, however, it seems to be not possible to investigate the existence of the internal rate of return analytically. One should solve the polynomial of high order numerically in most of cases and see whether there exists an internal rate of return or not.

Frequently economic models are assumed to consist of continuous functions of time. Suppose that $f(t)$ is a continuous function of time t representing a cost and return stream. As in the case of a general discrete finite stream it seems to be no general condition for the existence of the internal rate of return for $f(t)$ which can be derived analytically. There is, however, an illuminating simple case susceptible to analytical investigation, which indeed warns us to be careful in theorizing by use of supposition of the existence of the internal rate of return for any continuous cost-return stream. Consider the case that

$$f(t)=A \sin kt, \ A<0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (12)$$

be a cost-return stream. Then we want to find the rate of discount r which satisfies

$$\int_0^t A \sin kt\, e^{-rt} dt=0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (13)$$

This equation, however, cannot hold for any positive real rate r as we see from Figure 2.

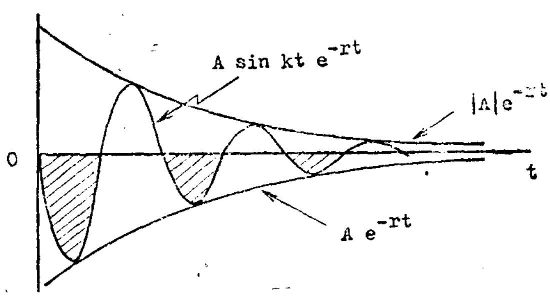

Figure 2

Figure 2 shows clearly that the negative area cannot be offset by the positive area whatever the time period may be. This implies that equation (13) does not have a positive real rate of return.

3. Construction of the Aggregate Investment Demand Schedule.

We have seen that the aggregate investment demand curve cannot be derived from individual investors' marginal efficiency of capital or marginal efficiency of investment schedules. It is therefore, natural to approach the problem from viewpoint of maximizing the capitalized value of the ownership capital.

Suppose that a representative firm faces a sufficiently large number of investment options each of which is characterized by an uncertain cost and return stream

$$\begin{pmatrix} R_1\ R_2\ R_3 \ldots\ldots R_n \\ C_1\ C_2\ C_3 \ldots\ldots C_n \end{pmatrix}$$

where the subscript stands for time period. We can, then, find the expected value of the ownership of the firm as the expected capitalized value of the net income stream,

$$V=\frac{1}{r_t}\sum_{i=1}^{m}\sum_{t=1}^{n}(R_t^i-C_t^i)$$

where R_t^i and C_t^i are the expected return and cost from the ith investment option in time period t respectively, and r_t the expected market rate of interest in time period t. Associated with V is the risk denoted by σ arising from uncertainties of r_t, R_t and C_t. On the other hand, a particular set of ordered pairs (V,σ) which form an opportunity frontier can be found for a given level of investment in each time period. Thus, the optimal investment decision problem is equivalent to maximizing V given a level of risk σ subject to the opportunity frontier. If we recognize the behavior of investors that they take investment options only when they are assured that the options will yield return at least not smaller than their required minimum return called safety level, and if we can identify σ with the standard deviation of V, the feasible set is bounded by the opportunity frontier and by the lower confidence limit

$$V-\lambda\sigma\geq L$$

where λ is a scalar and L the safety level for a given size of investments. Restricting our analysis to the present time $t=0$, the firm's optimal investment decision problem can be reduced to finding an optimal point n Figure 3.

In Figure 3, I_j represents the firm's indifference map as a risk averter' $F^i(V, \sigma)=0$ the opportunity frontier for a given level of market rate r_i, and the shaded area the feasible set. If the market rate of interest is, say, r_1, then a certain level of total investments will occur at $t=0$ and the subsequent investments will be planned thereafter corresponding to the optimum.

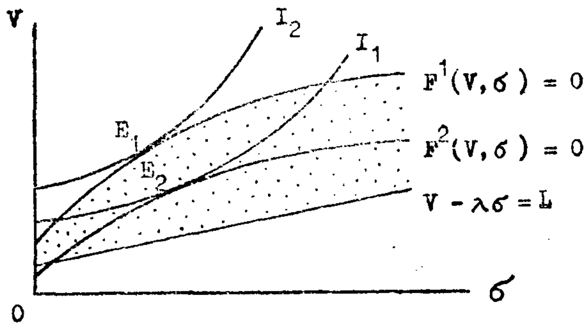

Figure 3

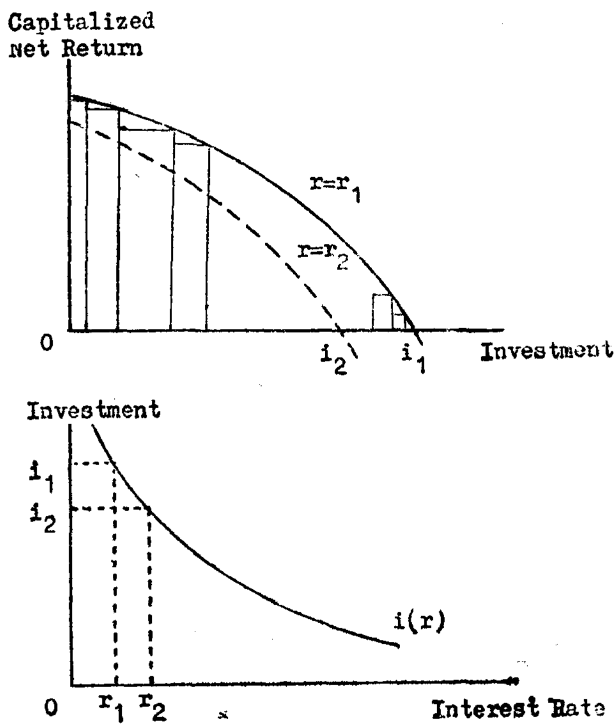

Figure 4

point E_1. If the market rate rises to r_2, then opportunity frontier will shift downward to $F^2(V,\sigma)=0$ and E_2 becomes the optimum point. Total investments at E_2 must be smaller than that one corresponding to E_1 because some of marginal investment options will begin to show negative net return as the market rate employed as a discount rate rises from r_1 to r_2. Consequently, the new opportunity frontier can be reached with smaller investments as those options with negative net return at r_2 are dropped out from the feasible set. The question whether a greater risk is taken by the firm as the market rate rises depends, of course, upon the degree of risk aversion of the firm. The summary is shown in Figure 4. Now, derivation of the aggregate investment demand schedule of the economy is completed by simply adding individual firms' schedules vertically. The aggregate investment demand schedule so derived can easily be fitted into the macroeconomic system and determine the equilibrium investment demand given the rate of interest of the money market.

4. Conclusion

We have shown first that the internal rate of return does not exist necessarily for any discrete or continuous cost-return flow. Therefore, before we can discuss the relevance of the internal rate of return in deriving the investment demand schedule, it may not exist at all. The aggregate investment demand schedule can be, however, logically derived from individual firms' schedules seeking optimum investment decisions as risk averters. Therefore, the usual aggregate investment demand schedule founded upon the concept of MEC or MEI must be replaced by the one we have presented here.

References

A. A. Alchian, "The Rate of Interest, Fisher's Rate of Return over Cost and Keynes' Internal Rate of Return," *American Economic Review* (December, 1955), pp, 938-43.

W.J. Baumol, "An Expected Gain-Confidence Limit Criteria for Portfolio Selection," *Management Science* (October, 1963), pp. 174-82.

Irving Fisher, *The Theory of Interest* (New York:Augustus M. Kelly), 1961.

Jack Hirschleifer, "On the Theory of Optimal Investment Decision," *Journal of Political Economy* (August, 1958), pp.329-52.

J.M. Keynes, *The General Theory of Employment, Interest, and Money* (New York: Harcourt, Brace and Co.), 1936.

Kwan Chi Oh, "Two Essays on the Economics of Kye," *the Journal of the Korean Statistical Society* (June, 1974), pp. 31-57.

J. H. Lorie and L.J. Savage, "Three Problems in Rationing Capital," *Journal of Business* (October, 1955), pp. 236-9.

A.P. Lerner, "On the Marginal Productivity of Capital and Marginal Efficency of Investment," *Journal of Political Economy*, 61 (1953), pp.1-14.