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ON THE CONVERGENCE OF LOGIT EQUILIBRIUM IN ORDER STATISTIC GAMES

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This paper identifies the conditions under which rational players can reach an inefficient equilibrium as if they make choices ignoring their own influences on the game outcome in a logit equilibrium model of order statistic game. It is shown that, if the number of players increases sufficiently faster than the noise parameter, the game outcome depends entirely on the prespecified order-statistic and the number of players, and inefficient outcome could result.

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I. INTRODUCTION

In an order-statistic game, a group of symmetric players simultaneously choose the levels of costly efforts. Each player's payoff is increasing in the prespecified order statistic of his own and others' efforts and decreasing in the distance between the resulting order statistic and his own effort. In these games, any configuration in which all players choose the same effort is a strict, symmetric, pure-strategy equilibrium, and these equilibria are Pareto-ranked. Other things equal, the closer subjects' efforts were to the order statistic, the higher their payoff, with all players preferring equilibria with higher efforts to those with lower efforts. However, there is a tension between the higher payoffs of the Pareto-efficient equilibrium and its greater fragility, which makes it riskier to

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play for the efficient equilibrium when others' responses are not perfectly predictable. These games capture important aspects of coordination problems in economic environments and resemble a number of economic models, including Roussau's stag-hunt game and the models of Keynesian coordination failure in Bryant (1983) and Cooper and John (1988).

Anderson et. al. (2001) and Yi (2003) showed that a noisy response model has a potential to describe those experimental results but they could not explain the differences across sessions where identical games are played. Recently, Yi (2005) showed that if players made choices without considering their own influences on the order statistic, an inefficient outcome could result. As a matter of fact, the possibility of inefficient outcome has a potential to explain the history dependence that appeared in the experiments as well as the inefficient outcome. However, it is not clear whether the "price-taking" behavior can be justified with rational players. The present paper identifies a condition under which price-takers and rational players reach the same outcome in noisy equilibrium framework.

Anderson et. al. (2001) and Yi (2003) analyzed McKelvey and Palfrey's (1995) notion of "quantal response equilibrium"(QRE) model of order statistic games in which players choose their strategies stochastically, with strategies that have higher expected payoffs chosen with higher probabilities retaining most of the parsimony of an equilibrium analysis. QRE allows a wide class of probabilistic choice rules to be substituted for perfect maximizing behavior in an equilibrium context. In applications of QRE, the probabilistic choice rule is often represented by a specific distribution, logit distribution and the associated QRE is called a "logit equilibrium," in which the amount of strategic uncertainty is measured by a single parameter. A "limiting logit equilibrium," the limit of logit equilibrium as the noise vanishes, which is a Nash equilibrium in the game without noise, is often compared to other notions of equilibrium.

In the standard QRE model of an order statistic game, the key to achieve full efficiency is the players' awareness of their own influences on the resulting order statistic. If players behaved as price takers or believed that the resulting order statistic was determined independent of their own effort choices, players would try to place their choice to the expected resulting order statistic as close as possible and thus each individual player has no incentive to raise his effort at all. Such a incentive structure can be thought of as a part of sources of history dependence and inefficiency, and it would be of interest whether price-taking behavior could be justified as a rational behavior. The analysis shows that if the number of players increases faster enough relative to the noise parameter, the limiting logit equilibrium depends entirely on the prespecified order-statistic and the number of players, and inefficient outcome could result. After all, a large number of players is not sufficient for price-taking behavior.

The rest of the paper is organized as follows. Section 2 introduces a class of order statistic games with bounded, continuous strategy spaces, and defines the notions of logit equilibrium with a brief summary of Yi's (2003) main result on standard logit equilibrium. Section 3 identifies the conditions that could lead the play to inefficient outcome. Section 4 concludes.

Ⅱ. ORDER STATISTIC GAMES AND LOGIT EQUILIBRIUM

Consider an n-person normal form game where the set of effort choices available to player i is denoted by $x_i \in X_i = [0, \bar{x}]$, $i = 1, \ldots, n$ with $X = \times_i X_i$. Let σ_i denote the set of all probability measures on X_i with $\sigma = \times_i \sigma_i$. The probability of player i playing a x_i is denoted by $\sigma_i(x_i)$. Since Yi (2003, 2004) showed that the logit equilibrium is symmetric in every order statistic game considered here, the player index i is dropped throughout the paper.

A commonly studied game in the literature is the order-statistic coordination games with Pareto-ranked equilibria. In this game, each of players chooses simultaneously among pure strategies called efforts, with increasing cost in the effort level, with a finite maximum effort level $\bar{x}$. In this game, the higher the value of the prespecified order statistic is, each player's payoff gets higher but the choice of the higher effort level accompanies a larger cost. Therefore, there is a tension between the higher payoffs of the Pareto efficient equilibrium and its greater fragility, which makes it riskier to play for the efficient equilibrium when others' response are not perfectly predictable.

Each player is assumed to have a risk-neutral preference. Let $u(x, m)$ be a player's payoff when he plays x and the prespecified order statistic is m. In this paper, I use a specific functional form which has been used in Van Huyck et al. (1991, 2001).

$$u(x, m_{j:n}) = am_{j:n} - b(m_{j:n} - x)^2 + c, \quad a, b, c \geq 0$$

where $m_{j:n}$ is the j^{th} order statistic of n choices. Clearly, the higher the $m_{j:n}$ and the smaller the distance between $m_{j:n}$ and x_i, the higher a player's payoff. If $m_{j:n}$ is the median, this payoff function represents Keynes' (1936, p156) average opinion problem in newspaper beauty contests and in stock markets. When no confusion will arise, $u(x,m)$ is denoted by $u(x)$.

In this paper, I focus on a specialized version of QRE where the probability density of a player's choosing x is a function of the expected payoff $\pi^e(x)$ and the density of each choice is an increasing function of the expected payoff for that choice:

$$f(x) = \frac{\exp(\lambda \pi^e(x))}{\int_0^x \exp(\lambda \pi^e(y))\, dy} \tag{1}$$

where $0 \le \lambda \le \infty$ measures the amount of noise, or equivalently, the degree of rationality. This functional form is called a logit function where the odds are determined by the exponential transformation of the utility times a given non-negative constant λ. A logit equilibrium for λ is attained when the distribution of behavior of all players is consistent with their logit responses so that in equilibrium f are mutually "noisy best responses."

To complete the probabilistic choice rule, Equation (1), we need an explicit form of the expected payoff, $\pi^e(x)$. Let $F(x)$ denote the cumulative distribution function associated with $f(x)$. Let $G_{j:n-1}(x)$ be the cumulative distribution function of j^{th} order statistic of choices drawn from distribution, F. Let $g_{j:n-1}(x)$ be the associated probability density function. As Yi (2003) showed, the part of expected payoff that is relevant to logit equilibrium (only the part that depends on x) is

$$\pi(x) = a\left[x - \int_0^x G_{1:n-1}(y)\, dy\right] + 2b\left[\int_0^x (y-x) G_{1:n-1}(y)\, dy\right]. \tag{2}$$

in the minimum game and when $2 \le j \le n-1$,

$$\begin{aligned}\pi_i(x_i) = {} & 2bE(m_{j-1:n-1})x - bx^2 \\ & + \int_0^x (a - 2by + 2bx)(G_{j-1:n-1}(y) - G_{j:n-1}(y))\, dy. \end{aligned} \tag{3}$$

The effort density function is constructed by substituting Equations (2) and (3) into Equation (1).

In every order statistic game including the minimum game with $a>0$, Yi (2003) showed that the limiting logit equilibrium is the most efficient Nash equilibrium, $x=\bar{x}$ for all i. The intuition behind this result is that a sufficiently small increase in one's effort the benefit always dominates the cost. That is, at $x^*=E(m_{j:n}\mid x^*)$, raising an effort level always can increase the expected value of $m_{j:n}$ while the associated marginal penalty is negligible as the penalty increases quadratically. This implies that if players are aware of their own influence on the expected order statistic, this small "tilt" in favor of higher efforts tips the balance of the benefit and the cost in favor of a more efficient equilibrium.

In many game situations, however, as the number of players is sufficiently large, it is usual that individual players ignore their own influences on the summary statistic. Although it is not the case in the logit equilibrium model of the order statistic games, it is still possible that players behave as price takers if the number of player increases fast enough relative to the noise parameter λ.

Ⅲ. THE LIMIT OF LOGIT EQUILIBRIUM

The main result in Yi (2003) is that the limiting outcome of an order statistic game is efficient if n is finite and players take into account the influences of their own choices on the resulting order statistic. Although it is more difficult for players to achieve full efficiency with larger n and smaller j, since the marginal expected benefit always dominates the marginal expected cost, the efficiency result does not depend on the values of n and j. However, if the number of players increases without a bound, the influence of each individual player's choice on the resulting order statistic should diminish. Therefore, if λ does not increase sufficiently fast, raising effort level would be risky because there is little chance to increase the resulting order statistic. In average opinion problems with increasing numbers of participants, it could make a big difference in efficiency of the outcome how fast they learn to make fewer mistakes.

The following analysis does not consider minimum and maximum games because in any circumstances, every single player could affect the resulting order statistic and no one would ignore his own influence on the resulting minimum or maximum.

Proposition 1 *In order statistic games with* $2 \leq j \leq n-1$, *if* n_t *and* λ_t *go to infinity with* $\frac{a\lambda_t}{n_t} \to \infty$ *and* $\frac{j_t}{n_{t+1}} \equiv q$ *where* $q = \frac{j}{n+1}$, *logit equilibrium converges to the most efficient Nash equilibrium.*

Proof. Choosing $n_t - t(n+1)$ makes j_t and n_t integers. When $j \geq 2$, Since $\frac{\partial f(x)}{\partial \pi} = \frac{\partial f(x)}{\partial \pi}\frac{\partial \pi(x)}{\partial \pi}$, by integrating both sides,

$$
\begin{aligned}
f_t(\bar{x}) &\geq f_t(0) + 2\lambda_t b(E_t(m_{j_t-1:n_t-1}) - E_t(x)) \\
&\quad + \lambda_t a \int_0^{\bar{x}} (G_{j_t-1:n_t-1}(y) - G_{j_t-1:n_t-1}(y)) f_t(y)\, dy \\
&= f_t(0) + 2\lambda_t b(E_t(m_{j_t-1:n_t-1}) - E_t(x)) \\
&\quad + \lambda_t a \int_0^{\bar{x}} \binom{n}{j_t} [F_t(y)]^{j_t-1} [1 - F_t(y)]^{n_t - j_t} f_t(y)\, dy \\
&= f_t(0) + 2\lambda_t b(E_t(m_{j_t-1:n_t-1}) - E_t(x)) + \frac{\lambda_t a}{n_t} G_{j_t:n_t}(\bar{x}) \\
&= f_t(0) + \lambda_t \left[2bE_t(m_{j_t-1:n_t-1}) - E_t(x)) + \frac{a}{n_t} \right]
\end{aligned}
$$

Since the variance of x, denoted by s^2, is order of λ^{-2}, and $-s\sqrt{\frac{n-j}{j}} \leq E_t(m_{j_t-1:n_t-1} - E_t(x)) \leq s\sqrt{\frac{j-1}{n-j+1}}$ (Wolkowicz and Styan, 1979) or $-s\sqrt{\frac{1-q}{q}} \leq E_t(m_{j_t-1:n_t-1} - E_t(x)) \leq s\sqrt{\frac{q}{1-q}}$, $E_t(m_{j_t-1:n_t-1} - E_t(x))$ is, at least, of order $O(\lambda^{-1})$ and by the assumption $\frac{a}{n_t}$ is of order $o(\lambda^{-\varepsilon})$ for some $0 < \varepsilon < 1$.

Therefore, $f_t(\bar{x})$ diverges and the result follows. ■

Proposition 1 implies that if λ grows fast enough so that $\frac{\partial E(m_{j:n})}{\partial x}$ remains big enough, the limiting logit equilibrium should be efficient. Therefore, a large number of players is not sufficient for a justification for the price-taking behavior in order statistic games. For an intuition, consider the following:

$$
\begin{aligned}
\frac{\partial \pi^e(x = E(m_{j:n}))}{\partial x} &= a\frac{E(m_{j:n})}{\partial x} \\
&\quad - b\frac{\partial[E((m_{j:n} \mid x = E(m_{j:n})) - E(m_{j:n}))^2]}{\partial x_i}
\end{aligned}
$$

$$= a\frac{E(m_{j:n})}{\partial x} - b\frac{\partial var(m_{j:n} \mid x = E(m_{j:n}))}{\partial x}$$

When a player increases the effort level slightly above $E(m_{j:n})$, the expected payoff is affected through two ways, the expected value and the variance of the order statistic. For a given n no matter how large it is, if λ is sufficiently large, one's choice has a little influence on the variance while it has a relatively large effect on the expected value, which is essentially what Yi (2003) showed. Because of such a feature of expected payoff, under the condition stated in Proposition 1, the net benefit of raising an effort level remains positive. However, if n grows sufficiently faster than λ, the effect of an effort choice on the variance would become larger since a player's choice could barely affect the expected value so that players have no incentive to raise their effort level.

Proposition 2 *In order statistic games with* $2 \le j \le n-1$, *if* n_t *and* λ_t *go to infinity with* $n_t > \lambda_t^{2+\varepsilon}$ *for any* $\varepsilon > 0$ *and* $\frac{j_t}{n_{t+1}} \equiv q$ *where* $q = \frac{j}{n+1}$, *logit equilibrium effort density of a order statistic game converges to a point-mass at 0,* $\frac{\bar{x}}{2}$, *and* $\bar{x}$ *when* $q < \frac{1}{2}$, $q = \frac{1}{2}$, *and* $q < \frac{1}{2}$, *respectively.*

Sketch of the proof. For the proof, let's first define "competitive" logit equilibrium. If players ignore the own influences on the resulting order statistic such that $\frac{\partial E(m_{j:n})}{\partial x} = 0$, then a player makes a effort choice based on the expectation of $E_{ct}(m_{j_t:n_t})$ that depends only on other players' choices, where the subscript ct denote the associated expectation of the j_t^{th} order statistic with n_t players. In the proof, it is assumed that $E_{ct}(m_{j:n}) = E(m_{j:n})$. With finite number of players, $E_c(m_{j:n}) = E(m_{j:n})$ does not make a sense but as long as $E(m_{j-1:n-1}) \le E_{ct}(m_{j:n}) \le E(m_{j:n-1})$, the exact value does not matter in the limit. Then the result follows from the following three lemmas, whose proofs are in Appendix.

Lemma 1. A logit equilibrium effort density converges uniformly on $[0, \bar{x}]$ to the competitive logit equilibrium effort density as λ and n go to infinity.

Lemma 2. A competitive logit equilibrium effort density converges uniformly to that of the q-quantile game, where there is a continuum of players and the payoff is determined by the quantile.

Lemma 3 (Yi, 2004). The logit equilibrium of a q-quantile game converges to a point-mass at 0, $\frac{\bar{x}}{2}$, and $\bar{x}$ when $q<\frac{1}{2}$, $q=\frac{1}{2}$, and $q>\frac{1}{2}$, respectively.

Ⅳ. CONCLUDING REMARK

Several models have been proposed to explain the experimental results of Van Huyck et al.'s (1991, 2001) order statistic games such as noisy response (Anderson et al. 2001), experimentation (Van Huyck et al. 2001), and strategic teaching (Camerer et al., 2002). Since noisy response model of QRE is not comparable with the inefficiency results in the experiments, the present paper examines whether it can be justified in the limit of increasing number of players, and shows that it requires infinite number of players. However, the analysis suggests that the key condition for the inefficient outcome is the players ignorance of their own influences on the resulting order statistic. This suggests that a QRE has a potential to explain the inefficient outcome even with a finite number of players if they behave as price takers as they do in daily lives.

APPENDIX

Proof of Lemma 1. Let $q=\frac{j}{n+1}$ and (j_t, n_t) be increasing sequences such that $\frac{j_t}{n_t}+1\equiv q$, $t=1,2,\ldots$, with $n_t=t(j_t+1)$. Then j_t and n_t are integers. Then the associated expected payoff is

$$E_{ct}(\pi_{ct}(x))=aE_{ct}(m_{j_t:n_t})-b(E_{ct}(m_{j_t:n_t})-x)^2+c$$

and the relevant part of the expected payoff to the logit response function is

$$\pi_c^e(x)=b[E_c(m_{j:n})x-x^2].$$

As each player believes that the resulting order statistic is determined independent of his own choice, the object becomes how closely he can place the effort choice to the resulting order statistic. Since $\frac{\partial f(x)}{\partial \pi}=\frac{\partial f(x)}{\partial \pi}\frac{\partial \pi(x)}{\partial \pi}$, by integrating both sides from 0 to x, the corresponding competitive equilibrium effort density satisfies

$$f_{ct}(x)=f_{ct}(0)+2\lambda_t b\left[E_t(m_{j_t:n_t})F_{ct}(x)-\int_0^x yf_{ct}(y)dy\right] \tag{4}$$

A logit equilibrium effort density is

$$\begin{aligned}f_t(x)=f_t(0)&+2\lambda_t b\left[E_t(m_{j_t-1:n_t-1})F_c(x)-\int_0^x yf_t(y)dy\right]\\&+2\lambda_t b\int_0^x\int_0^y(G^t_{j_t-1:n_t-1}(z)-G^t_{j_t:n_t-1}(z))dzf_t(y)+\frac{a_t\lambda}{n_t}G^t_{j_t:n_t}(x)\end{aligned} \tag{5}$$

and the last term vanishes under the assumption. For the convergence, it is sufficient to show that $\lambda_t\mid E_t(m_{j_t-1:n_t-1})-E_t(m_{j_t:n_t-1})\mid\rightarrow 0$ because, then, from Equation (4) or (5), $f_t(x)$ converges to either

$$f_t(x)\rightarrow f_t(0)+2\lambda_t b\left[E_t(m_{j_t-1:n_t-1})F_c(x)-\int_0^x yf_t(y)dy\right]$$
$$f_t(x)\rightarrow f_t(0)+2\lambda_t b\left[E_t(m_{j_t:n_t-1})F_c(x)-\int_0^x yf_t(y)dy\right]$$

Since $E_t(m_{j_t-1:n_t}) < E_t(m_{j_t-1:n_t-1}) < E_t(m_{j_t:n_t}) < E_t(m_{j_t:n_t-1}) < E_t(m_{j_t+1:n_t})$, by sandwich theorem, it is sufficient to show $\lambda_t \mid E_t(m_{j_t-1:n_t}) - E_t(m_{j_t+1:n_t}) \mid \to 0$ for the convergence.

In the proof, the expected value of each order statistic is approximated by Taylor series expansion. The precision in terms of n is shown in David and Johnson (1954), but for the precision in terms of λ, following exercise is necessary.

The probability integral transformation, $u = F(x)$, transforms the order statistic $m_{j:n}$ from a continuous population with distribution function $F(x)$ into the uniform order statistic $U_{j:n}$ on $[0,1]$. Hence, by inverting the above transformation, we have

$$m_{j:n} = F^{-1}(U_{j:n}) = Q(U_{j:n})$$

By Taylor's theorem, there exists a $\tilde{q} \in [\min[U_{j:n}, q], \max[U_{j:n}, q]]$ such that

$$m_{j:n} = Q(q) + Q'(q)(U_{j:n} - q) + \frac{1}{2} Q''(\tilde{q})(U_{j:n} - q)^2. \tag{6}$$

The central moments of uniform order statistics are

$$E(U_{j:n}) = q,\ E(U_{j:n} - E(U_{j:n}))^2 = \frac{1-q}{n+2}.$$

By taking expectation on both sides of Equation (6) and using the values of central moments,

$$E(m_{j:n}) = Q(q) + \frac{q(1-q)}{2(n+2)} Q''(\tilde{q}) \tag{7}$$

and in a logit equilibrium,

$$Q'(q) = \frac{1}{f(Q(q))},\quad Q''(\tilde{q}) = \frac{f'(Q(\tilde{q}))Q(\tilde{q})}{f(Q(\tilde{q}))^2} = \frac{\lambda\pi'(Q(\tilde{q}))}{f(Q(\tilde{q}))^2}$$

Therefore,

$$E_t(m_{j_t-1:n_t}) - E_t(m_{j_t+1:n_t}) = Q_t\left(q - \frac{1}{n_t+1}\right) - Q_t\left(q - \frac{1}{n_t+1}\right) + \frac{q(1-q)}{2(n_t+2)}\left[Q_t''(\hat{q}) - Q_t''(\check{q})\right] \tag{8}$$

where

$$\hat{q} \in [\min[F_t(E(m_{j_t-1:n_t})), q], \max[F_t(E(m_{j_t-1:n_t})), q]]$$
$$\check{q} \in [\min[F_t(E(m_{j_t-1:n_t})), q], \max[F_t(E(m_{j_t-1:n_t})), q]]$$

Since $F_t(Q_t(q)) > 0$, $f_t(Q_t(q))$, $f_t(E_t(m_{j_t:n_t}))$ is strictly positive for all j_t and n_t and $\pi'(x)$ is bounded. Thus both $Q''(\hat{q})$ and $Q''(\check{q})$ are of order $O(\lambda)$ and $O(n^0)$. Therefore, under the assumption that $n > \lambda^{2+\varepsilon}$, if $\lambda_t\left[Q_t\left(q - \frac{1}{n_t+1}\right) - Q_t\left(q + \frac{1}{n_t+1}\right)\right] \to 0$, $\lambda_t[E_t(m_{j_t-1:n_t}) - E_t(m_{j_t+1:n_t})] \to 0$. Since

$$Q_t\left(q + \frac{1}{n_t+1}\right) = Q_t(q) + \frac{1}{n_t+1} Q_t'(q) + \frac{1}{2(n_t+1)^2} Q_t''(\bar{q})$$
$$Q_t\left(q - \frac{1}{n_t+1}\right) = Q_t(q) - \frac{1}{n_t+1} Q_t'(q) + \frac{1}{2(n_t+1)^2} Q_t''(\check{q})$$

where $q \le \bar{q} \le q + \frac{1}{n_t+1}$ and $q + \frac{1}{n_t+1} \le \check{q} \le q$, we have

$$Q_t\left(q + \frac{1}{n_t+1}\right) - Q_t\left(q - \frac{1}{n_t+1}\right) = \frac{2}{n_t+1} \frac{1}{f_t(Q(q))}$$

Using the same argument above, one can show that $Q_t''(\bar{q})$ and $Q_t''(\check{q})$ are of order $O(\lambda)$, and thus the logit equilibrium converges uniformly to the competitive logit equilibrium. ■

Proof of Lemma 2. For the result, it is sufficient to show that, for every $\varepsilon > 0$, there exists a $T > 0$ such that $|E_{t_1}(m_{j_{t_1}:n_{t_1}}) - E_{t_2}(m_{j_{t_2}:n_{t_2}})| < \varepsilon$ for all $T < t_1 < t_2$. Then, using identical steps in the proof of Lemma 1, one can show that $\lambda\,|E_{t_1}(m_{j_{t_1}+1:n_{t_1}}) - E_{t_1}(m_{j_{t_1}-1:n_{t_1}})| \to 0$ as $T \to \infty$. From Equation (7),

$\lambda \mid E_{ct}^{-1}(q) - E_t(m_{j_t:n_t}) \mid \to 0$ as $t \to \infty$ and $f_{ct}(x)$ converges uniformly to the corresponding quantile game.

Suppose $E_{t_1}(m_{j_{t_1}:n_{t_1}}) < E_{t_2}(m_{j_{t_2}:n_{t_2}})$ for a given λ. Since f_{ct} is uniformly distributed when $\lambda = 0$, $E_t(m_{j:n_t}) = \bar{x}\frac{j_t}{n_t+1}$ and $E_{t_1}(m_{j_{t_1}+1:n_{t_1}}) > E_{t_2}(m_{j_{t_2}:n_{t_2}})$. Since $E_t(m_{j_t:n_t})$, and f_{ct_1} and f_{ct_2} are identical.

$$Q_{ct}\left(q + \frac{1}{n_{t_1}+1}\right) = Q_{ct}(q) + \frac{1}{f_{ct}(Q_{ct}(q))} + \frac{1}{2(n_t+1)^2} Q_{ct}''(\hat{q}).$$

By substituting this into Equation (7),

$$E_{t_1}(m_{j_{t_1}+1:n_{t_1}}) = Q_{ct,j_{t_1}+1:n_{t_1}}(q) + \frac{1}{(n_{t_1}+1) f_{ct}(Q_{ct,j_{t_1}+1:n_{t_1}}(q))} \tag{9}$$
$$+ \frac{1}{2(n_t+1)^2} Q''_{ct,j_{t_1}+1:n_{t_1}}(\hat{q}) + \frac{1}{2(n_t+1)^2} Q''_{ct,j_{t_1}+1:n_{t_1}}(\check{q})$$

$$E_{t_1}(m_{j_{t_2}:n_{t_2}}) = Q_{ct,j_{t_2}+1:n_{t_2}}(q) + \frac{1}{2(n_t+1)^2} Q''_{ct,j_{t_2}+1:n_{t_2}}(\check{q})) \tag{10}$$

Following identical argument before, one can show that $f_{ct_1}(Q_{ct,j_{t_1}+1:n_{t_1}}(\hat{q}))$, $f_{ct_1}(Q_{ct,j_{t_1}+1:n_{t_1}}(\check{q}))$, and $f_{ct_2}(Q_{ct,j_{t_2}+1:n_{t_2}}(\check{q}))$ are strictly positive. Since f_{ct_1} and f_{ct_2} are identical, $Q_{ct,j_{t_1}+1:n_{t_1}}(q) = Q_{ct,j_{t_2}+1:n_{t_2}}(q)$. Comparing the orders of the terms in the right hand sides of Equations (9) and (10) shows that there exists a T such that for every $T < t_1 < t_2$, $E_{t_1}(m_{j_{t_1}+1:n_{t_1}}) \geq E_{t_2}(m_{j_{t_2}+1:n_{t_2}})$.

Similarly, $E_{t_1}(m_{j_{t_1}-1:n_{t_1}}) \leq E_{t_2}(m_{j_{t_2}:n_{t_2}})$ and we have $E_{t_1}(m_{j_{t_1}-1:n_{t_1}}) \leq E_{t_2}(m_{j_{t_2}:n_{t_2}}) \leq E_{t_1}(m_{j_{t_1}+1:n_{t_1}})$. By combining this with $E_{t_1}(m_{j_{t_1}-1:n_{t_1}}) \leq E_{t_1}(m_{j_{t_1}:n_{t_1}}) \leq E_{t_1}(m_{j_{t_1}+1:n_{t_1}})$, we have $\mid E_{t_1}(m_{j_{t_1}:n_{t_1}}) - E_{t_2}(m_{j_{t_2}:n_{t_2}}) \mid \leq E_{t_1}(m_{j_{t_1}+1:n_{t_1}}) - E_{t_1}(m_{j_{t_1}-1:n_{t_1}})$. ■

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