

COMPETITIVE EQUILIBRIUM WITH SHORT-SELLING AND NONTRANSITIVE PREFERENCES*

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The notion of arbitrage is predominantly used as a conceptual framework in finance and economics for studying equilibrium as well as pricing relations of asset markets. The consequences of equilibrium analysis based upon arbitrage, however, do not apply to the case with nontransitive preferences simply because the literature assumes the transitivity of preferences. We propose a weak notion of arbitrage which provides a sufficient condition for the existence of equilibrium of the economy where preferences need be neither transitive nor complete and the consumption sets need not be bounded from below.

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I. INTRODUCTION

The consumption set need not be bounded from below in an asset market economy where unlimited short sales are allowed. One important example is the capital asset pricing model developed by Sharpe (1964), Lintner (1965) and Markowitz (1952). The existence of a Walrasian equilibrium with unbounded-from-below choice sets was initially addressed in Hart (1974) who introduced a condition on preferences eventually known as a no arbitrage condition. To

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generalize the condition of Hart (1974) on preferences, different arbitrage notions have been introduced in the literature (see for example, Hammond (1983), Page (1987), Werner (1987), Chichilnisky (1995), Dana *et al.* (1999), Page *et al.* (2000), Allouch (2002), among others). The arbitrage conditions are not only sufficient but also necessary for the existence of a Walrasian equilibrium in certain cases. It is important to note that all the equilibrium existence results with unbounded consumption sets make use of the assumption of transitivity of preferences. Thus, the consequences of the literature do not apply to the case with nontransitive preferences.

The transitivity of preference has been regarded as an essential element for individual rationality. Such belief may not hold in the face of decision making under uncertainty.¹ Moreover, when agents such as households, firms, and financial institutions behave as group decision makers, the nontransitivity of preferences frequently occurs due to the aggregation problem of individuals' preferences. Such a phenomenon cannot be dismissed as a consequence of irrational choices. Indeed, substantial evidence of the nontransitivity of aggregate or group preferences is reported in the economics, psychology and politics literature. They do some justice to the development of general equilibrium theory for economies with bounded-below consumption sets and nontransitive preferences in 1970's.²

The purpose of this paper is to prove the existence of a competitive equilibrium in an exchange economy with unbounded consumption sets and nontransitive preferences. Consequently, the paper generalizes not only Gale and Mas-Colell (1975, 1979) to the economies with unbounded consumption sets but also Allouch (2002) to the economies with nontransitive preferences. To this end, we introduce a notion of 'weak bounded arbitrage' which is weaker than the CPP condition of Allouch (2002).

This paper is organized as follows. In the next section, we describe the economy where consumers have nontransitive preferences. The main theorem is stated and proved in Section III, which is followed by concluding remarks.

¹ For example, Grether and Plott (1979) and Loomes *et al.* (1991) detect the preference reversal phenomenon through experiments, which is not congruent to the transitivity of preferences. Preference reversals arise when an agent prefers lottery A to lottery B but sets a higher selling price on B than on A.

² See Gale and Mas-Colell (1975, 1979) and Shafer and Sonnenschein (1975).

II. THE MODEL

We consider an exchange denoted by $E = (X_i, \succeq_i, e_i)_{i \in I}$, where each component will be defined below. Let $I = \{1, 2, \dots, m\}$ denote the set of consumers. The consumption set X_i in which consumer i chooses his consumptions is a subset of the Euclidean space and need not be bounded from below. The preferences of agent i are denoted by \succeq_i , which is reflexive but may be incomplete or nontransitive. For a point $x_i \in X_i$, we define the upper contour set (weakly preferred set) $R_i(x_i)$ and the strict upper contour set (preferred set), respectively, as follows: $R_i(x_i) = \{x'_i \in X_i : x'_i \succeq_i x_i\}$ and $P_i(x_i) = \{x'_i \in X_i : x'_i \succ_i x_i\}$. Consumer i is endowed with a consumption bundle e_i . An array $x = (x_1, \dots, x_m)$ of consumptions is called an allocation of the economy E . We set $X = \prod_{i \in I} X_i$. The set X contains all the allocations of E . As usual, the open budget set of agent i is defined as $B_i(p) = \{x_i \in X_i : p \cdot x_i < p \cdot e_i\}$. We define notions of equilibrium for the exchange economy E .

DEFINITION 2.1 :

1. A *competitive equilibrium* for E is a pair $(p, x) \in (\mathbb{R}^{\ell} \setminus \{0\}) \times X$ such that (i) $x_i \in \text{cl } B_i(p)$ for all $i \in I$,³ (ii) $P_i(x_i) \cap \text{cl } B_i(p) = \emptyset$ for all $i \in I$, (iii) $\sum_{i \in I} (x_i - e_i) = 0$.
2. A *quasi-equilibrium* for E is a pair $(p, x) \in (\mathbb{R}^{\ell} \setminus \{0\}) \times X$ such that (i) $x_i \in \text{cl } B_i(p)$ for all $i \in I$, (ii) $P_i(x_i) \cap B_i(p) = \emptyset$ for all $i \in I$, (iii) $\sum_{i \in I} (x_i - e_i) = 0$.

An allocation is said to be feasible when it satisfies the third condition of competitive equilibrium and to be individually rational when it is contained in $R(e) := \prod_{i \in I} R_i(e_i)$. Let A denote the individually rational feasible allocations, i.e.,

$$A = \{x \in X : x_i \in R_i(e_i), \forall i \in I \text{ and } \sum_{i \in I} (x_i - e_i) = 0\}.$$

³ For a set S , $\text{cl } S$ denotes the closure of S and $\text{co } S$ is the convex hull of S .

We define the *augmented preference correspondence* $\widehat{P}_i: X_i \rightarrow 2^{X_i}$ by

$$\widehat{P}_i(x_i) = \{(1-\alpha)x_i + \alpha x_i' : \alpha \in (0, 1], x_i' \in \text{co}P_i(x_i)\}.$$

It is obvious that \widehat{P}_i is convex-valued. Moreover we have $P_i(x_i) \subset \text{co}P_i(x_i) \subset \widehat{P}_i(x_i)$ for all $x_i \in X_i$. Note that the reference set of $\widehat{P}_i(x_i)$ is not $P_i(x_i)$ but $\text{co}P_i(x_i)$, which differs from those of Gale and Mas-Colell (1975, 1979) and Allouch (2002). We make the following assumptions.

ASSUMPTIONS : For every $i \in I$,

- A1. X_i is a closed convex set in \mathbb{R}^{ℓ} .
- A2. $x_i \notin \widehat{P}_i(x_i)$ for every $x_i \in X_i$.
- A3. $\widehat{P}_i(x_i) \neq \emptyset$ for every $x_i \in X_i$.
- A4. \widehat{P}_i is lower hemicontinuous on X_i .⁴
- A5. $P_i(x_i)$ is open in X_i for every $x_i \in X_i$.⁵
- A6. $e_i \in \text{int}X_i$.

First of all, we do not assume that the preferences are either transitive or complete. We use the augmented preference correspondence \widehat{P}_i instead of P_i , which is in the same spirit as Allouch (2002). Since $P_i(x_i) \subset \text{co}P_i(x_i) \subset \widehat{P}_i(x_i)$, it may seem that our assumptions on P_i are weaker than those of Allouch (2002). However, it is the case that Assumptions A2 and A3 are equivalent to the corresponding Allouch's assumptions. It is straightforward that Assumption A2 is stronger than the corresponding Allouch's weak convexity assumption but they turn out to be equivalent. Indeed, if $x_i \notin \text{co}P_i(x_i)$, the definition of $\widehat{P}_i(x_i)$ implies that $x_i \notin \widehat{P}_i(x_i)$. In addition, one can notice that Assumption A2 implies that P_i is irreflexive, i.e., $x_i \notin P_i(x_i)$. Moreover, it is easy to see that Assumption A3 is equivalent to the nonsatiation assumption: $P_i(x_i) \neq \emptyset$, $\forall x_i \in X_i$. Consequently, only Assumption A4 is weaker than Allouch's corresponding assumption that P_i has lower open sections.⁶

⁴ A correspondence $\varphi: X \rightarrow 2^Y$ is said to be *lower hemicontinuous* if the set $\{x \in X: \varphi(x) \cap V \neq \emptyset\}$ is open in X for every open set in V in Y .

⁵ In this case, P_i is said to *have open upper sections* on X_i .

Assumptions A5 and A6 are sufficient for a quasi-equilibrium to become a competitive equilibrium.

III. THE MAIN RESULTS

Allouch (2002) introduces the compactness with partial preorder (CPP) condition to extend the condition of compact utility set of Dana *et al.* (1999) to the case with incomplete and transitive preference relations. Here we generalize Allouch's CPP condition to the case with incomplete and nontransitive preferences.

DEFINITION 3.1. (Allouch, 2002) : The exchange economy E satisfies *compactness with partial preorder* (CPP) if for every sequence $\{x^n\}$ in A , there exists a subsequence $\{x^{n_k}\}$ and a sequence $\{y^{n_k}\}$ in X such that $y^{n_k} \rightarrow y \in A$ and $y_i^{n_k} \in \tilde{P}_i(x_i^{n_k})$ for every n_k and for every $i \in I$, where $\tilde{P}_i(x_i) = \{(1-\alpha)x_i + \alpha x_i' : \alpha \in (0, 1], x_i' \in P_i(x_i)\}$.⁷

One can observe that $\tilde{P}_i(x_i)$ may not be convex-valued. It is obvious that $P_i(x_i) = coP_i(x_i) \subset \tilde{P}_i(x_i) = \hat{P}_i(x_i)$ as well as $\tilde{P}_i(x_i)$, and $\hat{P}_i(x_i)$ are convex, whenever $P_i(x_i)$ is convex. Moreover, one can easily verify that, when the preferences are locally nonsatiated,⁸ $P_i(x_i) = \tilde{P}_i(x_i) \subset coP_i(x_i) = \hat{P}_i(x_i)$. Consequently, we can deduce that, if the preferences are locally nonsatiated and Assumption A2 is satisfied, it holds true that $P_i(x_i) = \tilde{P}_i(x_i) = coP_i(x_i) = \hat{P}_i(x_i)$.

DEFINITION 3.2. : The exchange economy E satisfies *weak bounded arbitrage* (WBA) if for every sequence $\{x^n\}$ in A , there exists a subsequence $\{x^{n_k}\}$ and

⁶ A correspondence $\varphi: X \rightarrow 2^Y$ is said to have *lower open sections* if the set $\varphi^{-1}(y) := \{x \in X : y \in \varphi(x)\}$ is open in X . It is well known that if P_i has open lower sections, then P_i is lower hemicontinuous, which implies that coP_i is lower hemicontinuous. It is easy to show that coP_i is lower hemicontinuous if and only if \hat{P}_i is lower hemicontinuous.

⁷ It should be noticed that we use different notation from that in Allouch (2002).

⁸ The preference of consumer i is said to be *locally nonsatiated* if for every $\varepsilon > 0$, $B_\varepsilon(x_i) \cap P_i(x_i) \neq \emptyset$, $\forall x_i \in X$, where $B_\varepsilon(x_i)$ is an open ε -ball with center x_i .

a sequence $\{y^{n_k}\}$ in X such that $y^{n_k} \rightarrow y \in A$ and $\hat{P}_i(y^{n_k}) \subset \hat{P}_i(x_i^{n_k})$ for every n_k and for every $i \in I$.

Like the CPP condition, the WBA condition will be used to overcome the problem of unboundedness by requiring every sequence in A to be dominated by a preferred subsequence converging to an attainable allocation in terms of preferred sets. The reason why we employ the correspondence \hat{P}_i instead of \tilde{P}_i in the WBA condition is that it is already used to provide weaker assumptions than those of Allouch (2002). Moreover, the WBA condition is weaker than the CPP condition when the preferences are transitive and convex.

PROPOSITION 3.1. : Suppose that Assumption A2 holds and R_i is convex-valued for every $i \in I$. If the preference are transitive, the CPP condition implies the WBA condition.

PROOF : First, we claim that P_i is convex under Assumption A2. Suppose that for some $x_i \in X_i$, $\text{co}P_i(x_i) \neq P_i(x_i)$. Then there exists $z_i \in \text{co}P_i(x_i) \setminus P_i(x_i)$. Since $\text{co}P_i(x_i) \subset R_i(x_i)$, this implies that $z_i \in R_i(x_i) \setminus P_i(x_i)$. Thus, $P_i(z_i) = P_i(x_i)$ and therefore $\text{co}P_i(x_i) = \text{co}P_i(z_i)$. Recalling that $z_i \in \text{co}P_i(x_i)$, we have $z_i \in \text{co}P_i(z_i)$. Since $\text{co}P_i(z_i) \subset \hat{P}_i(z_i)$, this implies that $z_i \in \hat{P}_i(z_i)$, which contradicts Assumption A2.

Suppose the CPP condition holds and consider a sequence $\{(x^{n_k}, y^{n_k})\}$ in $X \times X$ which satisfies the CPP condition. Because P_i is convex-valued, $\tilde{P}_i(x_i) = \hat{P}_i(x_i)$ for every $x_i \in X_i$ and therefore we have only to show that $y^{n_k} \in \hat{P}_i(x_i^{n_k})$ implies that $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$ for each n_k . From the fact that R_i is convex-valued, it follows that $\hat{P}_i(x_i^{n_k}) \subset R_i(x_i^{n_k})$. Since $y_i^{n_k} \in \hat{P}_i(x_i^{n_k})$ by the CPP condition, we have $y_i^{n_k} \in R_i(x_i^{n_k})$. There will be two cases: either (i) $y_i^{n_k} \in P_i(x_i^{n_k})$ or (ii) $y_i^{n_k} \in R_i(x_i^{n_k}) \setminus P_i(x_i^{n_k})$. If $y_i^{n_k} \in P_i(x_i^{n_k})$, by transitivity, we have $P_i(y_i^{n_k}) \subset P_i(x_i^{n_k})$ and therefore $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$. Suppose that $y_i^{n_k} \in R_i(x_i^{n_k}) \setminus P_i(x_i^{n_k})$. Then $P_i(y_i^{n_k}) = P_i(x_i^{n_k})$ and therefore $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$. Hence the WBA condition holds. ■

When Assumption A3 is added to the assumptions of Proposition 3.1, a stronger result can be obtained as follows.

PROPOSITION 3.2. : Suppose that Assumptions A2-A3 are satisfied, R_i is convex-valued for every $i \in I$, and preferences are transitive. Then the CPP condition holds if and only if the WBA condition does.

PROOF : Due to Proposition 3.1., it suffices to show the sufficiency. Since Assumption A2 is satisfied, we know that $P_i(x_i) = coP_i(x_i) \subset \bar{P}_i(x_i) = \hat{P}_i(x_i)$ for every $x_i \in X_i$. Consider a sequence $\{(x^{n_k}, y^{n_k})\}$ in $X \times X$ which satisfies the WBA condition. Then under Assumption A3, there is a sequence $\{\hat{y}^{n_k}\}$ in X such that $\hat{y}_i^{n_k} \in \hat{P}_i(y_i^{n_k})$ for each n_k and for each $i \in I$. Now let us take a sequence $\{\tilde{y}^{n_k}\}$ converging to $y \in A$ such that for each $i \in I$ and for each n_k ,

$$\tilde{y}_i^{n_k} = \left(1 - \frac{1}{\max\{k+1, \|\hat{y}_i^{n_k}\|^2\}}\right) y_i^{n_k} + \left(\frac{1}{\max\{k+1, \|\hat{y}_i^{n_k}\|^2\}}\right) \hat{y}_i^{n_k} \in \hat{P}_i(y_i^{n_k}).$$

Since $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$ by the WBA condition and $\hat{P}_i(x_i^{n_k}) = \tilde{P}_i(x_i^{n_k})$, we have $\tilde{y}_i^{n_k} \in \tilde{P}_i(x_i^{n_k})$ for all n_k . Hence, the CPP condition is satisfied. ■

Allouch (2002) documents the relationships between various notions of arbitrage and the CPP condition. They can be easily enriched via Proposition 3.1. and 3.2. to incorporate the WBA condition under the transitivity of preferences.⁹

When the preferences are nonsatiated and representable by a strict quasi-concave and upper semicontinuous utility function, the CPP condition is a necessary and sufficient condition for the utility set $U := \{u \in \mathbb{R}^m : u = (u_1(x_1), \dots, u_m(x_m)), x \in A\}$ to be compact. By Proposition 3.2, the same is true for the WBA condition. An advantage of the WBA condition over the CPP condition, however, lies in the fact that the latter does not apply to the case with nontransitive preferences.

When proving the existence of quasi-equilibrium, we shall rely on the following fixed point theorem which is given by Gale and Mas-Colell (1975,

⁹ For details, see Allouch (2002).

1979).

THEOREM 3.1. (Gale and Mas-Colell, 1975, 1979) : Let T_k be a nonempty compact convex subset of the finite dimensional Euclidean space and $T = \prod_{k \in K} T_k$. Let $\varphi_k: T \rightarrow 2^{T_k}$ be lower hemicontinuous. Then there is $t^* \in T$ such that $t_k^* \in \text{co} \varphi_k(t^*)$ or $\varphi_k(t^*) = \emptyset$ for every k .

Take an increasing sequence $\{K_n\}$ of compact convex cubes with center 0 such that $\{e_1, \dots, e_m\} \subset K_1$. Let us define $X_i^n = X_i \cap K_n$ and $X^n = \prod_{i \in I} X_i^n$. Consider the sequence of truncated economies $\{E^n = (X_i^n, \succsim_i, e_i)_{i \in I}\}$. Let us define the correspondence $B_i^n: \Delta \rightarrow 2^{X_i^n}$ with

$$B_i^n(p) = \{x_i \in X_i^n : p \cdot x_i < p \cdot e_i + 1 - \|p\|\}.$$

Note that the correspondence $B_i^n: \Delta \rightarrow 2^{X_i^n}$ is lower hemicontinuous on Δ . Let $z = \sum_{i \in I} z_i$ where $z_i = x_i - e_i$. Let us define correspondences $\gamma_i^n: \Delta \rightarrow 2^{X_i^n}$, $\varphi_i^n: \Delta \times X^n \rightarrow 2^{X_i^n}$ for every $i \in I$, and $\varphi_0^n: \Delta \times X^n \rightarrow 2^\Delta$ as follows:

$$\begin{aligned} \gamma_i^n(p) &= \begin{cases} \{e_i\}, & \text{if } B_i^n(p) = \emptyset, \\ \text{cl } B_i^n(p), & \text{otherwise,} \end{cases} \\ \varphi_i^n(p, x) &= \begin{cases} \gamma_i^n(p), & \text{if } x_i \notin \text{cl } B_i^n(p), \\ \hat{P}_i(x_i) \cap B_i^n(p), & \text{otherwise,} \end{cases} \\ \varphi_0^n(p, x) &= \{q \in \Delta : q \cdot z > p \cdot z\}. \end{aligned}$$

The following Lemma and its proof are a slight modification of those of Allouch (2002).

LEMMA 3.1. : Under Assumptions A1-A5, the following hold.

1. For every $i \in I$ and every n , γ_i^n is lower hemicontinuous with nonempty convex values.
2. For every $i \in I \cup \{0\}$ and every n , φ_i^n is lower hemicontinuous.

PROOF : See Allouch (2002) for proving that γ_i^n is lower hemicontinuous for

every $i \in I$ and φ_0^n is lower hemicontinuous for every n . Assuming that γ_i^n is lower hemicontinuous for every $i \in I$ and every n , we will show φ_i^n is lower hemicontinuous for every $i \in I$ and n . For every $i \in I$ and every n , define $C_i^n = \{(p, x) \in \Delta \times X : x_i \notin c\ell B_i^n(p)\}$, which is open in $\Delta \times X$. It is obvious that φ_i^n is lower hemicontinuous on C_i^n since $\varphi_i^n(p, x) = \gamma_i^n(p)$ for all $(p, x) \in C_i^n$ and γ_i^n is lower hemicontinuous on Δ . On the other hand, for every $(p, x) \notin C_i^n$, $\varphi_i^n(p, x) = \hat{P}_i(x_i) \cap B_i^n(p)$. Observe that, by Lemma 4.2. of Yannelis (1987), the correspondence $\hat{P}_i \cap B_i^n : \Delta \times X \rightarrow 2^{X_i}$ with $(\hat{P}_i \cap B_i^n)(p, x) = \hat{P}_i(x_i) \cap B_i^n(p)$ is lower hemicontinuous since P_i is lower hemicontinuous and \hat{B}_i^n has open graph. Hence, φ_i^n is lower hemicontinuous. ■

Thanks to Lemma 3.1., we are ready to provide the existence theorem of quasi-equilibrium.

THEOREM 3.2. : Suppose that the economy E satisfies the WBA condition. Then under Assumptions A1-A4, E has a quasi-equilibrium.

PROOF : By Lemma 3.1. and Theorem 3.1, there exists $(p^n, x^n) \in \Delta \times X^n$ for each n such that (1) $x_i^n \in c\ell B_i^n(p^n)$, (2) $\hat{P}_i(x_i^n) \cap B_i^n(p^n) = \emptyset$, (3) $p^n \cdot z^n \geq p \cdot z^n, \forall p \in \Delta$, where $z^n := \sum_{i \in I} z_i^n$ with $z_i^n = x_i^n - e_i$. Since $\{p^{n_k}\}$ is bounded, without loss of generality, we can assume that $p^{n_k} \rightarrow p^*$ for some $p^* \in \Delta$.

We claim that for every $i \in I$, $x_i^n \in R_i(e_i)$ for sufficiently large n . Suppose otherwise, i.e., $e_i \in P_i(x_i^n)$ for some i . Since $e_i \in \text{int} X_i^n$ for sufficiently large n , we have $ae_i + (1-a)x_i^n \in X_i^n$ for a close to 1. Since $P_i(x_i^n) \subset \text{co} P_i(x_i^n)$, however, it holds that $e_i \in \text{co} P_i(x_i^n)$. Thus we have $ae_i + (1-a)x_i^n \in \hat{P}_i(x_i^n)$. Hence $ae_i + (1-a)x_i^n \notin c\ell B_i^n(p^n)$, a contradiction.

Next we need to show that $z^n = 0$ for every n . Suppose otherwise. Then from (3), it follows that $p^n \cdot z^n > 0$ and $\|p^n\| = 1$. But this and (1) implies $p^n \cdot z^n \leq 0$, a contradiction. Hence, $x^n \in A$ for sufficiently large n .

By the WBA condition, there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ and a sequence $\{y^{n_k}\}$ converging to $y \in A$ such that $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$ for every n_k

and for every $i \in I$. We will show that $\|p^*\| = 1$ and $p^* \cdot y_i = p^* \cdot e_i$, $\forall i \in I$. Since $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$, it follows that (2) implies $\hat{P}_i(y_i^{n_k}) \cap B_i^{n_k}(p^{n_k}) = \emptyset$. Since $\{y_i^{n_k}\}$ is bounded and $\hat{P}_i(y_i^{n_k}) = \emptyset$ according to Assumption A3, we have $y_i^{n_k} \in \text{int}K_{n_k}$ and $\hat{P}_i(y_i^{n_k}) \cap K_{n_k} \neq \emptyset$ for sufficiently large n_k . Thus, we can choose $\hat{y}_i^{n_k} \in \hat{P}_i(y_i^{n_k}) \cap \text{int}K_{n_k}$. Then $\lambda \hat{y}_i^{n_k} + (1-\lambda)y_i^{n_k} \in \hat{P}_i(y_i^{n_k}) \cap \text{int}K_{n_k}$ with $\lambda \in (0, 1]$. Since $\lambda \hat{y}_i^{n_k} + (1-\lambda)y_i^{n_k} \notin B_i(p^{n_k})$, we have $p^{n_k} \cdot (y_i^{n_k} - e_i) \geq 1 - \|p^{n_k}\|$ by taking $\lambda \rightarrow 0$. Taking limits for n_k , we obtain $p^* \cdot (y_i - e_i) \geq (1 - \|p^*\|)$. Since $y \in A$, it follows that $\|p^*\| = 1$ and $p^* \cdot y_i = p^* \cdot e_i$, $\forall i \in I$.

Finally, it suffices to show that $P_i(y_i) \cap B_i(p^*) = \emptyset$, $\forall i \in I$. Due to Assumption A3, we can take $z_i \in \hat{P}_i(y_i)$. By Assumption A4, there is a sequence $\{z_i^{n_k}\}$ such that $z_i^{n_k} \rightarrow z_i$ and $z_i^{n_k} \in \hat{P}_i(y_i^{n_k})$. Since $\hat{P}_i(y_i^{n_k}) \subset \hat{P}_i(x_i^{n_k})$, it holds that $z_i^{n_k} \in \hat{P}_i(x_i^{n_k})$. Passing to the limit, by (2) we have $p^* \cdot (z_i - e_i) \geq 0$. Thus, it must be the case that $\hat{P}_i(y_i) \cap B_i(p^*) = \emptyset$, which implies that $P_i(y_i) \cap B_i(p^*) = \emptyset$. Hence (p^*, y) is a quasi-equilibrium ■

In an exchange economy with transitive preferences, Allouch (2002) shows the existence of quasi-equilibrium without making nonsatiation assumption like A3 in an explicit way. But it should be noticed that the nonsatiation assumption is embedded in the CPP condition.¹⁰

Clearly, every competitive equilibrium of the economy E is its quasi-equilibrium. Conversely, every quasi-equilibrium of the economy E is its competitive equilibrium under certain conditions. Theorem 3.2 leads to the following theorem which is a main result of the paper. The proof is standard but is given for completion.

THEOREM 3.3. : Suppose that the economy E satisfies the WBA condition. Then, under Assumptions A1-A6, there exists a competitive equilibrium of E .

PROOF : Let $(p^*, x^*) \in \mathcal{A} \times X$ be a quasi-equilibrium obtained by Theorem 3.2. Suppose $x_i \in P_i(x_i^*)$ but $p^* \cdot x_i = p^* \cdot x^*$ for some $i \in I$. Since $e_i \in \text{int}X_i$ by

¹⁰ This is one of the distinctions between the WBA condition and the CPP condition.

Assumption A6, there is, a vector $\hat{x}_i \in X_i$ such that $p^* \cdot \hat{x}_i < p^* \cdot e_i$. This implies that $p^* \cdot [(1-\alpha)x_i + \alpha\hat{x}_i] < p^* \cdot e_i$ but $(1-\alpha)x_i + \alpha\hat{x}_i \in P_i(x_i^*)$ for sufficiently small $\alpha > 0$ due to Assumption A5. That is, $P_i(x_i^*) \cap B_i(p^*) \neq \emptyset$, which contradict the fact that (p^*, x^*) is a quasi-equilibrium. ■

IV. CONCLUDING REMARKS

We extend Allouch's model to show the existence of competitive equilibrium of the economy where consumption sets are need not to be bounded from below and preferences are neither transitive nor complete. The WBA condition is introduced, which is weaker than the CPP condition. Moreover, the set of assumptions are imposed on a weaker form of augmented preference correspondences.

Here we restrict ourselves to an exchange economy where one consumer's choices do not affect the preferences of other consumers. The result of the paper can be extended to the case where the preferences are interdependent among consumers or productions are involved. They would be topics for future research.

REFERENCES

- Allouch, N. (2002), "An Equilibrium Existence Result with Short Selling," *Journal of Mathematical Economics*, 37, 81-94.
- Chichilnisky, G. (1995), "Limited Arbitrage is Necessary and Sufficient for the Existence of a Competitive Equilibrium with or without Short Sales," *Economic Theory*, 5, 79-108.
- Dana, R.A., C. Le Van and F. Magnien (1999), "On the Different Notion of Arbitrage and Existence of Equilibrium," *Journal of Economic Theory*, 87, 169-193.
- Gale, D. and A. Mas-Colell (1975), "An Equilibrium Existence Theorem for a General Model without Ordered Preferences," *Journal of Mathematical Economics*, 2, 9-15.
- Gale, D. and A. Mas-Colell (1979), "Correction to an Equilibrium Existence Theorem for a General Model without Ordered Preferences," *Journal of Mathematical Economics*, 6, 297-298.
- Grether, D.M. and C.R. Plott (1979), "Economic Theory of Choice and the Preference Reversal Phenomenon," *American Economic Review*, 69, 623-638.
- Hart, O.D. (1974), "On the Existence of Equilibrium in a Securities Model," *Journal of Economic Theory*, 9, 293-311.
- Hammond, P.J. (1983), "Overlapping Expectations and Hart's Condition for Equilibrium in a Securities Model," *Journal of Economic Theory*, 31, 170-175.
- Lintner, J. (1965), "The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, 47, 13-37.
- Loomes, G., C. Starmer and R. Sugden (1991), "Observing Violations of Transitivity by Experimental Methods," *Econometrica*, 59, 425-439.
- Markowitz, H. (1952), "Portfolio Selection," *Journal of Finance*, 7, 77-91.
- Page, F.H. Jr. (1987), "On Equilibrium in Hart's Securities Exchange Model," *Journal of Economic Theory*, 41, 392-404.
- Page, F.H. Jr., M.H. Wooders and P.K. Monteiro (2000), "Inconsequential Arbitrage," *Journal of Mathematical Economics*, 34, 439-469.
- Shafer, W.J. and H. Sonnenschein (1975), "Equilibrium in Abstract Economies

without Ordered Preferences,” *Journal of Mathematical Economics*, 2, 345-348.

Sharpe, W.F. (1964), “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk,” *Journal of Finance*, 19, 425-442.

Werner, J. (1987), “Arbitrage and the Existence of Competitive Equilibrium,” *Econometrica*, 55, 1403-1418.

Yannelis, N.C. (1987), “Equilibria in Noncooperative Models of Competition,” *Journal of Economic Theory*, 41, 96-111.