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BAND SPECTRUM LEAST SQUARES IN FRACTIONAL COINTEGRATION MODELS WITH UNKNOWN FRACTIONAL INTEGRATION ORDERS*

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Band spectrum regression procedure in a bivariate model of fractional nonstationary cointegration is proposed. Both variables and cointegrating error in the system are assumed to be fractionally integrated processes. The proposed estimator can reduce bias by modifying a frequency domain regression, and it is just a simple least squares and easy to use. Unlike other available estimation procedures, the estimator is free from any preliminary estimation of short memory components and fractional parameter. It is also expected to be less volatile and more reliable, which can be confirmed by finite sample performances. A limited version of asymptotic theory will be developed and some simulation results will also be provided.

JEL Classification: C22
Keywords: Fractional Cointegration, Band Spectrum Regression, Unknown Long memory Parameters, Frequency Domain Least Squares

Ⅰ. INTRODUCTION

Recently, fractionally integrated long memory processes have received much attention as more general time series characterization, and a broader concept of nonstationarity based on fractional processes has been developed in the time series econometrics. Since numerous statistical inferences and analyses have recently established about long memory and fractionally integrated processes,

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analyzing cointegration among fractionally integrated processes can be of much interest as a generalization of standard cointegration.

The chance of a long run stable relationship among nonstationary (possibly stationary) variables indeed exists even though variables in a given model are not following unit roots. The slow decay of the effect of shocks in long memory processes with allowance for the eventual adjustment to an equilibrium level can be viewed as an important characteristic of fractionally integrated processes, and this can be incorporated in the econometric modeling. In fact, nonstationary fractional processes under particular fractional parameter ranges still have mean reverting property which is potentially useful for the modeling of long-run economic equilibrium.

Consider a linear combination of variable z_t of the form

$$\alpha z_t = u_t \tag{1}$$

where α is an unknown $p \times 1$ vector, and u_t has a lower integration order than z_t, not necessarily weakly stationary process. Then z_t is said to be fractionally cointegrated, and u_t is referred as to cointegrating error. The integration order of z_t is not necessarily one or integer, but can be any real number, for example, between 0.5 and 2 whereas standard cointegration requires z_t and u_t to be I(1) unit root and I(0). Fractional modeling can extend the concept of standard cointegration and pose lots of interest in applications of economic models. Lots of methods developed for AR based cointegration analysis presume the known order of integration like I(1) unit root variable z_t and I(0) weakly stationary variable u_t, but most of the established procedures seem to lose their properties when the presumed integration orders are not correct. Therefore, it is necessary to explore fractional cointegration models in which unknown (or possibly pre-specified) orders of integrations are allowed.

The analysis of fractional cointegration models has been an interesting issue along with statistical inferences of long memory fractional processes in the past decade, and different testings and estimations of fractional cointegration models are now available in the literature, for example, Robinson and Hualde (2003), Marinucci and Robinson (2001b), Davidson and de Jong (2002) among others. It is known that ordinary least squares (OLS) is consistent in nonstationary fractional cointegration models (see, for example, Robinson and Marinucci

(2003)), OLS, however, suffers from second order biases as shown, for example, in Kim and Phillips (2002) under specific model characterization. Therefore, alternative approaches other than OLS are desirable to improve the performance of estimation in the fractional cointegration models since the precise estimation of a fractional cointegration model like (1) is important in testings and empirical applications.

This paper proposes a band spectrum regression procedure in a bivariate model of fractional cointegration. Since Hannan's (1963) band spectrum regression, the procedure is a very useful apparatus that has been used in many regression models. Band spectrum regression is especially attractive to estimate cointegration relations which describe low frequency or long run relations among economic variables. In this paper, we propose a bias reduced modified version of band spectrum regression in a bivariate fractional model free from any preliminary estimates unlike other estimation methods available. The estimation method suggested in the present paper adopts the idea of bias reduction in standard cointegration model as will be explained later, and we will show that the bias reduction procedure in the standard cointegration still works when we extend the standard I(1)/I(0) cointegration into fractional modeling with I(d) variables. In the following section, we present the model and modified band spectrum regression with limited asymptotics of the proposed estimator. In the present paper, only consistency of proposed estimator will be provided, not limit distribution which has not been established yet. We also look at the finite sample performances of the estimator compared with those of OLS under different fractional cointegration model specifications. Complete asymptotics with limit distribution will be a part of subsequent work.

Ⅱ. MODEL AND ASSUMPTIONS

(1) Fractional Cointegration Model

We consider the following model for the bivariate observed series (y_{1t}, y_{2t}) as

$$y_{1t} = \beta y_{2t} + u_{1t}, \tag{2}$$

where y_{2t} and u_{1t} are defined by

$$
\begin{aligned}
&(1-L)^{d_1} u_{1t} = e_{1t}, \\
&(1-L)^{d_2} y_{2t} = e_{2t}, \quad \Delta y_{2t} = u_{2t}\,.
\end{aligned} \tag{3}
$$

The fractional differencing operator can be defined as

$$
(1-L)^{-d} = \sum_{k=0}^{t} \frac{(d)_k}{k} L^k, \tag{4}
$$

where $(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)}$ is Pochhammer's symbol for the forward factorial function and L is lag operator. The model (2) considered in the present paper is just a simple form of a bivariate fractional cointegration model that has been considered in the literature. It should be noted that the definition of fractionally integrated processes here is not the partial sum of stationary processes, but direct fractionally integrated processes defined by the filtering in (4). In fact there are two different types of nonstationary fractional processes in the literature, one can be defined as a partial sum of stationary processes and the other follows the fractional differencing operator as in (3). Fractional cointegration models similar to (3) have been considered in Marinucci (2003). Robinson and Marinucci (1998, 2001b), Robinson and Hualde (2003) and Tanaka (1999) while Chan and Terrin (1995), Dolado and Marmol (1996) and Jeganathan (1999, 2001) follow the definition using partial sums. Based on two different definitions of nonstationary fractional cointegration models, the limit behavior can also be characterized accordingly in the theoretical development (See Marinucci and Robinson (1999)). The difference, however, may not be important from a practical point of view because one can hardly tell which formulation can be applied to the data set in the given model.

(2) Assumptions

Since the nonstationary fractional cointegration model in (2) is the main focus of the present paper, the following restrictions on the fractional parameters should be imposed,

$$
0 \le d_1 < \frac{1}{2}, \frac{1}{2} < d_2 < 2\,. \tag{5}
$$

The restrictions on the fractional parameter naturally give the idea of fractional cointegrations, and the model given in (2) contains a fractional cointegration relationship between y_{1t} and y_{2t} since the order of integration in regression error should be strictly less than the order of integration in regressors y_{2t} under the (5). We therefore exclude the case of fractional stationary cointegration in which regressors follow the fractional order of integrations in the stationary range between 0 and $\frac{1}{2}$. Since a cointegration is trying to explain the comovement of economic variables in the long run, and most macro and financial economic variables have been considered to retain some nonstationarity properties even though they are not unit roots variables, nonstationary fractional cointegrations can be more practical and important related to the application of economic models than stationary cointegrations. Recently, a stationary cointegration model was considered by Nielsen (2003) based on the local Whittle estimation which was established by Robinson (1995b). The restrictions on fractional parameters in (5) are quite mild and commonly used. An additional restriction, however,

$$d_2 - d_1 > \frac{1}{2}$$

should be imposed to develop further asymptotics just like in almost all the models considered in the literature (e.g. Kim and Phillips (2002), Robinson and Marinucci (2001b, 2003), Robinson and Hualde (2003)).[1]

The short memory components of given fractional processes e_{1t} and e_{2t} are linear processes of the form

$$e_{it} = C_i(L)\varepsilon_{it} = \sum_{j=0}^{\infty} c_{ij}\varepsilon_{it-j}, \quad \sum_{j=0}^{\infty} j \mid c_{ij} \mid < \infty, \tag{6}$$

with $\varepsilon_{it} = iid(0, \sigma^2)$ with finite fourth moments for $i = 1, 2$. One summability assumption in (6) is necessary to develop the limit theory of sample covariance of y_{1t}, y_{2t}, and it covers stationary and invertible ARMA systems. Robinson and Hualde (2003) imposed the same summability condition to develop the limit behavior of their estimator. There is no need to assume uncorrelatedness between

[1] The restriction will be needed for the derivation of distributional results that are not covered in the present paper.

ε_{1t} and ε_{2t} and hence contemporaneous correlation in $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ is allowed. Generally, $e_t = (e_{1t}, e_{2t})$ is a bivariate covariance stationary unobservable process with spectral density matrix, $g(\lambda)$, satisfying

$$E(e_t e_{t+j}') = \int_{-\pi}^{\pi} e^{ij\lambda} g(\lambda) d\lambda$$

which is at least nonsingular and continuous for all frequencies.

Now define the partial sums $S_{1T} = \sum_{t=1}^{T} u_{1t}$ and $S_{2T} = \sum_{t=1}^{T} u_{2t} = \sum_{t=1}^{T} \Delta y_{2t}$. The limits

$$\lim_{T \to 0} T^{-(1+2d_1)} E(S_{1T} S_{1T}') := \Omega_{11}$$
$$\lim_{T \to 0} T^{-(1+2d_2)} E(S_{2T} S_{2T}') := \Omega_{22}$$
$$\lim_{T \to 0} T^{-(1+2d_1+d_2)} E(S_{1T} S_{2T}') := \Omega_{12}$$

will be called long run variance/covariance of the $u_{it} (i = 1, 2)$. The long run variance/covariance will be used in the transformation of the given cointegration model as will be seen in the next section.

Ⅲ. ESTIMATE OF COINTEGRATION VECTOR AND LIMIT BEHAVIOR OF BIAS REDUCED BAND SPECTRUM LEAST SQUARES

(1) Estimators in the frequency Domain

Note that

$$(1-L)^{d_2 - 1}(1-L) y_{2t} = e_{2t} \Rightarrow (1-L)^{d_2 - 1} u_{2t} = e_{2t},$$

we then transform (2) as follows.

$$y_{1t} = \beta y_{2t} + C \Delta y_{2t} + u_{1t} - \Omega_{12} \Omega_{22}^{-1} \Delta y_{2t} \tag{7}$$

where $C = \Omega_{12} \Omega_{22}^{-1}$. Ω_{12} and Ω_{22} denote the long run covariance of $u_{1t}, \Delta y_{2t}$ and the long run variance of Δy_{2t} defined in the previous section. Since our main focus is to estimate the cointegrating vector β, the estimation of C is of no use in the procedure. The main reason for this transformation can be clearer when the triangular structural model is considered like in Phillips (1991a, b), but simply adding Δy_{2t} in the regression model can reduce the biases of estimation of the cointegration vector β. In fact, the regression model (7) has been considered by Phillips (1991a) for the optimal inference of standard I(0)/I(1) cointegration as an extension of the triangular Error Correction Model (ECM) representation. He has shown that the maximum likelihood estimation of triangular structure of ECM in the standard cointegration is equivalent to the OLS in (7) when the error term is white noise. Therefore, without any numerical maximization, simple OLS will give MLE (Gaussian pseudo-ML estimate) in the given model when $d_1=0$, $d_2=1$, that is, standard I(1)/I(0) cointegration.

As noted, it is clear that the role of Δy_{2t} in the augmented regression in (7) as an additional regressor is to adjust the conditional mean and hence to remove the second order biases which present in the regression of y_{1t} on y_{2t}. However, the OLS in (7) is only valid as an optimal inference when the regression error is white noise which is very unlikely in the analysis of time series regression even in the standard I(1)/I(0) cointegration case. Other issues of optimal inferences in the standard cointegration have well been explored, for example, in Phillips (1991a) and Johansen (1988) using full system maximum likelihood estimation.

Frequency domain analysis can be an alternative in the cointegration models and all the well known advantages can be enjoyed in the spectral regression. In a simultaneous equations model with serially independent errors and triangular structure of coefficients, it is well known that MLE is equivalent to generalized least squares (GLS). Phillips (1991b) used this idea in a cointegration model using the fact that the ECM model can be transformed into frequency domain, keeping the structure of coefficient but making the regression errors asymptotically independent so that GLS can be applied in the frequency domain. In fact, this is the idea of efficient spectral regression in a stationary time series model. Another advantage of spectral regression is that it allows a nonparametric treatment of regression error in the regression model, and hence neither prior distributional assumption nor specification of system is required.

Frequency domain approach is more important in the analysis of fractionally integrated processes mainly because the behavior of spectral density of low frequency dominates even in the stationary processes and therefore the traditional asymptotic theories can not be applied into stationary long memory processes. As Granger (1966) pointed out 'the typical spectral shape of an economic variable', the definition of fractional processes is inevitably related to the abnormal behavior of spectral density near origin or low frequencies, and numerous inferences about the fractionally integrated processes are based on the spectral representation of given data. The same applies in the analysis of fractional cointegration models since the idea of cointegration itself is to characterize the relationship of variables in the low frequencies near origin which represents the long run equilibrium. Robinson and Marinucci (1998, 2003), Robinson and Hualde (2003) are all frequency domain analyses of fractional cointegrations whereas Dolado and Marmol (1996), Jeganathan (1999) and Kim and Phillips (2002) look at the aspect of the time domain estimation of fractional cointegrations.

This paper utilizes the idea of (7) in the frequency domain analysis, and we suggest a bias reduced modified band spectrum least squares (MBSLS) in a bivariate fractional cointegration model in (2) and (3). The approach proposed in this paper is therefore related to the usage of discrete time ECM formulation in the frequency domain. In view of the merits of spectral analysis in the fractionally integrated processes, frequency domain least squares is expected to perform well in the fractional cointegration analysis. In addition to that, the method suggested in this paper is very easy to use in practice; it is indeed simple OLS in the frequency domain without involving any numerical maximization which is necessary for many other estimation procedures in the literature. Other aspects of suggested estimation procedure in the present paper will be given in the following section.

(2) Bias Reduced Band Spectrum Regression

Now writing the discrete Fourier transforms (dft) of any given process a_t as

$$w_a(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda},$$

then we can transform the model (7) to evaluate in the frequency domain as

$$w_1(\lambda_s) = \beta w_2(\lambda_s) + C w_{\Delta y_2}(\lambda_s) + w_{1\cdot 2}(\lambda_s) \tag{8}$$

where $w_1(\lambda_s)$, $w_2(\lambda_s)$, $w_{\Delta y_2}(\lambda_s)$ and $w_{1\cdot 2}(\lambda_s)$ denote the dft's of $y_1, y_2, \Delta y_2$ and $u_1 - \Omega_{12}\Omega_{22}^{-1}\Delta y_2$.

The transformed regression model was first considered in Phillips (1988) as a subsystem of multivariate standard I(1)/I(0) cointegration model when no restriction exists in the parameter matrix of the system. In fact, it is just a frequency domain version of the optimality result of standard cointegration when there is no restriction on the cointegrating vectors which is the case in (8). Unlike time domain optimality conditions mentioned above, there is no need to assume the white noise error in the frequency domain. The additional term, $w_{\Delta y_2}(\lambda_s)$, to the original band spectrum regression is asymptotically independent of $w_{1\cdot 2}(\lambda_s)$ in the standard I(1)/I(0) cointegration because they are asymptotically normal and their covariance is zero, which makes the simple least squares efficient in standard cointegration (See Phillips (1988, 1991(C)).

We will use the model in (8) for the estimation of cointegration vector β in the fractional cointegration model in (2) to reduce the second order biases of least squares in the time domain regression. According to Phillips (1988), it is clear that the simple least squares in (8) is optimal in the sense that it achieves the same limit distribution in the full system maximum likelihood estimator in the standard cointegration model. Moreover, the least squares in (8) can be shown to have optimal convergence rate and shares the same limit behavior with Gaussian MLE especially when first differenced regressor y_{2t} and regression error u_{1t} share the same degree of integration order (See Kim and Phillips (2002)). Unlike I(1)/I(0) standard cointegration, the optimal theory for general fractional cointegration models is yet to be established although some results about optimal convergence rates of estimators in the cointegration system are known. We therefore suggest the least squares procedure of (8) for the estimation of more general fractional cointegration models.

Linear regression on (8) leads to the modified band spectrum regression estimator $\hat{\beta}$ as

$$\begin{aligned}\hat{\beta} = {} & [\hat{f}_{22}(0) - \hat{f}_{2\Delta}(0)(\hat{f}_{\Delta\Delta}(0))^{-1}\hat{f}_{\Delta 2}(0)]^{-1} \\ & [\hat{f}_{12}(0) - \hat{f}_{2\Delta}(0)(\hat{f}_{\Delta\Delta}(0))^{-1}\hat{f}_{\Delta 2}(0)]\end{aligned} \tag{9}$$

where $\hat{f}$ is a one sided average of m ordinates at origin. That is,

$$\hat{f}_{ij}(0)=\frac{1}{m}\sum_{s=1}^{m}w_i(\lambda_s)w_j(\lambda_s)^*=:\frac{1}{m}\sum_{s=1}^{m}I_{ij}(\lambda_s),\ \text{for}\ \ i,j=1,2,$$
$$\hat{f}_{i\Delta}(0)=\frac{1}{m}\sum_{s=1}^{m}w_i(\lambda_s)w_{\Delta y_2}(\lambda_s)^*=:\frac{1}{m}\sum_{s=1}^{m}I_{i\Delta}(\lambda_s),\ \text{for}\ \ i=1,2,$$
$$\hat{f}_{\Delta\Delta}(0)=\frac{1}{m}\sum_{s=1}^{m}w_{\Delta y_2}(\lambda_s)w_{\Delta y_2}(\lambda_s)^*=:\frac{1}{m}\sum_{s=1}^{m}I_{\Delta\Delta}(\lambda_s),\ \text{for}\ \ i=1,2,$$

where $*$ and $I(\lambda_s)$ denotes complex conjugate and the periodogram respectively. The number of ordinates m included in the regression can be chosen in such a way $\frac{m}{n^a}\rightarrow\infty$, for some $a>0$.[2] We therefore have

$$\begin{aligned}\hat{\beta}-\beta=\Big[&\sum_{s=1}^{m}w_2(\lambda_s)w_2(\lambda_s)^*\\ &-\sum_{s=1}^{m}w_2(\lambda_s)w_{\Delta y_2}(\lambda_s)^*\Big(\sum_{s=1}^{m}w_{\Delta y_2}(\lambda_s)w_{\Delta y_2}(\lambda_s)^*\Big)^{-1}\sum_{s=1}^{m}w_{\Delta y_2}(\lambda_s)w_2(\lambda_s)^*\Big]^{-1}\\ \Big[&\sum_{s=1}^{m}w_2(\lambda_s)w_{1\cdot 2}(\lambda_s)^*\\ &-\sum_{s=1}^{m}w_2(\lambda_s)w_{\Delta y_2}(\lambda_s)^*\Big(\sum_{s=1}^{m}w_{\Delta y_2}(\lambda_s)w_{\Delta y_2}(\lambda_s)^*\Big)^{-1}\sum_{s=1}^{m}w_{\Delta y_2}(\lambda_s)w_{1\cdot 2}(\lambda_s)^*\Big]^{-1}.\end{aligned}\qquad(10)$$

Our band spectrum least squares is just simple OLS in the frequency domain with one additional regressor which leads to reduce the bias of the original OLS estimator. As in (8), the additional regressor is just simple dft of first difference of y_{2t}.

Main Theorem :

For $d_1<\frac{1}{2}$, $\frac{1}{2}<d_2<2$, and under the assumptions in section 2,

$$\hat{\beta}\xrightarrow{p}\beta$$

As shown in the main theorem, our bias reduced modified band spectrum regression gives a consistent estimator for a quite wide range of fractional parameters in the regressor and the regression error. In fact, it holds for

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[2] The choice of m, that is, a does not matter in the consistency of suggested estimation procedure, it however is crucial to derive the limit distribution of the given estimator.

nonstationary error in the cointegration system with some restrictions on fractional parameters which will not be pursued in the paper.[3] Even though the limit distribution has not been established yet, and hence traditional statistical inferences can not be applied here, it has lots of merits in the estimation of cointegration vector and testings. It also can be shown that MBSLS can achieve the optimal convergence rates under some parametric restrictions even though we do not pursue any proof in the present paper.[4] The optimal rate of convergence in the fractional cointegration model is known, for example see table 1 in Robinson and Hualde(2003). The main and crucial advantage of MBSLS is that it is indeed free from any preliminary estimates for the estimation of cointegration vector and hence can reduce the variance as can be confirmed by the simulation in the next section.[5]

In fact, most of available estimation methods in the literature except OLS and frequency domain least squares (Robinson and Marinucci, 1998, 2003) inevitably involve several preliminary estimation steps. For example, Kim and Phillips (2002) requires preliminary estimation of fractional parameters as well as the long run variances of long memory processes, which clearly involves estimation of short memory components. Therefore, one should first use OLS to get residuals in the fractional cointegration model, and need to estimate long memory parameters of the cointegration residual as well as regressors. Once fractional parameters are estimated, one can estimate fractional cointegration vectors based on the augmented transformed model. Even though the estimation method in Kim and Phillips (2002) is a time domain approach, and hence familiar with practitioners, it entails several preliminary steps to get fractional cointegration vectors unless a specific condition for integration orders is satisfied.

The estimation procedure in Robinson and Hualde (2003) is more complicated since it involves fractional differencing with estimated fractional parameters. It

[3] Kim and Phillips (2002) developed a limit theory on fractional cointegration with nonstationary cointegration errors with $d_2 - d_1 > \frac{1}{2}$ condition.

[4] If $d_1 > d_2 - 1$ and $d_1 < \frac{1}{2}$, $1 < d_2 < \frac{3}{2}$ the rate of convergence of MBSLS is $n^{d_2 - d_1}$ as shown in (beta-2-1) in the appendix. Moreover, it can be shown in the same way that the same rate of convergence is applied for $\frac{1}{2} < d_1 < 1$, $1 < d_2 < \frac{3}{2}$ and $\frac{1}{2} < d_1 < \frac{3}{2}$, $\frac{3}{2} < d_2 < 2$ under $d_2 - 1 > d_1$.

[5] The estimation procedure does not require any preliminary estimates, but the parametric restrictions given in the paper especially $d_2 - d_1 > \frac{1}{2}$ is required to apply the method suggested in the paper.

first needs to transform the regression model by fractional differencing with estimated fractional parameters, and then implement generalized least squares (GLS) in the frequency domain in the bivariate fractional cointegration model just like in (2). Since they consider the GLS type approach in the frequency domain,[6] the procedure requires to estimate spectral density of cointegration errors which also needs the maximum likelihood estimation of short memory parameters, that is, ARMA(p,q) coefficients. Robinson and Hualde approach is theoretically appealing since it achieves mixed normal asymptotics, and hence one can use chi-square asymptotics in the inferences. It is, however, very sensitive to the correct estimation of fractional parameters because the method necessarily entails fractional differencing with estimated parameters before estimating cointegrating vector, and therefore it could have over or under differencing problem in practice.

In light of all complicated preliminary steps in many available estimation procedures for fractional cointegrating vectors, our method suggested in the present paper is certainly recommended in the sense that it does not use any preliminary estimates in the estimation of cointegrating vector, and hence can reduce the variance of cointegrating vector estimation.

Ⅳ. FINITE SAMPLE PERFORMANCES

As briefly mentioned before, there has been no optimal theory for the estimation of cointegrating vector in the fractional cointegration model, and therefore the finite sample performances of available estimators are indeed important. Even though the suggested estimator in the paper is simple to use, poor finite sample performances would deteriorate the conveniency of our MBSLS estimator. In fact, comparison among all available estimators in the literature would be recommended with different sets of fractional parameters and short memory components. This, however, would add considerable space to this already lengthy paper, and hence detailed simulation results and improvements on finite sample performances of different estimators will be left for subsequent work.

In this section, we will compare the finite sample performances of MBSLS,

[6] Robinson and Hualde (2003) also consider the GLS type of estimation in time domain, but the calculation in the time domain approach is way more complicated than frequency domain except white noise short memory case, which is very unlikely to observe in econometric models.

Robinson and Hualde (2003) (R-H here after), and OLS under different parametric values. Since both estimation procedures are simple least squares in the time domain and the frequency domain without any preliminary estimations, evaluating the finite sample performances of the two would be fair. Moreover, GLS-type R-H estimator should be compared to MBSLS. In fact, we can clearly see how the suggested MBSLS will reduce bias and variance in estimating fractional cointegration model.

There are several methods available for simulating fractionally integrated processes and some of these are reviewed in Beran (1993). The simplest method is to use fractional differencing operators, and fractionally integrated processes can be generated by simple linear filters. Another popular method suggested by Davis and Harte (1987) is based on the fast Fourier transform and uses the autocovariance sequence of a fractionally integrated process which can be computed by well known formulae, such as those given in Granger and Joyeux (1980) and Hosking (1981). In the present paper, we use this approach to generate vector fractionally integrated processes allowing for cross dependence in their components. The autocovariance function between two fractionally integrated processes $\{u_{1t}, u_{2t}\}$ can be written in the form

$$Cov(u_{1t}, u_{2t+k}) = \pi^{-1}\int_0^\pi \cos k\lambda \left(2\sin\frac{\lambda}{2}\right)^{-2\left(\frac{d_1+d_2}{2}\right)} d\lambda.$$

Then, the matrix autocovariance sequence $\gamma(k)$ is

$$\gamma(k) = \begin{pmatrix} a_{11}\sigma_{11} & a_{12}\sigma_{12} \\ a_{21}\sigma_{21} & a_{22}\sigma_{22} \end{pmatrix},$$

where

$$a_{ii} = \frac{\Gamma(k+d_i)\Gamma(1-2d_i)}{\Gamma(k+1-d_i)\Gamma(1-d_i)\Gamma(d_i)}, \quad i=1,2,$$

$$a_{12} = \frac{\Gamma\left(k+\frac{d_1+d_2}{2}\right)\Gamma(1-d_1-d_2)}{\Gamma\left(k+1-\frac{d_1+d_2}{2}\right)\Gamma\left(1-\frac{d_1+d_2}{2}\right)\Gamma\left(\frac{d_1+d_2}{2}\right)},$$

and σ_{ij} is the variance of the innovations e_{1t}, e_{2t}. Then, using the fast Fourier transform vector fractionally integrated processes can be generated just as in the method of Davis and Harte (1987).

The data generating process for simulation is the same as our original model in (2),

$$y_{1t} = \beta y_{1t} + u_{1t}$$

and y_{2t} can be generated by $\{u_{2t}\}$ For simplicity, we set $Var(\varepsilon_{1t}) = Var(\varepsilon_{2t}) = 1$, and $Cov(\varepsilon_{1t}, \varepsilon_{2t}) = 0.8$. Three sets of short memory components, ARMA(0,0), AR(1), and ARMA(1,1) are tried and $\{e_{1t}\}$, $\{e_{2t}\}$ are generated from $\{\varepsilon_{1t}\}$, $\{\varepsilon_{2t}\}$ The AR coefficient is set to 0.5 and MA coefficient to -0.5. The system is now estimated by OLS, R-H and MBSLS[7] and the results for the three procedures are compared. The number of replications is 15,000 and sample sizes of 50, 100 are used. The following six combinations of d_1, $d_2^*(=d_2-1)$ values were tried: $(0.2, 0.2)$, $(0.3, 0.1)$, $(0.4, 0.01)$, $(0.1, -0.2)$, $(0.4, -0.2)$, and $(0.4, 0.6)$. In fact, all the estimators considered here are not consistent for the case of $(0.4, -0.2)$, that is, when $d_1 = 0.4$ and $d_2 = 0.8$. However, we include the result in the appendix for a simple comparison of finite sample performances of the given estimators.

Tables 6.1-3 show the mean bias and the standard deviations of OLS, R-H, and MBSLS, and they clearly reveal that the finite sample performances of MBSLS are better than those of OLS in every case.[8] Using MBSLS clearly reduces the mean bias and standard deviation significantly. As mentioned before, When $d_1 = d_2^*(=d_2-1)$, MBSLS has much less dispersion than OLS. This case indeed has n-convergence rate which is shown in Robinson and Marinucci (2003). For other fractional parameter values, OLS suffers from second order biases and is generally less reliable than MBSLS. We also looked at the median bias and the interquartile range for OLS and MBSLS, and the difference between OLS and MBSLS is clear just like mean biases and standard deviations of the two estimators.

When we compare the R-H and MBSLS, R-H is slightly better than MBSLS when the short memory components are generated by ARMA(0,0), however the

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[7] A different set of m has been tried in the simulation, for example; $n^{0.3}$, $n^{0.5}$, $n^{0.7}$. The results are not quite different, here the case of $m = n^{0.5}$ is reported.

[8] The results for $n = 50$ cases are not reported here since they are very similar to the cases of $n = 100$.

performances of R-H procedure are definitely worse than MBSLS when they are generated by AR(1) or ARMA(1,1). This is naturally expected considering that R-H method involves the estimation of short memory parameters. Preliminary estimation of short memory parameters and fractional parameters make the variance of R-H estimator bigger, and hence yield the poor performances in general cases like AR(1) or ARMA(1,1). It can be expected that R-H estimator works well only when the short memory components follow ARMA(0,0) which is hardly justified in many applications.

Figures in appendix show the sampling distributions of MBSLS, R-H and OLS and they show clear differences between MBSLS, R-H and OLS.[9] The densities of MBSLS are centered much better than those of OLS and show less volatility. Even though sample size gets larger, differences are still apparent. When fractional parameters follow $(0.1, 0.01)$, the convergence rate is slower than other parametric values, and hence the finite sample performances for both estimators are poorer than the other two cases. The short memory component seems not to play a crucial role in evaluating finite sample performances of OLS and MBSLS whereas it does matter in the R-H estimation. However, better performances of MBSLS seem apparent in non-white noise short memory components as we can see in figures in appendix.

In terms of mean bias, neither MBSLS nor R-H dominates the other in three different cases, MBSLS performs better in some cases, especially when $d_1 = d_2^*(=d_2-1)$ but in some other cases, R-H does better. In terms of variance and interquartile range, however, MBSLS dominates R-H, and therefore, MSE's of FFM are smaller than those of R-H in every case in AR(1), ARMA(1.1). This finding can be seen clearly in the graphs. As expected, all three estimators seem not working for the case of $d_1 = 0.4$ and $d_2 = 0.8$ as shown in the figures.

V. CONCLUSION

This paper suggests an estimation procedure for bivariate fractionally cointegrating regression models. The estimator proposed in this paper is based upon least squares in the frequency domain with a simple modification which is known to be optimal under conventional $I(1)/I(0)$ cointegration. It is shown

[9] Figures for AR(1) only are included in the appendix due to space limits.

that the same estimation method is still working under general parametric values of fractional orders in variables. The main advantage of the estimator is that it can be used for a fractional cointegration model without preliminary estimates unlike other available methods in the literature. In simulation study, MBSLS can significantly reduce second order biases and variance compared to the performance of OLS. MBSLS performs better than the GLS type Robinson and Hualde (2003), especially when the short memory components are not white noise. It is robust not only to the presence of long range dependence in the regression errors, but to the different sets of short memory components. Simulations confirm that MBSLS estimator performs well and seems reliable.

Cointegrations represent long-run equilibrium relationships in economic models, but conventional cointegrations restrict too much in defining disequilibrium errors in the model. The fractional cointegrations encompass traditional cointegration models and extend the idea of equilibrium relationships, and therefore accurate and robust estimation of these relationships is clearly important in many empirical applications. This paper provides a simple and reliable estimation procedure free from any prior knowledge of fractional parameter or pre-estimation of a given model.

The conditions on the fractional parameters are not restrictive in practice, and hence can be used almost all applications except stationary cointegrations. The present paper, however, does not provide asymptotic distributional results of MBSLS that are left to subsequent work. Although simple sets of simulations are given in this paper, detailed comparison of the finite sample performances of all different estimation procedures in the literature is certainly recommended with different sets of fractional parameters and short memory components, which is also left in future work. Extension of suggested methods to vectors valued variables seems straightforward only when all the variables in the multivariate system share the same degree of fractional/integration orders. If fractional orders are different across the variables, especially in regression errors, considerable efforts should be made to deal with this problem. That might require different techniques and raise all complicated issues of estimation.

Ⅵ. APPENDIX

The limit behavior of MBSLS is quite complicated according to the fractional parameters in the model, and we therefore need to divide several different cases dependent upon the values of fractional parameters to prove the given theorem. We will present only one case in this paper to save space, that is, when $d_1 < \frac{1}{2}$, $\frac{1}{2} < d_2 < 1$. Proofs of the other parameter values are similar to the following two cases.

- $d_1 < \frac{1}{2}$ **and** $\frac{1}{2} < d_2 < 1$.

Note that (10) can be written as

$$\tilde{\beta} - \beta = \left[\frac{1}{m}\sum_{s=1}^{m} I_{22}(\lambda_s) - \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta 2}(\lambda_s)\right]^{-1}$$
$$\left[\frac{1}{m}\sum_{s=1}^{m} I_{2,1\cdot 2}(\lambda_s) - \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\right] \quad (11)$$

where $I_{2,1\cdot 2}(\lambda_s) = w_2(\lambda_s) w_{1\cdot 2}(\lambda_s)^*$ and $I_{\Delta,1\cdot 2}(\lambda_s) = w_{\Delta, y_2}(\lambda_s) w_{1\cdot 2}(\lambda_s)^*$ The following steps are needed proceed the analysis of the asymptotic behavior of the modified band spectrum estimator $\tilde{\beta}$.

Step (A)

By the theorem 4.1 in Phillips (1999), we have

$$\begin{aligned}\frac{m}{n^{2d_2}}\frac{1}{m}\sum_{s=1}^{m} I_{22}(l_s) &= \sum_{s=1}^{m}\frac{w_2(\lambda_s)}{n^{d_2}}\frac{w_2(\lambda_s)^*}{n^{d_2}} \\ &= \sum_{s=1}^{n}\frac{w_2(\lambda_s)}{n^{d_2}}\frac{w_2(\lambda_s)^*}{n^{d_2}} - \sum_{s=m+1}^{n}\frac{w_2(\lambda_s)}{n^{d_2}}\frac{w_2(\lambda_s)^*}{n^{d_2}} \\ &= \frac{1}{2p}\frac{1}{n}\sum_{s=1}^{n}\left(\frac{y_{2t}}{n^{d_2-\frac{1}{2}}}\right)^2 - \sum_{s=m+1}^{n}\frac{w_2(\lambda_s)}{n^{d_2}}\frac{w_2(\lambda_s)^*}{n^{d_2}}.\end{aligned}$$

If m is such that $\frac{m}{n^a} \to \infty$, it follows that

$$\sum_{s=m+1}^{n}\frac{w_2(\lambda_s)}{n^{d_2}}\frac{w_2(\lambda_s)^*}{n^{d_2}} = O_p\left(\frac{n}{m^{2d}}\right),$$

and then

$$\sum_{s=m+1}^{n} \frac{w_2(\lambda_s)}{n^{d_2}} \frac{w_2(\lambda_s)^*}{n^{d_2}} = o_p(1)$$

for $\alpha \geq \frac{1}{2d_2}$. Therefore, we have

$$\frac{m}{n^{2d_2}} \frac{1}{m} \sum_{s=1}^{m} I_{22}(\lambda_s) \rightarrow \frac{1}{2\pi} \int B_{d_2-\frac{1}{2}}(r)^2 dr.$$

Step (B)

To analyze the limit behavior of $\frac{1}{m} \sum_{s=1}^{m} I_{2\Delta}(\lambda_s)$, we need the following representation of the dft of a fractionally integrated processes

$$w_2(\lambda_s) = (1-e^{i\lambda_s})^{-d_2} w_{e_2}(\lambda_s) - \frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})} \frac{y_{2n}}{\sqrt{2\pi n}} + o_p\left(\frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})} \frac{y_{2n}}{\sqrt{2\pi n}} \right),$$

as shown in Phillips (1999). The fractional parameter of Δy_{2t} is $d_2 - 1 < 0$ for $d_2 < 1$ here. Then, we have

$$\begin{aligned} \frac{1}{m} \sum_{s=1}^{m} I_{2\Delta}(\lambda_s) &= \frac{1}{m} \sum_{s=1}^{m} w_2(\lambda_s) w_{\Delta y_2}(\lambda_s)^* \\ &= \frac{1}{m} \sum_{s=1}^{m} (1-e^{i\lambda_s})^{-d_2} w_{e_2}(\lambda_s) w_{\Delta y_2}(\lambda_s)^* \\ &\quad - \frac{1}{m} \sum_{s=1}^{m} \frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})} \frac{y_{2n}}{\sqrt{2\pi n}} w_{\Delta y_2}(\lambda_s)^* \\ &\quad + o_p\left(\frac{1}{m} \sum_{s=1}^{m} \frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})} \frac{y_{2n}}{\sqrt{2\pi n}} w_{\Delta y_2}(\lambda_s)^* \right) \end{aligned} \quad (12)$$

The limit behavior of first term in (12) can be obtained as

$$\begin{aligned} &\frac{1}{m} \sum_{s=1}^{m} (1-e^{i\lambda_s})^{-d_2} w_{e_2}(\lambda_s) w_{\Delta y_2}(\lambda_s)^* \\ &= \frac{1}{m} \sum_{s=1}^{m} (1-e^{i\lambda_s})^{-2d_2+1} w_{e_2}(\lambda_s) (1-e^{i\lambda_s})^{d_2-1} w_{\Delta y_2}(\lambda_s)^* \\ &\leq \left(\frac{1}{m} \sum_{s=1}^{m} |1-e^{i\lambda_s}|^{-4d_2+2} I_{e_2}(\lambda_s) \right)^{\frac{1}{2}} \\ &\left(\frac{1}{m} \sum_{s=1}^{m} |1-e^{i\lambda_s}|^{2(d_2-1)} I_{\Delta\Delta}(\lambda_s) \right)^{\frac{1}{2}}. \end{aligned} \quad (13)$$

By Robinson (1995), it follows that

$$\frac{1}{m}\sum_{s=1}^{m}|1-e^{i\lambda_s}|^{-2(d_2-1)}I_{\Delta\Delta}(\lambda_s)=O_p(1).$$

Moreover, the collection of $I_{e_2}(\lambda_s)$ are independently distributed, and then

$$\left(\frac{1}{m}\sum_{s=1}^{m}|1-e^{i\lambda_s}|^{-4d_2+2}I_{e_2}(\lambda_s)\right)^{\frac{1}{2}}=\begin{cases}O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right), & \text{if } d_2<\frac{3}{4}\,.\\ O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right), & \text{if } d_2>\frac{3}{4}\end{cases}$$

From (13), we have

$$\frac{1}{m}\sum_{s=1}^{m}(1-e^{i\lambda_s})^{-d_2}w_{e_2}(\lambda_s)w_{\Delta y_2}(\lambda_s)^*=\begin{cases}O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right), & \text{if } d_2<\frac{3}{4}\,.\\ O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right), & \text{if } d_2>\frac{3}{4}\end{cases}\tag{14}$$

The order of the second term in (12) can be analyzed as

$$\begin{aligned}&\frac{1}{m}\sum_{s=1}^{m}\frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})}\frac{y_{2n}}{\sqrt{2\pi n}}w_{\Delta y_2}(\lambda_s)^*\\&=\frac{y_{2n}e^{i\lambda_s}}{\sqrt{2\pi n}^{\,d_2-\frac{1}{2}}}n^{d_2-1}\frac{1}{m}\sum_{s=1}^{m}(1-e^{i\lambda_s})^{-d_2}(1-e^{i\lambda_s})^{d_2-1}w_{\Delta y_2}(\lambda_s)^*\\&\le\left|\frac{y_{2n}}{\sqrt{2\pi n}^{\,d_2-\frac{1}{2}}}\right|n^{d_2-1}\left(\frac{1}{m}\sum_{s=1}^{m}|1-e^{i\lambda_s}|^{-2d_2}\right)^{\frac{1}{2}}\\&\left(\frac{1}{m}\sum_{s=1}^{m}|1-e^{i\lambda_s}|^{2(d_2-1)}I_{\Delta\Delta}(\lambda_s)\right)^{\frac{1}{2}}\\&=O_p(1)\,O(n^{d_2-1})O\left(\frac{n^{d_2}}{\sqrt{m}}\right)O_p(1)=O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right).\end{aligned}\tag{15}$$

By (14) and (15), we have

$$\begin{aligned}\frac{1}{m}\sum_{s=1}^{m}I_2\Delta(\lambda_s)&=O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right)+O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right)\\&=O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right)+O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}m^{2d_2-\frac{3}{2}}\right)\end{aligned}$$

$$= O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right) \text{ for } d_2 < \frac{3}{4}, \tag{16}$$

$$\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s) = O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right) \text{ for } d_2 > \frac{3}{4} \tag{17}$$

Step (C)

Since the fractional parameter of Δy_{2t} is negative, we need the following to obtain the order of $\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)$

$$\begin{aligned}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s) &= \frac{1}{m}\sum_{s=1}^{m}\lambda_s^{2(d_2-1)} I_{\Delta\Delta}(\lambda_s)\lambda_s^{-2(d_2-1)} \\ &\le \frac{1}{m}\sum_{s=1}^{m}\left|\lambda_s^{2(d_2-1)} I_{\Delta\Delta}(\lambda_s)\right| \sum_{s=1}^{m}\left|\lambda_s^{-2(d_2-1)}\right| \\ &= O_p\left(\sum_{s=1}^{m}\left|\lambda_s^{-2(d_2-1)}\right|\right) = \left(\left(\frac{n}{m}\right)^{2d_2-2} m\right).\end{aligned} \tag{18}$$

Now, from step (B) and step(C), for $d_2 < \frac{3}{4}$

$$\begin{aligned}&\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta 2}(\lambda_s) \\ &= O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right)\left(O_p\left(\left(\frac{n}{m}\right)^{2d_2-2} m\right)\right)^{-1} O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right) \\ &= O_p\left(\left(\frac{n}{m}\right)^{2d_2}\frac{1}{m}\right)\end{aligned}$$

and hence

$$\begin{aligned}&\frac{m}{n^{2d_2}}\left[\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta 2}(\lambda_s)\right] \\ &= O_p(m^{-2d_2}) = o_p(1).\end{aligned} \tag{19}$$

Similarly, for $d_2 < \frac{3}{4}$

$$\begin{aligned}&\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta 2}(\lambda_s) \\ &= O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right)\left(O_p\left(\left(\frac{n}{m}\right)^{2d_2-2}\right)\right)^{-1} O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right) \\ &= O_p(n^{2d_2} m^{-2d_2-4}) = o_p(1),\end{aligned}$$

and hence

$$\frac{m}{n^{2d_2}}\left[\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta 2}(\lambda_s)\right]$$
$$= O_p(m^{-2d_2-3}) = o_p(1), \text{ for } d_2 < \frac{3}{4}. \tag{20}$$

Therefore, it follows that

$$\frac{1}{m}\sum_{s=1}^{m} I_{22}(\lambda_s) - \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta 2}(\lambda_s)$$
$$= O_p\left(\frac{n^{2d_2}}{m}\right) + o_p\left(\frac{n^{2d_2}}{m}\right) \tag{21}$$

from (19), (20), and step (A). Note that (21) holds for all $\frac{1}{2} < d_2 < 1$ and for both $d_1 + d_2 < 1$ and $d_1 + d_2 > 1$.

Step (D)

Observe that

$$\frac{1}{m}\sum_{s=1}^{m} w_2(\lambda_s)w_{1\cdot 2}(\lambda_s)^* = \frac{1}{m}\sum_{s=1}^{m} w_2(\lambda_s)[w_1(\lambda_s)^* - \Omega_{12}\Omega_{22}^{-1}w_{\Delta y_2}(\lambda_s)^*]$$
$$= \frac{1}{m}\sum_{s=1}^{m} w_2(\lambda_s)w_1(\lambda_s)^* - C\frac{1}{m}\sum_{s=1}^{m} w_2(\lambda_s)w_{\Delta y_2}(\lambda_s)^* \tag{22}$$

By the results in Robinson and Marinucci (2003), we have

$$\frac{1}{m}\sum_{s=1}^{m} w_2(\lambda_s)w_1(\lambda_s)^* = \begin{cases} O_p\left(\left(\frac{n}{m}\right)^{d_1-d_2}\right), & \text{for } d_1 + d_2 < 1, \\ O_p\left(\frac{n^{d_1+d_2}}{m}\right), & \text{for } d_2 > \frac{3}{4} \end{cases}$$

and from (16) and (17) in step(B)

$$\frac{1}{m}\sum_{s=1}^{m} w_2(\lambda_s)w_{\Delta y_2}(\lambda_s)^* = \begin{cases} O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right), & \text{if } d_2 < \frac{3}{4} \\ O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right), & \text{if } d_2 > \frac{3}{4} \end{cases}$$

Now consider the following four cases,

(i) $d_1+d_2<1,\ \ d_2<\frac{3}{4}$

$$\left(\frac{n}{m}\right)^{d_1+d_2}=\left(\frac{n}{m}\right)^{2d_1-1}\left(\frac{n}{m}\right)^{d_1+1-d_2},$$

(ii) $d_1+d_2<1,\ \ d_2>\frac{3}{4}$

$$\left(\frac{n}{m}\right)^{d_1+d_2}=\left(\frac{n^{2d_1-1}}{m^{\frac{1}{2}}}\right)\left(\frac{n^{d_1+1-d_2}}{m^{d_1+d_2-\frac{1}{2}}}\right),$$

(iii) $d_1+d_2>1,\ \ d_2<\frac{3}{4}$

$$\frac{N^{d_1+d_2}}{m}=\left(\frac{n}{m}\right)^{2d_2-1}\left(\frac{n^{d_1+1-d_2}}{m^{2-2d_2}}\right),$$

(iv) $d_1+d_2>1,\ \ d_2>\frac{3}{4}$

$$\frac{N^{d_1+d_2}}{m}=\left(\frac{n^{2d_1-1}}{m^{\frac{1}{2}}}\right)\left(\frac{n^{d_1+1-d_2}}{m^{\frac{1}{2}}}\right).$$

Here, we can not tell which term in (22) dominates in each of these four cases. The dominant term changes on the values of m, d_1, and d_2. For example, if $\frac{n^{d_1+1-d_2}}{m^{2-2d_2}}\to\infty$, i.e., $\frac{d_1+1-d_2}{2-2d_2}>\alpha$ such that $\frac{m}{n^{\alpha}}\to\infty$ then the first term in (22) dominates the second term in (22) in case $d_1+d_2>1$, $d_2<\frac{3}{4}$. It then can be deduced that

$$\frac{1}{m}\sum_{s=1}^{m}w_2(\lambda_s)w_{1\cdot 2}(\lambda_s)^*=O_p\left(\frac{n^{d_1+d_2}}{m}\right).$$

We can only guess that the first term may be dominant on the second term in (22) in most cases, but can not exclude the opposite according to some extreme values of m, d_1, and d_2. However, the consistency of the modified band spectrum estimator holds regardless of what values m, d_1, and d_2 may take as we can see below.

Step (E).

Now it remains to analyze the limit behavior of $\frac{1}{m}\sum_{s=1}^{m}I_{\Delta,1\cdot 2}(\lambda_s)$ in (11). Note that

$$\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)=\frac{1}{m}\sum_{s=1}^{m} w_{\Delta y_2}(\lambda_s)\left[w_1(\lambda_s)^* - \Omega_{12}\Omega_{22}^{-1} w_{\Delta y_2}(\lambda_s)^*\right]$$
$$=\frac{1}{m}\sum_{s=1}^{m} w_{\Delta y_2}(\lambda_s)w_1(\lambda_s)^* - C\frac{1}{m}\sum_{s=1}^{m} w_{\Delta y_2}(\lambda_s)w_{\Delta y_2}(\lambda_s)^*. \quad (23)$$

The first term in (23) can be written as

$$\frac{1}{m}\sum_{s=1}^{m} w_{\Delta y_2}(\lambda_s)w_1(\lambda_s)^*$$
$$\leq\left(\frac{1}{m}\sum_{s=1}^{m} w_{\Delta y_2}(\lambda_s)w_{\Delta y_2}(\lambda_s)^*\right)^{\frac{1}{2}}\left(\frac{1}{m}\sum_{s=1}^{m} w_1(\lambda_s)w_1(\lambda_s)^*\right)^{\frac{1}{2}}.$$

We have the following

$$\frac{1}{m}\sum_{s=1}^{m} w_1(\lambda_s)w_1(\lambda_s)^* = \frac{n}{m}O_p\left(\frac{m}{n}\right)^{1-2d_1} = O_p\left(\frac{m}{n}\right)^{2d_1} \quad (24)$$

as in step (D) shown in Robinson and Marinucci (2003). From (18) and (24), it follows that

$$\frac{1}{m}\sum_{s=1}^{m} w_{\Delta y_2}(\lambda_s)w_1(\lambda_s)^* = \left[O_p\left(\frac{n}{m}\right)^{2d_2-2} m\right]^{\frac{1}{2}}\left[O_p\left(\frac{m}{n}\right)^{2d_1}\right]^{\frac{1}{2}}$$
$$= O_p\left(\frac{n^{d_1+d_2-1}}{m^{d_1+d_2-\frac{3}{2}}}\right).$$

Therefore, we have

$$\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s) = O_p\left(\frac{n^{d_1+d_2-1}}{m^{d_1+d_2-\frac{3}{2}}}\right) + O_p\left(\left(\frac{n}{m}\right)^{2d_2-2} m\right)$$
$$O_p\left(\left(\frac{n}{m}\right)^{2d_2-2} m\frac{n^{d_1+1-d_2}}{m^{d_1+d_2-\frac{3}{2}}}\right) + O_p\left(\left(\frac{n}{m}\right)^{2d_2-2} m\right).$$

As in the case in step (D), here we can not tell which term is dominant in (23). That is, If $\frac{n^{d_1+1-d_2}}{m^{d_1-d_2+\frac{3}{2}}}\rightarrow\infty$ i.e., $\frac{d_1+1-d_2}{d_1-d_2+\frac{3}{2}} > \alpha$ such that $\frac{m}{n^{\alpha}}\rightarrow\infty$ the first term in (23) dominates the second term in (23) and hence

$$\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)=O_p\left(\frac{n^{d_1+d_2-1}}{m^{d_1+d_2-\frac{3}{2}}}\right). \tag{25}$$

Then, from (16), (17) in step (B), (18) in step (C), and (25) in step (E), we have

$$\begin{aligned}
&\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\\
&=O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right)\left(O_p\left(\left(\frac{n}{m}\right)^{2d_2-2}m\right)\right)^{-1}O_p\left(\frac{n^{d_1+d_2-1}}{m^{d_1+d_2-\frac{3}{2}}}\right)\\
&=O_p\left(\frac{n^{d_1+d_2}}{m^{d_1+d_2-\frac{1}{2}}}\right),
\end{aligned} \tag{26}$$

for $d_2<\frac{3}{4}$ Similarly, for $d_2>\frac{3}{4}$ we have

$$\begin{aligned}
&\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\\
&=O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right)\left(O_p\left(\left(\frac{n}{m}\right)^{2d_2-2}m\right)\right)^{-1}O_p\left(\frac{n^{d_1+d_2-1}}{m^{d_1+d_2-\frac{3}{2}}}\right)\\
&=O_p\left(\frac{n^{d_1+d_2}}{m^{d_1-d_2+2}}\right).
\end{aligned} \tag{27}$$

However, if $\frac{n^{d_1+1-d_2}}{m^{d_1-d_2+\frac{3}{2}}}\to\infty$ i.e., $\frac{d_1+1-d_2}{d_1-d_2+\frac{3}{2}}<\alpha$ such that $\frac{m}{n^{\alpha}}\to\infty$ the second term in (23) dominates the first term in (23) and hence

$$\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\sim\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s),$$

which implies that

$$\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)=O_p\left(\left(\frac{n}{m}\right)^{2d_2-2}m\right). \tag{28}$$

Therefore, it can be easily deduced that

$$\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\Big(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\Big)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s) \sim \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s), \tag{29}$$

the order of which is given in (16) and (17) in step (B).

Now, we can deduce the order of the second term in (11) from the results in (26), (27), (28), and (29) in step (E), and the results in step (D) as follows.

The first case is

$$\Big[\frac{1}{m}\sum_{s=1}^{m} I_{2,1\cdot 2}(\lambda_s) - \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\Big(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\Big)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\Big]$$
$$= \begin{cases} O_p\Big(\Big(\frac{n}{m}\Big)^{d_1+d_2}\Big) + O_p\Big(\Big(\frac{n}{m}\Big)^{d_1+d_2}\Big), & \text{for } d_1+d_2<1, \\ O_p\Big(\frac{n^{d_1+d_2}}{m}\Big) + O_p\Big(\frac{n^{d_1+d_2}}{m}\Big), & \text{for } d_1+d_2>1 \end{cases} \tag{30}$$

which holds when

$$\frac{1}{m}\sum_{s=1}^{m} I_{2,1\cdot 2}(\lambda_s) \sim \frac{1}{m}\sum_{s=1}^{m} I_{12}(\lambda_s) \tag{31}$$

The second case is

$$\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s) \sim \Big[\frac{1}{m}\sum_{s=1}^{m} I_{2,1\cdot 2}(\lambda_s) - \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\Big(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\Big)^{-1}$$
$$\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\Big] = \begin{cases} O_p\Big(\Big(\frac{n}{m}\Big)^{2d_2-1}\Big), & \text{if } d_2<\frac{3}{4}, \\ O_p\Big(\frac{n^{2d_2-1}}{m}\Big), & \text{if } d_2>\frac{3}{4} \end{cases} \tag{32}$$

and if we have

$$\frac{1}{m}\sum_{s=1}^{m} I_{2,1\cdot 2}(\lambda_s) \sim \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s) \tag{33}$$

and

$$\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\Big(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\Big)^{-1}\frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s) = o_p\Big(\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\Big). \tag{34}$$

The third case is

$$\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s) \sim \left[\frac{1}{m}\sum_{s=1}^{m} I_{2,1\cdot 2}(\lambda_s) - \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1} \frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s)\right] = \begin{cases} O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right), & \text{if } d_2 < \frac{3}{4}, \\ O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right), & \text{if } d_2 > \frac{3}{4} \end{cases} \tag{35}$$

and if we have (33) and

$$\frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s)\left(\frac{1}{m}\sum_{s=1}^{m} I_{\Delta\Delta}(\lambda_s)\right)^{-1} \frac{1}{m}\sum_{s=1}^{m} I_{\Delta,1\cdot 2}(\lambda_s) \sim \frac{1}{m}\sum_{s=1}^{m} I_{2\Delta}(\lambda_s). \tag{36}$$

Finally, according to the results in (21) and (30), the asymptotic order of the modified band spectrum estimator in (11) can be written as

$$\begin{aligned}\hat{\beta}-\beta &= \left[O_p\left(\frac{n^{2d_2}}{m}\right)\right]^{-1}\left[O_p\left(\frac{m}{n}\right)^{d_1+d_2}\right] \\ &= O_p\left(\left(\frac{m}{n}\right)^{d_2-d_1}\frac{1}{m^{2d_2-1}}\right) = o_p(1)\end{aligned}$$

for $d_1+d_2<1$ Similarly, for $d_1+d_2>1$ it can be deduced that

$$\begin{aligned}\hat{\beta}-\beta &= \left[O_p\left(\frac{n^{2d_2}}{m}\right)\right]^{-1}\left[O_p\left(\frac{n^{d_1+d_2}}{m}\right)\right] \\ &= O_p\left(\left(\frac{1}{n}\right)^{d_2-d_1}\right) = o_p(1)\end{aligned}$$

under the condition (31).

The second case is

$$\begin{aligned}\hat{\beta}-\beta &= \left[O_p\left(\frac{n^{2d_2}}{m}\right)\right]^{-1}\left[O_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right)\right] \\ &= O_p\left(\frac{m^{2d_2-2}}{n}\right) = O_p\left(\frac{m}{n}\frac{1}{m^{2d_2-1}}\right) = o_p(1),\end{aligned}$$

or

$$\hat{\beta}-\beta=\left[O_p\left(\frac{n^{2d_2}}{m}\right)\right]^{-1}\left[O_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right)\right]$$
$$=O_p\left(\frac{m^{\frac{1}{2}}}{n}\right)=o_p(1),$$

which comes from (21) and (32) under the condition (33) and (34).

The final case is

$$\hat{\beta}-\beta=\left[O_p\left(\frac{n^{2d_2}}{m}\right)\right]^{-1}\left[o_p\left(\left(\frac{n}{m}\right)^{2d_2-1}\right)\right]$$
$$=o_p\left(\frac{m^{2d_2-2}}{n}\right)=o_p\left(\frac{m}{n}\frac{1}{m^{2d_2-1}}\right)=o_p(1),$$

or

$$\hat{\beta}-\beta=\left[O_p\left(\frac{n^{2d_2}}{m}\right)\right]^{-1}\left[o_p\left(\frac{n^{2d_2-1}}{\sqrt{m}}\right)\right]$$
$$=o_p\left(\frac{m^{\frac{1}{2}}}{n}\right)=o_p(1),$$

according to the results in (21) and (35) under the condition (33) and (36).

In each case, we conclude that the modified MBSLS is still consistent, but the order of convergence is slightly different and dependent upon on the values which m, d_1, and d_2 take. These complicated orders of convergence come from the limit behavior of the covariance between process X_{1t} with negative fractional parameter d_1 and X_{2t} with positive fractional parameter d_1, which has not been established yet.

Appendix 6.1. Tables and Graphs[10]

MODEL A: AR(1)

[Table 6.1] Mean Bias, Standard Deviation(SD), MSE of $(\hat{\beta}-\beta)$, $n=100$

		$n=100$		
		OLS	R-H	MBSLS
$d_1=0.2,\ d_2^*=0.2$	Mean Bias	0.0070	0.0054	0.0000
	SD	0.0242	0.0374	0.0196
	MSE	0.0006	0.0014	0.0004
		OLS	R-H	MBSLS
$d_1=0.3,\ d_2^*=0.1$	Mean Bias	0.0381	0.0235	0.0269
	SD	0.0440	0.0503	0.0357
	MSE	0.0034	0.0031	0.0020
		OLS	R-H	MBSLS
$d_1=0.4,\ d_2^*=0.1$	Mean Bias	0.1022	0.1108	0.0853
	SD	0.0738	0.0753	0.0641
	MSE	0.0159	0.0179	0.0114
		OLS	R-H	MBSLS
$d_1=0.1,\ d_2^*=-0.2$	Mean Bias	0.1097	0.0464	0.0689
	SD	0.0617	0.0569	0.0490
	MSE	0.0158	0.0054	0.0072
		OLS	R-H	MBSLS
$d_1=0.4,\ d_2^*=-0.2$	Mean Bias	0.3322	0.3686	0.2844
	SD	0.1513	0.1478	0.1465
	MSE	0.1332	0.1577	0.1023
		OLS	R-H	MBSLS
$d_1=0.4,\ d_2^*=0.6$	Mean Bias	-0.0009	0.0135	-0.0127
	SD	0.0294	0.0541	0.0253
	MSE	0.0009	0.0031	0.0008

[10] In the table $d_2^*=d_2-1$, and d2 in the graph indicates $d_2^*=d_2-1$ which follows the notation in section 4.

MODEL B: ARMA(1,1)

[Table 6.2] Mean Bias, Standard Deviation(SD), MSE of $(\hat{\beta}-\beta)$, $n=100$

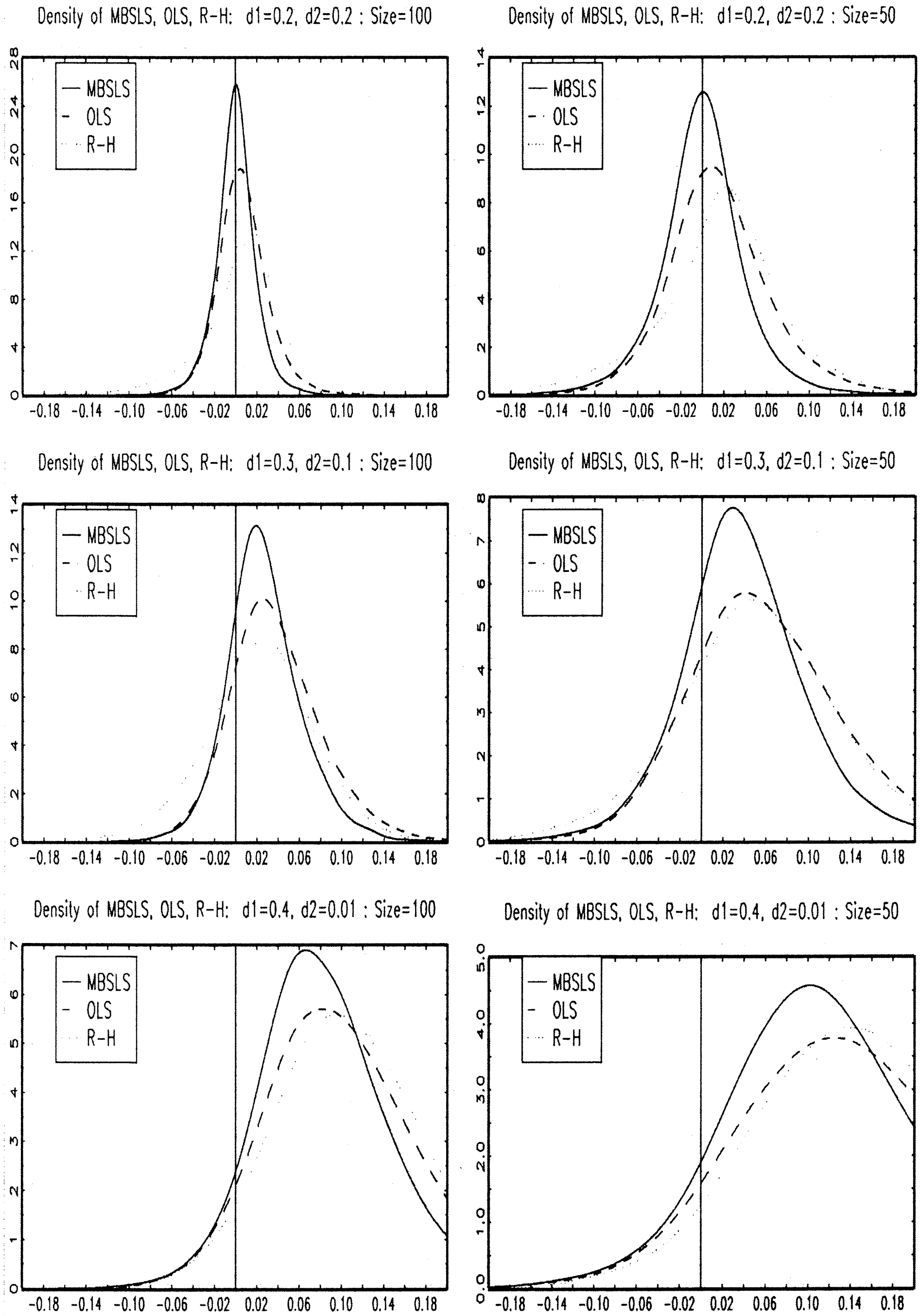
		$n = 100$		
		OLS	R-H	MBSLS
$d_1 = 0.2,\ d_2^* = 0.2$	Mean Bias	0.0060	-0.0008	-0.0001
	SD	0.0236	0.0624	0.0192
	MSE	0.0006	0.0039	0.0004
		OLS	R-H	MBSLS
$d_1 = 0.3,\ d_2^* = 0.1$	Mean Bias	0.0363	-0.0136	0.0268
	SD	0.0435	0.0966	0.0360
	MSE	0.0032	0.0095	0.0020
		OLS	R-H	MBSLS
$d_1 = 0.4,\ d_2^* = 0.1$	Mean Bias	0.0989	0.0098	0.0852
	SD	0.0725	0.1095	0.0636
	MSE	0.0150	0.0121	0.0113
		OLS	R-H	MBSLS
$d_1 = 0.1,\ d_2^* = -0.2$	Mean Bias	0.1004	-0.0581	0.0694
	SD	0.0595	0.1253	0.0494
	MSE	0.0136	0.0191	0.0073
		OLS	R-H	MBSLS
$d_1 = 0.4,\ d_2^* = -0.2$	Mean Bias	0.2611	0.1431	0.2331
	SD	0.1095	0.1294	0.1043
	MSE	0.0801	0.0372	0.0652
		OLS	R-H	MBSLS
$d_1 = 0.4,\ d_2^* = 0.6$	Mean Bias	-0.0014	0.0038	-0.0061
	SD	0.0128	0.0369	0.0109
	MSE	0.0002	0.0014	0.0002

MODEL C: ARMA(0,0)

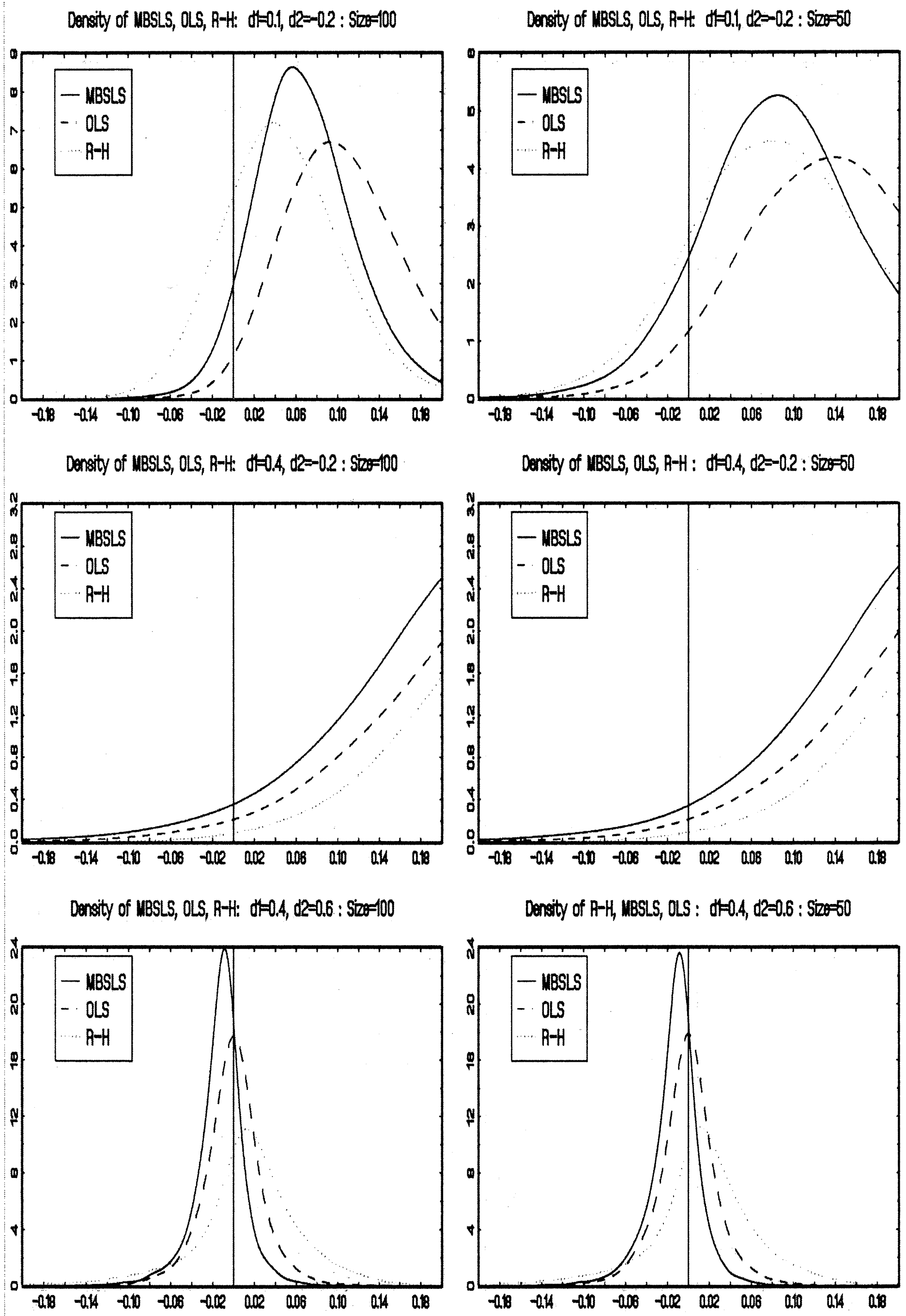
[Table 6.3] Mean Bias, Standard Deviation(SD), MSE of $(\hat{\beta}-\beta)$, $n=100$

		$n=100$		
		OLS	R-H	MBSLS
$d_1=0.2,\ d_2^*=0.2$	Mean Bias	0.0155	-0.0009	0.0003
	SD	0.0276	0.0207	0.0193
	MSE	0.0010	0.0004	0.0004
		OLS	R-H	MBSLS
$d_1=0.3,\ d_2^*=0.1$	Mean Bias	0.0541	0.0115	0.0277
	SD	0.0502	0.0330	0.0355
	MSE	0.0054	0.0012	0.0020
		OLS	R-H	MBSLS
$d_1=0.4,\ d_2^*=0.1$	Mean Bias	0.1286	0.0738	0.0880
	SD	0.0808	0.0539	0.0632
	MSE	0.0231	0.0083	0.0118
		OLS	R-H	MBSLS
$d_1=0.1,\ d_2^*=-0.2$	Mean Bias	0.1859	0.0387	0.0741
	SD	0.0845	0.0459	0.0493
	MSE	0.0417	0.0036	0.0079
		OLS	R-H	MBSLS
$d_1=0.4,\ d_2^*=-0.2$	Mean Bias	0.3326	0.2728	0.2427
	SD	0.1137	0.0931	0.1021
	MSE	0.1236	0.0831	0.0693
		OLS	R-H	MBSLS
$d_1=0.4,\ d_2^*=0.6$	Mean Bias	-0.0011	-0.0002	-0.0062
	SD	0.0128	0.0105	0.0109
	MSE	0.0002	0.0001	0.0002

[Figure 1] Densities of OLS, R-H, and MBSLS, Size : 50,100; ARMA(1,0)



[Figure 2] Densities of OLS, R-H, and MBSLS, Size : 50,100; ARMA(1,0)



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