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# LOGIT SOLUTION AND EQUILIBRIUM SELECTION IN TWO-PERSON GAMES

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*This paper studies the properties of logit solution as a limit of logit equilibrium, where mistake probabilities depend in plausible ways on their expected costs. This dependence sometimes allows logit solution to select an equilibrium that differs from the risk-dominant equilibrium in 2×2 games. This paper also identifies a sufficient condition for logit solution to follow risk-dominance in 2×2 games.*

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## I. INTRODUCTION

Although Nash equilibrium is the most widely used concept in economics, when there are multiple equally plausible Nash equilibria, game theory has been unsuccessful in predicting which Nash equilibrium will occur. Several static notions of refinement have offered partial predictions, such as risk-dominance by Harsanyi and Selten (1988), global games by Carlsson and van Damme (1993) and $p$-dominance by Morris, Rob, and Shin (1995), assuming best responses with some degree of strategic uncertainty. Although those attempts provide clear answers

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for two-person or 2×2 games, the generalization to larger classes of games is not obvious.

McKelvey and Palfrey (1995) proposed a quantal response equilibrium model (QRE) to explain various experimental data. In a laboratory environment, decision errors or mistakes seem pervasive and provide a natural motivation to adopt QRE (see Chen, Friedman and Thisse (1997) for possible interpretations of QRE.) McKelvey and Palfrey (1995, 1998) have shown that QRE is successful in explaining some of the departures from Nash equilibrium, the dynamics, and limiting outcomes observed in experimental data. Although QRE is a static model, one can describe dynamics in terms of "learning by doing." after changing a parameter value. However, little is known about the theoretical properties of a QRE and about the possibility of using a QRE as a selection device. This paper characterizes the implications of QRE as a selection device in normal-form games.

QRE is defined as an equilibrium in which players choose their strategies stochastically, with strategies that have higher expected payoffs chosen with higher probabilities. In a QRE, players rationally expect noise in each other's strategy choice and choose strategies with probabilities that are increasing in their expected payoffs based on the distribution of others' strategies. Thus a QRE does not describe an exact optimizing behavior by assuming a stochastic decision rule, but maintains most of the parsimony of an equilibrium analysis.

QRE allows a wide class of probabilistic choice rules to be used in an equilibrium context. In most applications, QRE is simplified to a logit response function to determine a "logit equilibrium," in which the amount of strategic uncertainty is measured by a single parameter. As the uncertainty vanishes, the stochastic response converges to the best response and the limit point of a sequence of logit equilibrium converges to a Nash equilibrium. Therefore, the limit points can be used to refine the set of Nash equilibria. McKelvey and Palfrey (1995) have shown that the logit equilibrium correspondence which starts from a centroid of strategy simplex, which I call principal branch, defines a unique selection in generic finite games. Following McKelvey and Palfrey (1995), the limit point of the principal branch is referred to as "logit solution" of the game.

The main result is that logit solution coincides with risk-dominant equilibrium (or *p*-dominant equilibrium) in symmetric 2×2 games, but not in general. Since in logit equilibrium, the relative choice probabilities are proportional to the difference in the associated expected payoffs, an increase in the magnitude of payoffs reduces the mistake probability and thus the selection is not invariant to linear payoff transformations. The selection, therefore, could be different even in games with an identical best-reply structure.

The rest of this paper is organized as follows. In Section 2, the notion of logit solution is introduced. Section 3 identifies a sufficient condition for the equivalence between logit solution and *p*-dominant equilibrium that coincides with risk-dominant equilibrium in 2×2 games. Section 4 is the conclusion.

## II. LOGIT EQUILIBRIUM AND LOGIT SOLUTION

Consider a finite $n$-person normal-form game. Players are indexed by $i \in \{1, 2, \cdots, n\}$ and $S_i$ is the finite set of actions available to player $i$. Let $\Sigma = X_i \Sigma_i$ be the set of mixed strategy profiles with $\sigma_i \in \Sigma$ and $\sigma^* \in \Sigma$ denote the mixed-strategy Nash equilibrium. $\sigma_i(s_i)$ is the probability of $s_i \in S_i$ being played by player $i$ and player $i$'s expected payoff is denoted by $\pi_i(\sigma)$.

When imperfect responses are modeled as logit responses, the odds are determined by the exponential transformation of the expected payoff times a given non-negative constant $\lambda$. For a given $\lambda \geq 0$, player $i$ plays pure strategy $s_i$ with probability:

$$\sigma_i(s_i) = \frac{\exp(\lambda \pi_i(s_i, \sigma_{-i}))}{\sum_{s_i \in S_i} \exp(\lambda \pi_i(s_i, \sigma_{-i}))} \tag{1}$$

where $\lambda$ measures the propensity of mistakes. As $\lambda \to \infty$, the probability of the choice having the highest expected payoff becomes one so that the choice behavior becomes the best response; for $\lambda = 0$, all choices have equal probability. IWe assume that, $\lambda$ is identical across players and is common knowledge.

For a given $\lambda$, since $\sigma_i$ is continuous in $\sigma_{-i}$, $\sigma$ has a fixed point. A logit equilibrium for $\lambda$ is defined as the fixed point and shall be denoted by $\sigma^{\lambda}$. Logit solution is the limit point of a sequence of logit equilibria beginning at the centroid strategy (the unique solution when $\lambda = 0$) and continuing for larger values of $\lambda$. Since a logit equilibrium exists for every $\lambda \geq 0$ and the principal branch is upper hemi-continuous, a logit solution exists in every finite game and is uniquely defined in generic finite games.

The selection process of logit solution is similar to Harsanyi and Selten's (1988) tracing procedure, in which the strategy choice is a weighted average of a prior strategy such as the centroid strategy and best responses. That is, for a given $0 \leq t \leq 1$, a strategy choice is $t\sigma^t + (1-t)\sigma^0$ where $\sigma^0$ is the prior strategy and $\sigma^t$ is the best response to $t\sigma^t + (1-t)\sigma^0$. A selection is defined in the limit of $t \to 1$. In both models, the propensity of mistakes is determined by a exogenous parameter and a selection is made in the limit of vanishing mistake probability. In a tracing procedure, however, since the probabilities of sub-optimal strategies being played depend only on the best reply structure, the selection is invariant to linear transformations of payoffs. When the prior strategy is the centroid strategy, the tracing procedure always selects a risk-dominant equilibrium.

In the analysis, logit solution is compared to *p*-dominant equilibrium. In 2×2 games, since *p*-dominant equilibrium coincides with risk-dominant equilibrium and the value of *p* summarized the most of information about best-reply structure, that would give a better idea on the nature of logit solution.

The notion of *p*-dominance is proposed by Morris, Rob, and Shin (1995). An action profile $s$ is *p*-dominant if, for every probability distribution $\sigma$ such that $\sigma_i(s_i) \geq p$ for $i = 1, 2,$ $\pi_i(s_i, \sigma_{-i}) > \pi_i(s_i,$ $\sigma_{-i})$ for all $s_i \neq s_i$. That is, a *p*-dominant action pair is one that remains a best response for every possible strategy mixture which assigns probability of at least *p* to that strategy pair. If there exists a *p* for which an action profile is a unique *p*-dominant action pair, the action pair is defined as *p*-dominant equilibrium. Note that, by definition, *p*-dominant equilibrium is strict and the notion is relevant only when there is a strict equilibrium.

## III. LOGIT SOLUTION IN 2×2 GAMES

Consider the following 2×2 game.

**[Figure 1]** 2×2 Game

| | $L_2$ | $R_2$ |
|---|---|---|
| $L_1$ | $a_1, a_2$ | $b_1, c_2$ |
| $R_1$ | $c_1, b_2$ | $d_1, d_2$ |

Letting $u_i = a_i - c_i$ and $v_i = d_i - b_i$, the best-reply structure of 2×2 games can be fully characterized by $u_i$ and $v_i$. Without loss of generality, the main analysis focuses on the class of games where $u_i v_i > 0$ with $u_i > 0$ for $i = 1, 2$ and $u_1 u_2 > v_1 v_2 > 0$ so that $L \equiv (L_1, L_2)$ and $R \equiv (R_1, R_2)$ are strict equilibria and $L$ is *p*-dominant equilibrium.[1]

First of all, it should be noted that *p*-dominance is invariant to positive linear transformations of payoffs because the selection depends only on $u_i / v_i$. In contrast, the logit equilibrium depends on levels of $u_i$ and $v_i$ as well as the ratio. When a player's expected payoffs are compared, the level can be rescaled by changing the value of $\lambda$. However, because of the equilibrium condition on logit equilibrium strategies, players' logit equilibrium strategies depend not only on their own $\lambda$ but also on others' $\lambda$ s. When different scales are applied to each player's payoffs, say, multiplied by $\alpha_i > 0$, it is equivalent to introducing an individualized $\lambda_i$ to the original game, with $\lambda_i = \alpha_i \lambda$. Given $\lambda$, a sufficiently small $\alpha_i$ would keep player $i$ to play a strategy close to the centroid strategy for a sufficiently large $\lambda$ for which his opponent's logit equilibrium strategy is close to the associated best response. As a consequence, the logit solution is likely to be the equilibrium

[1] Note that $L$ is a *p*-dominant action pair for $p > \max_i \sigma_i^*(L_i)$.

corresponding to the best response of the player with a bigger $\alpha_i$ to the centroid strategy. Because of this property, logit solution is different from $p$-dominance even in 2×2 games. In some special cases, however, both select the same equilibrium.

**Proposition 1.** In 2×2 games, if $u_i u_2 > v_1 v_2$ and $u_1 - v_1 > v_2 - u_2$, then $L$ is the logit solution.

**Proof.** First, consider a game where $L$ is a $p$-dominant action pair for $p < \frac{1}{2}$. Then, it is sufficient to show that a $\frac{1}{2}$-dominant action pair is a best reply in every logit equilibrium along the principal branch. Let $s$ be a $\frac{1}{2}$-dominant action pair. Since $s$ is the best reply to $\sigma^0$, $\sigma_i^\lambda(s_i) > \frac{1}{2}$ for a sufficiently small $\lambda$. If it is not a best reply to some logit equilibria on the principal branch, then as $\lambda$ increases, there exists a $\lambda'$ such that $\sigma_i^{\lambda'}(s_i^{'}) = \frac{1}{2}$ and $\sigma_i^{\lambda'}(s_i^{'}) > \frac{1}{2}$ for $s_i^{'} \neq s_i$. That contradicts to that $s$ is a $\frac{1}{2}$-dominant action pair.

Now consider a game where $L$ is a unique $p$-dominant action pair for some $p > \frac{1}{2}$. Since $u_1 u_2 > v_1 v_2$, without loss of generality, suppose that $\sigma_2^*(L_2) < \sigma_1^*(R_1) < \frac{1}{2} < \sigma_1^*(L_1) < \sigma_2^*(R_2)$, where $\sigma^*$ is the strictly mixed-strategy Nash equilibrium strategy. Then $L$ is $p$-dominant action pair for $p > \sigma_1^*(L_1)$ and $\sigma_1^*(L_1) + \sigma_2^*(L_2) < 1$. To reach a contradiction, suppose that the principal branch converges to either $R$ or $\sigma^*$. Letting $d\pi_i = \pi_i(L_i, \sigma_j^\lambda) - \pi_i(R_i, \sigma_j^\lambda)$, from Eq.(1),

$$\sigma_1^\lambda(L_1) + \sigma_2^\lambda(L_2) = \frac{1}{1+\exp(-\lambda d\pi_1)} + \frac{1}{1+\exp(-\lambda d\pi_2)}$$
$$= \frac{2+\exp(-\lambda d\pi_1)+\exp(-\lambda d\pi_2)}{1+\exp(-\lambda d\pi_1)+\exp(-\lambda d\pi_2)+\exp(-\lambda d\pi_1 - \lambda d\pi_2)}$$

Since $d\pi_1 + d\pi_2 = \frac{(u_1-v_1)+(u_2-v_2)}{2} > 0$ for $\lambda = 0$, by continuity there exists a small $\lambda$ such that $\sigma_1^\lambda(L_1) + \sigma_2^\lambda(L_2) > 1$. Therefore, if the principal branch converges to $R$ or $\sigma^*$, there should exist a $\lambda > 0$ for which $\sigma_1^\lambda(L_1) + \sigma_2^\lambda(L_2) = 1$.

Under the hypothesis, if $\sigma_1^\lambda(L_1) \leq \frac{1}{2}$ and $\sigma_2^\lambda(L_2) \geq \frac{1}{2}$, $R_1$ and $L_2$

are weakly better choices and we have $\sigma_2^\lambda(R_2) \ge \sigma_2^*(R_2)$ and $\sigma_1^\lambda(R_1) \ge \sigma_1^*(R_1)$ . However, since $\sigma_2^\lambda(R_2) = 1-\sigma_2^\lambda(L_2) \ge \sigma_2^*(R_2) = 1-\sigma_2^*(L_2)$ or $\sigma_2^\lambda(L_2) \le \sigma_2^*(L_2) < \frac{1}{2}$, that contradicts to $\sigma_2^\lambda(L_2) \ge \frac{1}{2}$. And the same argument applies to the case of $\sigma_1^\lambda(L_1) \ge \frac{1}{2}$ and $\sigma_2^\lambda(L_2) \le \frac{1}{2}$. Therefore, if $\sigma_1^\lambda(L_1) + \sigma_2^\lambda(L_2) < 1$ for a sufficiently small $\lambda$ , the principal branch converges to *L*. Q.E.D.

In games with a *p*-dominant equilibrium for $p < \frac{1}{2}$, both players "prefer" the same equilibrium in the sense of risk dominance as *L* is mutually best responses as long as $\sigma_i(L_i) > \frac{1}{2}$ . As *p*-dominant equilibrium is invariant to monotone payoff transformations, logit equilibrium coincides with *p*-dominant equilibrium in such games. In symmetric games or in games with $\frac{1}{2}$-dominant action pair, since we have $\max_i \sigma_i^*(L_i) < \frac{1}{2}$ or $\frac{v_i}{u_i+v_i} < \frac{1}{2}$ for $i = 1,2$, $u_1u_2 > v_1v_2$ if and only if $u_1 + u_2 > v_1 + v_2$ . Otherwise, however, $u_1u_2 > v_1v_2$ does not imply $u_1 + u_2 > v_1 + v_2$. Moreover, analogies of logit solution with different noisy responses produce the same selection as long as $\sigma_i^\lambda(s_i)$ is non-decreasing in $\pi_i(s_i, \sigma_{-i})$.

By contrast, in games with *p*-dominant equilibrium for $p > \frac{1}{2}$, players "prefer" different equilibria and the conflict is resolved by the relative attractiveness of each equilibrium to each player. Note that $\sigma_1^*(L_1) + \sigma_2^*(L_2)$ with $\sigma_1^*(L_1) > \frac{1}{2} > \sigma_2^*(L_2)$, where $\sigma^*$ is the mixed-strategy Nash equilibrium, implies that player 1 is more strongly attracted to *L* than the other player to *R* when there is a sizable strategic uncertainty. Moreover, since $\frac{\partial \pi_i(\sigma)}{\partial \sigma_i(R_i)} = -u_i\sigma_j(L_j) + v_i\sigma_j(R_j)$ , $i \ne j$ , and the condition $u_1 - v_1 > v_2 - u_2$ implies that player 1's deviation to $R_1$ is more costly than player 2's deviation to $L_2$, player 2 tends to make more mistakes for a given $\lambda$. Therefore, even if *R* is more preferred equilibrium to player 2, since player 1 has a stronger incentive to choose $L_1$, it becomes better for player 2 to play $L_2$ rather than $R_2$.[2]

[2] Another sufficient condition by Turocy (2005) is that, for all $\lambda$, $\partial\sigma_i(s_i)/\partial\lambda > \partial\sigma_i(s_i^{'})/\partial\lambda$ if

To some extent, logit solution can be thought of as an equilibrium that is easier to change from the other equilibrium, while $p$-dominance selects one that is harder to change to the other equilibrium. In 2×2 games, since there are at most two strict equilibria, there is no big difference in the selection criterion except the ways to determine mistake probabilities. However, in many-action games, since there are multiple ways to deviate, there is a fundamental difference between those two notions. Logit solution considers strictly mixed choice probability over equilibrium and non-equilibrium strategies, while $p$-dominance relies on pair-wise comparisons of equilibrium strategies. Logit solution is more sensitive to differences in payoffs. To see this more clearly, consider the following game.

**[Figure 2]** A 3×3 Game

| | $L_2$ | $M_2$ | $R_2$ |
|---|---|---|---|
| $L_1$ | 10, 10 | 0, 0 | 0, 0 |
| $M_1$ | 0, 0 | $10-\varepsilon$ , $10-\varepsilon$ | 9, 9 |
| $R_1$ | 0, 0 | 9, 9 | $10-2\varepsilon$ , $10-2\varepsilon$ |

In the game above with $\varepsilon > 0$, $L$ is the $p$-dominant equilibrium while $M$ is the logit solution. Since $L$ pairwise-risk-dominates $M$ and $R$, it requires the largest propensity of mistakes to upset $L$. However, for a sufficiently small $\varepsilon$, it might be sensible to choose $M_i$ as it involves less risk than $L_i$ with some degree of strategic uncertainty. Because of such a feature, it is difficult to characterize logit solution in many-action games. However, if a equilibrium is the hardest to upset with any kind of mistakes in many-action games, the same idea of Proposition 1 can be applied directly.

**Proposition 2.** In two-person $k_1 \times k_2$ games, if there is $p$-dominant action pair for $p \le \frac{1}{k}$ where $k = \max(k_1, k_2)$, the logit solution is $p$-

$\pi_i(s_i) > \pi_i(s_i^{'})$. In fact, this condition is equivalent to the existence of a $p$-dominant action pair for $p < \frac{1}{2}$, which is a special case of the condition in Proposition 1.

dominant equilibrium.

The proof is direct from the first part of the proof of Proposition 1 with $k$ in place of 2. For $\lambda = 0$, by definition, $\frac{1}{k}$-dominant action pair is a mutually best response. As $\lambda$ increases, the $\frac{1}{k}$-dominant equilibrium action pair remains a mutually best response along the whole principal branch.

## IV. CONCLUDING REMARKS

Since the logit equilibrium model well describes experimental data, it has become a standard tool in experimental literature, but little is known about its properties as a selection device. Although the present result is limited to a class of two-person normal-form games, it explains how a coordination problem is resolved when payoffs depend on mistakes, and the present study would help understand choice behavior in a wider class of coordination games.

## References

Carlsson, Hans and Eric Van Damme (1993), "Global Games and Equilibrium Selection," *Econometrica*, 61, pp. 989-1018.

Chen, Hsiao-Chi, James Friedman and Jacques-Francois Thisse (1997), "Boundedly Rational Nash Equilibrium: A Probabilistic Choice Approach," *Games and Economic Behavior*, 18, pp. 32-54.

Harsanyi, John and Reinhard Selten (1998), *A General Theory of Equilibrium Selection in Games*, Cambridge: MIT Press.

Michihiro Kandori, George Mailath, and Rafael Rob (1993), "Learning, Mutation, and Long Run Equilibria in Games," *Econometica*, 61, pp. 29-56.

Michihiro Kandori and Rafael Rob (1998), "Bandwagon Effects and Long Run Technology Choice," *Games and Economic Behavior*, 22, pp. 30-60.

McKelvey, Richard and Thomas Palfrey (1995), "Quantal Response Equilibria For Normal-Form Games," *Games and Economic Behavior*, 10, pp. 6-38.

Morris, Stephen, Rafael Rob, and Hyun Song Shin (1995), "*p*-dominance and Belief Potential," *Econometrica*, 63, pp. 145-157.

Turocy, Theodore (2005), "A Dynamic Homotopy Interpretation of the Logitstic Quantal Response Equilibrium Correspondence," *Games and Economic Behavior*, 51, pp. 243-263.