THE KOREAN ECONOMIC REVIEW
Volume 23, Number 2, Winter 2007

# EQUILIBRIUM IN PRODUCTION ECONOMIES WITH NONTRANSITIVE AND SATIABLE PREFERENCES *

GUANGSUG HAHN **

*It is shown that an equilibrium exists in a production economy when preferences are allowed to be nontransitive and satiated. In addition to the standard assumptions, a new condition is introduced to handle satiation. The conventional excess demand approach is modified to take into account the effect of satiation on equilibrium.*

JEL Classification: C62, D51
Keywords: Nontransitive Preferences, Satiation, Production Economies, Competitive Equilibrium

## I. INTRODUCTION

In standard Arrow-Debreu general equilibrium models, nonsatiation assumption is usually imposed to get an equilibrium existence result, because otherwise, no equilibrium may exist. However, the satiation of preferences arises as a natural phenomenon when preferences on financial assets are considered, in particular, in capital asset pricing models or when the choice set is compact (*e.g.*, choice variables are probability distributions on a finite set). There exist interesting attempts to address the existence issue with satiable preferences, which are roughly divided

---

*Received for publication: Nov. 26, 2006. Revision accepted: July 20, 2007.*
* The author appreciates the comments from the participants at the 12th International Conference of the Korean Economic Association, August 2006. He is also grateful to Dongchul Won and Nicholas C. Yannelis for their helpful comments. But usual disclaimer applies.
** Division of Humanities and Social Sciences, POSTECH, KOREA, Email: econhahn@postech.ac.kr

into two approaches. The first one is to employ a weaker notion of equilibrium, and the second one is to impose conditions on economies with taking advantage of the conventional notion of equilibrium. An example of the former approach is the study by Mas-Colell (1992). This study introduces 'equilibrium with slack' or dividend equilibrium, in which agents are given positive dividend from outside to attain an equilibrium. While this equilibrium is weakly efficient, slack variables are considered inconsistent with decentralized markets. In Cornet *et al.* (2003), the notion of dividend equilibrium is applied to production economies with a continuum of agents. Polemarkchakis and Siconolfi (1993) enforce consumers to spend all the income in the weak competitive equilibrium, even though a consumer would be happier otherwise. Indeed, a weak competitive equilibrium may be inefficient and the enforcement is an unnatural device in competitive markets.

In contrast, Kajii (1996) and Won and Yannelis (2005), who follow the second line of research, keep the conventional competitive equilibrium as their equilibrium concepts. It is assumed by Kajii (1996) that agents are endowed with a positive amount of fiat money which does not affect agents' welfare directly. However, a consumer is allowed to trade his endowments with money, but may end up with having money of no further use in a competitive equilibrium, which is an uninteresting aspect of the model.[1] Instead of injecting outside money into the economy, Won and Yannelis (2005) impose a condition on the primitives of an exchange economy while preserving the notion of standard competitive equilibrium. In fact, their condition characterizes the state of income distribution that is sufficient for the existence of a competitive equilibrium.

Even though the transitivity axiom is necessary to the utility representation of preference relation, in reality, an individual's preferences may not satisfy the transitivity axiom. It is well known that transitive preferences fail to explain important anomalous phenomena such as preference reversal.[2] As shown in May (1956), we can easily construct an example where preferences do not exhibit transitivity. Consider preferences on consumptions in $\mathbb{R}^3$. Suppose a person prefers a

[1] On the other hand, in a dynamic monetary economy with infinite horizon, this idea seems plausible for dealing with economies with satiable preferences.

[2] Loomes *et al.* (1991) detect the preference reversal phenomenon through experiments.

consumption bundle to another if two components of the former are greater than the two corresponding components of the latter. Then, he prefers (1, 2, 3) to (3, 1, 2) or (3, 1, 2) to (2, 3, 1). But he does not prefer (1, 2, 3) to (2, 3, 1), which means that his preferences are not transitive. Moreover, when a household as group decision maker is a consumer, then its preferences may not be transitive, due to the aggregation problem of individuals' preferences. Considering the possibility of nontransitive preferences, Sonnenschein (1971) introduces a model of demand without transitivity, which is incorporated by Mas-Colell (1974), Gale and Mas-Colell (1975, 1979), and Shafer and Sonnenschein (1975) into general equilibrium theory for economies with nontransitive preferences satisfying nonsatiation.

The purpose of this paper is to show the existence of equilibrium in a production economy where preferences are allowed to be nontransitive and satiated. We extend the approach of Won and Yannelis (2005) to a production economy by adapting their condition.[3] Unlike Won and Yannelis (2005), the standard excess demand approach is modified to take into account the effect of satiation on equilibrium.

This paper is organized as follows. In the next section, a production economy is introduced where preferences need not be transitive and are possibly satiated. In Section 3, we provide an equilibrium existence theorem for the production economy.

## II. THE MODEL

We consider a production economy with $n$ consumers, indexed by $i \in I = \{1, \ldots, n\}$, and $m$ firms, indexed by $j \in J = \{1, \ldots, m\}$. Let $\mathbb{R}^{\ell}$ be the commodity space of the production economy. The set, $X_i \subset \mathbb{R}^{\ell}$, is the consumption set for consumer $i$, and $Y_j \subset \mathbb{R}^{\ell}$ is the production set for firm $j$. Let $X = \Pi_{i \in I} X_i$ and $Y = \Pi_{j \in J} Y_j$. Let $Y_0 = \sum_{j \in J} Y_j$ denote the aggregate production set. Consumer $i \in I$ owns $\theta_{ij}$ share of firm $j \in J$. The preferences of agent $i \in I$ are denoted by $\succsim_i$, which is

[3] Even though Won and Yannelis (2005) deal with an exchange economy, their framework is much more general than mine in other respects, since they assume that consumption sets are unbounded and preferences are nonordered and interdependent.

reflexive and complete. We define a upper contour correspondence $R_i : X_i \to 2^{X_i}$ by $R_i(x_i) = \{x_i' \in X_i : x_i' \succsim_i x_i\}$. For a point $x_i \in X_i$, we define the preference correspondence $P_i : X_i \to 2^{X_i}$ by $P_i(x_i) = \{x_i' \in X_i : x_i' \succ_i x_i\}$, which is the set of consumption bundles of consumer *i* being preferred to $x_i$. Observe $P_i(x_i) = R_i(x_i) \setminus R_i^{-1}(x_i)$.[4] Recall that $\succsim_i$ (or $R_i$) is complete if $P_i(x_i) = X_i \setminus R_i^{-1}(x_i)$ for every $x_i \in X_i$. Consumer *i* is endowed with a consumption bundle $e_i \in X_i$. Thus the production economy is defined as $\mathcal{E} := ((X_i, \succsim_i, e_i), (\theta_{ij}), (Y_j))$.

Let $\Delta := \overline{B}_1(0)$ be the set of price vectors, where $\overline{B}_1(0)$ is the closed ball centered at 0 with radius 1. The profit function $\pi_j : \Delta \to \mathbb{R}$ of firm *j* is defined by $\pi_j(p) := \sup p \cdot Y_j$. The income function $M_i : \Delta \to \mathbb{R}$ of consumer *i* is defined by $M_i(p) := p \cdot e_i + \sum_{j \in J} \theta_{ij} \pi_j(p)$. Let us define the budget correspondence $\mathcal{B}_i : \Delta \to 2^{X_i}$ by $\mathcal{B}_i(p) := \{x_i \in X_i : p \cdot x_i \le M_i(p)\}$ and the open budget correspondence $\mathcal{B}_i^\circ : \Delta \to 2^{X_i}$ by $\mathcal{B}_i^\circ(p) := \{x_i \in X_i : p \cdot x_i < M_i(p)\}$.

Let us define the attainable set by $A := \{(x, y) \in X \times Y : \sum_{i \in I}(x_i - e_i) = \sum_{j \in J} y_j\}$, and let $A_X$ be the projection of A onto $X$, $\widetilde{X}_i$ be the projection of A onto $X_i$ (for every $i \in I$), and $\widetilde{Y}_i$ be the projection of *A* onto $Y_i$ (for every $i \in J$). A *competitive equilibrium* for $\mathcal{E}$ is a triple $(p, x, y) \in (\mathbb{R}^\ell \setminus \{0\}) \times A$, such that (1) $x_i \in \mathcal{B}_i(p)$ for all $i \in I$, (2) $P_i(x_i) \cap \mathcal{B}_i(p) = \emptyset$ for all $i \in I$, and (3) $p \cdot y_j = \sup p \cdot Y_j$ for all $j \in J$. Let us define the *augmented preference correspondence* $\widetilde{P}_i : X_i \to 2^{X_i}$ by $\widetilde{P}_i(x_i) = \{(1-\alpha)x_i + \alpha x_i' : \alpha \in (0, 1],\ x_i' \in P_i(x_i)\}$ and similarly $\widetilde{R}_i : X_i \to 2^{X_i}$ by $\widetilde{R}_i(x_i) = c\ell \widetilde{P}_i(x_i)$.[5] It is obvious that $P_i(x_i) \subset \widetilde{P}_i(x_i)$ for all $x_i \in X_i$. We need the following assumptions for the main results.

**Assumptions:** For every $i \in I$ and $j \in J$,

[4] For any correspondence $\varphi_i : X \to 2^Y$, in general, the *lower section* $\varphi^{-1} : Y \to 2^X$ of $\varphi$. is defined by $\varphi^{-1}(x) = \{x \in X : y \in \varphi(x)\}$. For instance, $R_i^{-1}(x_i) = \{y_i \in X_i : x_i \in R_i(y_i)\}$.

[5] For a set $S$ in $\mathbb{R}^\ell$, $c\ell S$ denotes the *closure* of $S$, $\operatorname{int} S$ the *interior* of $S$, $coS$ the *convex hull* of $S$, and $\overline{co}S$ the *closed convex hull* of $S$.

**A1.** $X_i$ is closed, convex, and bounded from below in $\mathbb{R}^{\ell}$.
**A2.** $x_i \notin coP_i(x_i)$ for every $x_i \in X_i$.
**A3.** $P_i$ is lower hemicontinuous on $X_i$.
**A4.** $P_i$ is open-valued on $X_i$.
**A5.** $\succsim_i$ is complete.
**A6.** $0 \in Y_j$.
**A7.** $Y_0$ is a closed convex set in $\mathbb{R}^{\ell}$.
**A8.** $Y_0 \cap (-Y_0) = \{0\}$.
**A9.** $(-\mathbb{R}^{\ell}_{+}) \subset Y_0$.
**A10.** $M_i(p) >$ inf $p \cdot X_i$ for every $p \in \Delta \setminus \{0\}$.

Most of the assumptions follow those of Gale and Mas-Colell (1975, 1979). It is assumed that the preferences are complete but need not be transitive. Assumption **A2** is a weak convexity assumption and implies that $P_i$ is irreflexive, *i.e.*, $x_i \notin P_i(x_i)$.[6] It is also noted that, if this assumption is satisfied, then $\widetilde{P}_i$ and $\widetilde{R}_i$ are convex-valued. Assumption **A3** is suggested by Gale and Mas-Colell (1979). One can notice that Assumption **A4** implies that $\widetilde{P}_i$ is open-valued on $X_i$. Assumptions **A6** to **A10** are standard in production economies. In particular, Assumptions **A8** (irreversibility) and **A9** (free disposal) are needed to guarantee that each attainable production set is compact.[7] It is also noted that our assumptions on production are mostly stronger than those of Cornet *et al.* (2003), who adopt a different notion of equilibrium from this paper, while our assumptions on preferences are not.

## III. EQUILIBRIUM EXISTENCE

To handle the difficulty with satiation, for each $x \in A_X$, we define the following index sets: $I(x) = \{i \in I : P_i(x_i) \neq \emptyset\}$ and $I^s(x) = I \setminus I(x_i)$. That is, $I^s(x)$ denotes the set of agents who are satiated at the allocation $x \in A_X$, at which agents in $I(x)$ are not satiated.

[6] Due to this, we do not have to assume the irreflexivity of preferences.
[7] Just after Example 3.3 in Section 3, we will come back to this point.

**DEFINITION 3.1:** A production economy $\mathcal{E}$ is said to admit the price support at satiation (PSS) if each $x \in A_X$ with $I^s(x) \neq \emptyset$ has the property that whenever there exists $p \in \mathbb{R}^\ell$ such that $p \cdot x_i' \geq p \cdot x_i$ for all $i \in I(x)$ and $x_i' \in P_i(x_i)$, it holds that $p \cdot x_i \geq M_i(p)$ for all $i \in I^s(x)$.[8]

Let $x \in A_X$ be any allocation with $I^s(x) \neq \emptyset$. The economy $\mathcal{E}$ admits PSS whenever $p \in \mathbb{R}^\ell$ supports $x_i$ for all $i \in I(x)$. The price $p$ keeps the value of the satiation consumption $x_i$, not less than $M_i(p)$, for each $i \in I^s(x)$. In fact, the role of the PSS condition is to exclude the cases where the satiation consumption bundles for some consumer may be optimal inside the budget set at candidate equilibrium prices. As exemplified below, the presence of satiation consumptions may cause the non-existence of equilibrium. To ensure the existence of equilibrium with satiable preferences, it turns out indispensable to impose the PSS condition on the economies under study.

**EXAMPLE 3.1:** For simplicity, we consider an economy where there is a single firm with $Y = -\mathbb{R}^2_+$, and there are two consumers whose endowments and shares of the firm are given by $(e_1, \theta_1) = ((1/2, 1), 1/2)$ and $(e_2, \theta_2) = ((3/2, 1), 1/2)$, respectively, with $X_i = \mathbb{R}^2_+$ for every $i$ =1, 2. The utility functions of consumers are given by:

$$u_1(a,b) = 3(a+b) - (a^2 + b^2),$$
$$u_2(a,b) = (a+b) - \frac{1}{2}(a^2 + b^2).$$

It is easy to check that consumer 1 is satiated at $s_1 = (3/2, 3/2)$, while consumer 2 is satiated at $s_2 = (1, 1)$. One can verify that there is no equilibrium in this economy. An equilibrium price must be a price supporting some efficient allocation. In this economy, there is a unique

[8] A similar condition for an exchange economy can be found in Won and Yannelis (2005). On the other hand, the related assumption (LNS (*i*), p.871) of Martins-da-Rocha (2003) can be rewritten in our setting as: For every $i \in I^s(x)$, it holds that $x_i \geq e_i + \sum_{j \in J} \theta_{ij} y_j$, $\forall y_j \in Y_j$, $\forall j \in J$. Moreover, he adopts the free disposal assumption and therefore prices are nonnegative. In this case, the PSS condition trivially holds because for every $p \in \mathbb{R}^\ell_+$, it holds that $p \cdot x_i \geq M_i(p)$ for every $i \in I^s(x)$.

normalized price, $p = (1/2, 1/2)$, that supports efficient allocations. Therefore, the optimal production choice of the firm must be $y = (0,0)$. Observe that the price, $p$, supports the preferred set $P_1(x_1)$ of consumer 1 at $x_1 = (1, 1)$ but makes the value of $e_2$ greater than that of $s_2$. This fact implies that this economy does not satisfy the PSS condition.[9]

When the PSS condition is satisfied in an economy, we can find a competitive equilibrium in the economy, as the following example illustrates.

**EXAMPLE 3.2:** We consider an economy which has the same environments as the economy in Example 3.1. [except that two consumers' endowments and shares of the firm are given by $(e_1, \theta_1) = ((1, 3/2), 1/2)$ and $(e_2, \theta_2) = ((1, 1/2), 1/2)$ ]. Consumer 1 is satiated at $s_1 = (3/2, 3/2)$, while consumer 2 is satiated at $s_2 = (1, 1)$. Recall that since there is a unique normalized price $p = (1/2, 1/2)$ that supports efficient allocations, the optimal production choice of the firm must be $y = (0, 0)$. Observe that the price, $p$, supports the preferred set $P_1(x_1)$ of consumer1 at $x_1 = (1, 1)$ but makes the value of $e_2$ not greater than that of $s_2$. Hence, the PSS condition is satisfied. Moreover, one can show that there is an equilibrium $(p^*, x^*, y^*)$, where:

$$p^* = (1/2, 1/2),\, x^* = (x_1^*, x_2^*) = ((5/4, 5/4), (3/4, 3/4)),\, y^* = (0, 0).$$

While the PSS condition is indispensable to the equilibrium existence, as shown above, it may not be a necessary condition, as shown below.

**EXAMPLE 3.3:** Consider an economy where there is a single firm with $Y = -\mathbb{R}^2_+$, and there are two consumers whose endowments and shares of the firm are given by $(e_1, \theta_1) = ((1/9, 17/12), 1/2)$ and $(e_1, \theta_1) = ((17/9, 7/12), 1/2)$, respectively, with $X_i = \mathbb{R}^2_+$, for every $i = 1, 2$. The

9 We can construct an example with a nontrival production set. Indeed, we take $Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \leq \ln(-y_1 + 1)\}$ instead of $Y = -\mathbb{R}^2_+$. Moreover, we assume $(\theta_1, \theta_2) = (1/6, 5/6)$ while keeping the other characteristics of the economy the same as before. By similar arguments to those in Example 3.1, one can show that no equilibrium exists and the PSS condition fails.

utility functions of consumers are given by:

$$u_1(a,b) = \min\{4a - 1/2, 3b\},$$
$$u_2(a,b) = (a+b) - (a^2 + b^2).$$

Consumer 1 is satiated nowhere, while consumer 2 is satiated at $s_2 = (1/2, 1/2)$. One can verify that there is a competitive equilibrium $(p^*, x^*, y^*)$ where:

$$p^* = (3, 4), \quad x^* = (x_1^*, x_2^*) = ((4/5, 9/10), (6/5, 11/10)), \quad y^* = (0,0).$$

Now we take the allocation $(x, y)$, where $x = (x_1, x_2) = ((1/2, 1/2), (3/2, 3/2))$ and $y = (0, 0)$. Then the price $p = (3, 1)$ supports the preferred set $P_1(x_1)$ of consumer 1 at $x_1$. But the value of $e_2$ becomes greater than that of $s_2$ at the price $p$. Therefore, this economy does not satisfy the PSS condition.

One may recall that $\sum_{j\in J} \overline{coY_j} = Y_0$ [See Debreu (1982)]. Now we consider the economy $\overline{\mathcal{E}} = ((X_i, \succsim_i, e_i), (\theta_{ij}), (\overline{coY_j}))$. Consider an attainable allocation $(x, y)$ (*i.e.*, $\sum_i (x_i - e_i) = \sum_j y_j$). Since $X_i$ is bounded from below, there is $b_i \in \mathbb{R}^\ell$ such that $x_i \geq b_i$, $\forall x_i \in X_i$. Therefore, one has $\sum_j y_j \geq \sum_i (b_i - e_i)$. Due to Lemma A.2. of Smale (1982), Assumptions **A8** and **A9** imply that the set $\widetilde{Y}_j$ of the attainable productions of producer *j* in the economy $\overline{\mathcal{E}}$ is bounded for every $j \in J$. Since $\mathcal{E}$ and $\overline{\mathcal{E}}$ has the same total production set $Y_0$, it is true that $x_i$ is an attainable consumption in $\mathcal{E}$, if and only if $x_i$ is an attainable consumption in $\overline{\mathcal{E}}$. Therefore, it is easy to see that, for every $i \in I$, the set of $\widetilde{X}_i$ of the attainable consumptions of consumer *i* in the economy $\overline{\mathcal{E}}$ is bounded, since $X_i$ is bounded from below. Consequently, in the economy $\overline{\mathcal{E}}$, $\widetilde{X}_i$ is compact for every $i \in I$, and $\widetilde{Y}_j$ is compact for every $j \in J$.

Take a compact convex set $K$ in $\mathbb{R}^\ell$, such that $\widetilde{X}_i \subset \operatorname{int} K$, $\forall i \in I$ and $\widetilde{Y}_j \subset \operatorname{int} K$, $\forall j \in J$. Define $\hat{X}_i = X_i \cap K$, $\forall i \in I$ and $\hat{Y}_j = \overline{coY_j} \cap K$, $\forall j \in J$ and let $\hat{X} = \prod_{i\in I} \hat{X}_i$ and $\hat{Y} = \prod_{j\in J} \hat{Y}_i$. For preferences on $\hat{X}_i$, we define $\hat{R}_i(x_i) = \widetilde{R}_i(x_i) \cap K$ and $\hat{P}_i(x_i) = \widetilde{P}_i(x_i) \cap K$. We consider the production economy $\hat{\mathcal{E}} = ((\hat{X}_i, \hat{R}_i, e_i), (\theta_{ij}), (\hat{Y}_j))$. Observe that an allocation $(x, y)$ is attainable for the economy $\overline{\mathcal{E}}$, if and only if it is so for the economy $\hat{\mathcal{E}}$. Therefore, the attainable set of consumer $i$ in $\hat{\mathcal{E}}$ is again $\widetilde{X}_i$.

It also needs to be noted that since $\widetilde{X}_i \subset \operatorname{int} K$ (for every $i \in I$) and $\widetilde{Y}_j \subset \operatorname{int} K$ (for every $j \in J$), the economy $\mathcal{E}$ satisfies the PSS condition, if and only if the economy $\hat{\mathcal{E}}$ satisfies the PSS condition.

We denote $\hat{Y}_0 = \sum_{j\in J} \hat{Y}_j$. Define the profit function $\hat{\pi}_j : \Delta \to \mathbb{R}$ of firm $j$ by $\hat{\pi}_j(p) = \sup p \cdot \hat{Y}_j$. Since $\hat{Y}_j$ is compact, one sees that $\hat{\pi}_j$ is well defined and is continuous on $\Delta$. Let $\hat{M}_i(p) = p \cdot e_i + \sum_{j\in J} \theta_{ij}\hat{\pi}_j(p)$, $\forall i \in I$. Then $\hat{M}_i$ is also continuous on $\Delta$. Let us define the supply correspondence $\eta_j : \Delta \to 2^{\hat{Y}_j}$ of firm $j$ is defined by $\eta_j(p) := \{y_j \in \hat{Y}_j : p \cdot y_j = \sup p \cdot \hat{Y}_j\}$ and define the total supply correspondence $\eta : \Delta \to 2^{\hat{Y}_0}$ by $\eta(p) = \sum_{j\in J} \eta_j(p)$. It is obvious that correspondences $\eta_j$'s and $\eta$ are upper hemicontinuous with nonempty compact convex values.[10]

To proceed, for every $i \in I$, we introduce a set:

$$C_i = \left\{ x_i \in \hat{X}_i \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{(p^n, x_i^n)\} \text{ in } (\Delta \setminus \{0\}) \times \hat{X}_i \\ \text{converging to } (0, x_i) \text{ such that } p^n \cdot x_i^n \le \hat{M}_i(p^n) + (1 - \| p^n \|). \end{array} \right\}.$$

Given this set, we can define the modified budget correspondence $\hat{\mathcal{B}}_i : \Delta \to 2^{\hat{X}_i}$ by:

$$\hat{\mathcal{B}}_i(p) = \begin{cases} \{x_i \in \hat{X}_i : p \cdot x_i \le \hat{M}_i(p) + (1 - \| p \|)\}, & \text{if } p \in \Delta \setminus \{0\}, \\ C_i, & \text{if } p = 0 \end{cases}$$

and the modified *open* budget correspondence $\hat{\mathcal{B}}_i^\circ : \Delta \to 2^{\hat{X}_i}$ by:

$$\hat{\mathcal{B}}_i^\circ(p) = \hat{X}_i \setminus c\ell(\hat{X}_i \setminus \hat{\mathcal{B}}_i(p)).$$

The individual demand correspondence $\xi_i : \Delta \to 2^{\hat{X}_i}$ of consumer $i \in I$ is defined by:

---

[10] See Lemma 1 of Debreu (1982).

$$\xi_i(p) = \{x_i \in \hat{\mathcal{B}}_i(p) : \hat{P}_i(x_i) \cap \hat{\mathcal{B}}_i(p) = \emptyset\}.$$

One can easily verify the following claims.

**LEMMA 3.1:** For every $i \in I$, the following hold.

(1) Under Assumption **A10**, the correspondence $\hat{\mathcal{B}}_i$ is continuous with nonempty compact convex values on Δ.

(2) Under Assumption **A10**, the correspondence $\hat{\mathcal{B}}_i^\circ$ has open graph with nonempty convex values on Δ.

(3) Under Assumptions **A4** and **A10**, the correspondence $\xi_i$ can be expressed as:

$$\xi_i(p) = \{x_i \in \hat{\mathcal{B}}_i(p) : \hat{P}_i(x_i) \cap \hat{\mathcal{B}}_i^\circ(p) = \emptyset\}.$$

Then, by the following lemma, the correspondence $\xi_i$ is upper hemicontinuous with nonempty compact convex values for every $i \in I$.

**Lemma 3.2:** Under Assumptions **A1-A4** and **A10**, the individual demand correspondence $\xi_i$ is upper hemicontinuous with nonempty compact values. Furthermore, if **A5** is assumed, then $\xi_i$ is convex-valued.[11]

PROOF: Take a sequence $\{(p^n, x_i^n)\}$ with $x_i^n \in \xi_i(p^n)$ which converges to $(p, x_i)$. Then, by (3) of Lemma 3.1, $x_i^n \in \hat{\mathcal{B}}_i(p^n)$ and $\hat{P}_i(x_i^n) \cap \hat{\mathcal{B}}_i^\circ(p^n) = \emptyset$. To show $\hat{P}_i(x_i) \cap \hat{\mathcal{B}}_i^\circ(p) = \emptyset$, suppose to the contrary that there exists a choice $y_i \in \hat{P}_i(x_i^n) \cap \hat{\mathcal{B}}_i^\circ(p^n)$. Since $\hat{P}_i$ is lower hemicontinuous, and $\hat{\mathcal{B}}_i^\circ$ has an open graph, the correspondence $\hat{P}_i \cap \hat{\mathcal{B}}_i^\circ : X_i \times \Delta \to 2^{X_i}$ is lower hemicontinuous, where it is defined by $(\hat{P}_i \cap \hat{\mathcal{B}}_i^\circ)(x_i, p) = \hat{P}_i(x_i) \cap \hat{\mathcal{B}}_i^\circ(p)$.[12] Thus, there exists a sequence $\{y_i^n\}$, such that $y_i^n \in \hat{P}_i(x_i^n) \cap \hat{\mathcal{B}}_i^\circ(p^n)$, a contradiction.

Thus, it is obvious that $\xi_i$ has compact values. To show that $\xi_i$ is nonempty-valued, fix $p$, and define $\psi_i : \hat{X}_i \to 2^{\hat{X}_i}$ by $\psi_i(x_i) :=$

---

[11] To my best knowledge, the current version of maximum theorem is new to the literature. The closest result is that of Walker (1979) but he assumes that $P_i$ has open graph in our context.

[12] This is due to Lemma 4.2. of Yannelis (1987).

$\hat{P}_i(x_i) \cap \hat{\mathcal{B}}_i^{\circ}(p)$ and $U_i := \{x_i \in \hat{X}_i : \hat{P}_i(x_i) \cap \hat{\mathcal{B}}_i^{\circ}(p) \neq \emptyset\}$. Observe that $U_i$ is open since $\hat{P}_i$ is lower hemicontinuous and $\hat{\mathcal{B}}_i^{\circ}$ has open graph in $\hat{X}_i$. Then $\psi_i|_{U_i} : U_i \to 2^{\hat{X}_i}$ is lower hemicontinuous with nonempty convex values. According to Michael (1956), there is a continuous function $f_i : U_i \to \hat{X}_i$, such that $f_i(x_i) \in \psi_i(x_i)$ on $U_i$. Let us define the correspondence $\varphi_i : \hat{X}_i \to 2^{\hat{X}_i}$ by:

$$\phi_i(x_i) = \begin{cases} \{f_i(x_i)\}, & \text{if } x_i \in U_i, \\ \hat{\mathcal{B}}_i(p), & \text{if } x_i \notin U_i. \end{cases}$$

Then $\varphi_i$ is upper hemicontinuous with nonempty compact convex values. By Kakutani's fixed point theorem, There is a point $x_i^* \in \varphi_i(x_i^*)$ If $x_i^* \in U_i$, we have $x_i^* \in f_i(x_i^*) \in \psi_i(x_i^*) \subset \hat{P}_i(x_i^*)$, a contradiction. Therefore, $x_i^* \in \hat{\mathcal{B}}_i(p)$ and $\hat{P}_i(x_i^*) \cap \hat{\mathcal{B}}_i^{\circ}(p) = \emptyset$, *i.e.*, $x_i^* \in \xi_i(p)$. Hence $\xi_i$ is nonempty-valued.

In addition, suppose that Assumption **A5** holds. To show that $\xi_i$ is convex-valued, take $x_i^1, x_i^2$ in $\xi_i(p)$. Let $x_i^{\alpha} = \alpha x_i^1 + (1-\alpha) x_i^2$ with $\alpha \in [0,1]$. Suppose that there exists some $y_i \in \hat{P}_i(x_i^{\alpha}) \cap \hat{\mathcal{B}}_i(p)$ for some $\alpha \in [0,1]$. Since $y_i \in \hat{\mathcal{B}}_i(p)$, we have $y_i \notin \hat{P}_i(x_i^1)$ and $y_i \notin \hat{P}_i(x_i^2)$. One can notice that if $\succsim_i$ is complete, then $\hat{R}_i$ is complete.[13] Thus, we have $y_i \in X_i \setminus \hat{P}_i(x_i^1) = \hat{R}_i^{-1}(x_i^1)$ and $y_i \in X_i \setminus \hat{P}_i(x^2) = \hat{R}_i^{-1}(x_i^2)$. That is, $x_i^1$ and $x_i^2$ belong to $\hat{R}_i(y_i)$. By virtue of convexity of $\hat{R}_i$, one obtains $x_i^{\alpha} \in \hat{R}_i(y_i)$ [*i.e.*, $y_i \in \hat{R}_i^{-1}(x_i^{\alpha})$ ]. This implies that $y_i \notin X_i \setminus \hat{R}_i^{-1}(x_i^{\alpha}) = \hat{P}_i(x_i^{\alpha})$, a contradiction. Thus, it is shown that $\hat{P}_i(x_i^{\alpha}) \cap \hat{\mathcal{B}}_i(p) = \emptyset$ for all $\alpha \in [0,1]$, and therefore, one can conclude that $\xi_i$ is convex-valued.

Q.E.D

The excess demand correspondence $Z : \Delta \to 2^{\mathbb{R}^{\ell}}$ is defined by:

$$\zeta(p) = \sum_{i \in I} (\xi_i(p) - \{e_i\}) - \eta(p).$$

Then by Lemma 3.2, the correspondence $\zeta$ is upper hemicontinuous with nonempty compact convex values. Furthermore, $p \cdot z \leq n(1 - \| p \|)$

[13] That is, $y_i \notin \hat{P}_i(x_i)$ if and only if $x_i \in \hat{R}_i(y_i)$ for every $x_i, y_i \in \hat{X}_i$.

for every $z \in \zeta(p)$.

Since $\Delta$ is compact, the upper hemicontinuity of the correspondence $\zeta$ on $\Delta$ implies the existence of a compact and convex subset $Z$ of $\mathbb{R}^\ell$ such that $\zeta(\Delta) \subset Z$. Define the correspondence $\mu : Z \to 2^\Delta$ of the 'auctioneer' by:

$$\mu(z) = \{p \in \Delta : p \cdot z = \sup \Delta \cdot z\},$$

which is the set of maximizers of the function $p \mapsto p \cdot z$ in $\Delta$. It is straightforward that the correspondence $\mu$ is upper hemicontinuous with nonempty compact convex values.

**LEMMA 3.3:** Under Assumptions **A1-A10**, there exist $x_i^* \in \xi_i(p^*)$, $\forall\, i \in I$ and $y_j^* \in \eta_j(p^*)$, $\forall\, j \in J$ such that $z^* = \sum_{i \in I}(x_i^* - e_i) - \sum_{j \in J} y_j^* = 0$.

PROOF: We consider the correspondence $\Psi : \Delta \times Z \to 2^{\Delta \times Z}$ by:

$$\Psi(p, z) = \mu(z) \times \zeta(p).$$

The set $\Delta \times Z$ is nonempty, compact, and convex. Moreover, the correspondence $\Psi$ is upper hemicontinuous with nonempty compact convex values. It follows from Kakutani's fixed point theorem that $\Psi$ has a fixed point $(p^*, z^*)$, such that $p^* \in \mu(z^*)$ and $z^* \in \zeta(p^*)$. The relation $p^* \in \mu(z^*)$ means that $p^* \cdot z^* \geq p \cdot z^*$, $\forall p \in \Delta$, and the relation $z^* \in \zeta(p^*)$ implies that $p^* \cdot z^* \geq n(1 - \| p^* \|)$. We will show that $z^* = 0$. Suppose to the contrary that $z^* \neq 0$. Then it holds that $\| p^* \| = 1$ and $p^* \cdot z^* > 0$. However, one knows from the second relation that $p^* \cdot z^* \leq 0$, which is a contradiction. Thus one obtains $z^* = 0$.

Hence, there exist $x_i^* \in \xi_i(p^*)$, $\forall\, i \in I$ and $y_j^* \in \eta_j(p^*)$, $\forall\, j \in J$, such that $z^* = \sum_{i \in I}(x_i^* - e_i) - \sum_{j \in J} y_j^* = 0$.

Due to Lemma 3.3, we are ready to provide the equilibrium existence theorem. Q.E.D

**THEOREM 3.2:** If the PSS condition is satisfied, under Assumptions

**A1-A10**, the production economy $\mathcal{E}$ has a competitive equilibrium.

PROOF: Take the fixed point $(p^*, x^*, y^*)$ obtained from Lemma 3.3. We will show that this is indeed a competitive equilibrium.

We claim that $p^* \cdot y_j^* = \sup p^* \cdot \hat{Y}_j$, for every $j \in J$. Suppose to the contrary that, for some $j \in J$, there is $y_j \in Y_j$, such that $p^* \cdot y_j > p^* \cdot y_j^*$. As we noted, $y_j^* \in \widetilde{Y}_j \subset \text{int}\, K$. One can find $\alpha \in (0,1)$, such that $y_j^\alpha := \alpha y_j^* + (1-\alpha) y_j \in K$. Since $y_j^\alpha \in \overline{co} Y_j$, we have $y_j^\alpha \in \hat{Y}_j$. But $p^* \cdot y_j^\alpha > p^* \cdot y_j^*$, which contradicts the fact that $p^* \cdot y_j^* = \sup p^* \cdot \hat{Y}_j$. This implies that $\hat{\pi}_i(p^*) = \pi_i(p^*)$ and hence $\hat{M}_i(p^*) = M_i(p^*)$.

We will show that $\| p^* \| = 1$ and $p^* \cdot x_i^* = M_i(p^*)$, $\forall\, i \in I$. This will be done by proving the following three claims.

CLAIM 1: $p^* \cdot x_i^* = M_i(p^*) + (1 - \| p^* \|)$, $\forall\, i \in I(x^*)$.

PROOF: Since $x_i^* \in \hat{\mathcal{B}}_i(p^*)$, we have $p^* \cdot x_i^* \le M_i(p^*) + (1 - \| p^* \|)$. We need to show $p^* \cdot x_i^* \ge M_i(p^*) + (1 - \| p^* \|)$. Since $i \in I(x^*)$, we can take $x_i \in P_i(x_i^*)$. Then $\alpha x_i + (1-\alpha) x_i^* \in \hat{P}_i(x_i^*)$ for $\alpha$ close to 0. Therefore, one has $p^* \cdot [\alpha x_i + (1-\alpha) x_i^*] > M_i(p^*) + (1 - \| p^* \|)$. As $\alpha \to 0$, we have $p^* \cdot x_i^* \ge M_i(p^*) + (1 - \| p^* \|)$. This proves that $p^* \cdot x_i^* = M_i(p^*) + (1 - \| p^* \|)$, $\forall\, i \in I(x^*)$. Q.E.D

CLAIM 2: $p^* \cdot x_k^* \ge M_k(p^*)$, $\forall\, k \in I^s(x^*)$.

PROOF: We will show that $P_i(x_i^*) \cap \hat{\mathcal{B}}_i(p^*) = \emptyset$, $\forall\, i \in I(x^*)$. Suppose to the contrary that for some $i \in I(x^*)$, there is some $x_i \in P_i(x_i^*) \cap \hat{\mathcal{B}}_i(p^*)$. Since $x_i^* \in \text{int}\, K$, it is easy to see that $x_i^\alpha := (1-\alpha) x_i^* + \alpha x_i \in \hat{X}_i$ for $\alpha$ sufficiently close to 0 and therefore $x_i^\alpha \in \hat{P}_i(x_i^*)$. Moreover, observe that $x_i^\alpha \in \hat{\mathcal{B}}_i(x_i^*)$. Thus $x_i^\alpha \in \hat{P}_i(x_i^*) \cap \hat{\mathcal{B}}_i(p^*)$, a contradiction.

Now, for each $i \in I(x^*)$, take $x_i \in P_i(x_i^*)$. Since $P_i(x_i^*) \cap \hat{\mathcal{B}}_i(p^*) = \emptyset$, one has $x_i \notin \hat{\mathcal{B}}_i(p^*)$, *i.e.*, $p^* \cdot x_i > p^* \cdot M_i(p^*) + (1 - \| p^* \|) = p^* \cdot x_i^*$. From the PSS condition, we obtain $p^* \cdot x_k^* \ge M_k(p^*)$ for every $k \in I^s(x^*)$. Q.E.D

CLAIM 3: $\| p^* \| = 1$ and $p^* \cdot x_i^* \ge M_i(p^*)$, $\forall\, i \in I$.

PROOF: Claim 1 and Claim 2 provide us:

$$p^* \cdot x_i^* = M_i(p^*) + (1 - \| p^* \|), \quad \forall\, i \in I(x^*),$$
$$p^* \cdot x_k^* \geq M_i(p^*), \qquad\qquad \forall\, k \in I^s(x^*).$$

Summing up these over $I$ gives $p^* \cdot z^* \geq | I(x^*) | \cdot (1 - \| p^* \|)$ with equality holding when $I(x^*) = I$, where $| I(x^*) |$ denotes the number of consumers in $I(x^*)$. Since $p^* \cdot z^* = 0$ and $\| p^* \| \leq 1$, it follows that that $\| p^* \| = 1$ and $p^* \cdot x_k^* = M_i(p^*)$, $\forall\, i \in I$. Q.E.D

To complete the proof, we need to show that $P_i(x_i^*) \cap \mathcal{B}_i\,(p^*) = \emptyset$, $\forall\, i \in I$. It trivially holds for all $i \in I^s(x^*)$ because $P_i(x_i^*) = \emptyset$. Therefore, we will show that $P_i(x_i^*) \cap \mathcal{B}_i\,(p^*) = \emptyset$, $\forall\, i \in I(x^*)$. Suppose to the contrary that for some $i \in I(x^*)$, there is some $x_i \in P_i(x_i^*) \cap \mathcal{B}_i\,(p^*)$. Since $x_i^* \in \operatorname{int} K$, it is easy to see that $x_i^\alpha := (1-\alpha)x_i^* + \alpha x_i \in \hat{X}_i$ for $\alpha$ sufficiently close to 0 and therefore $x_i^\alpha \in \hat{P}_i(x_i^*)$. Moreover, observe that $x_i^\alpha \in \hat{\mathcal{B}}_i\,(p^*)$. Thus $x_i^\alpha \in \hat{P}_i(x_i^*) \cap \hat{\mathcal{B}}_i\,(p^*)$, a contradiction. Therefore, we have $P_i(x_i^*) \cap \mathcal{B}_i\,(p^*) = \emptyset$, $\forall\, i \in I(x^*)$. Hence $(p^*, x^*, y^*)$ constitutes a competitive equilibrium.

## IV. CONCLUDING REMARKS

The PSS condition plays a crucial role in showing an equilibrium existence. In fact, this condition is an adaptation of the condition S5 in Won and Yannelis (2005) to a production economy, which is quite classical, except that preferences may be satiable. However, the economy of Won and Yannelis (2005) is much more general than the one in this paper, in that the consumption sets are unbounded, and preferences are non-ordered and satiable. This means that neither paper encompass the other. The completeness of preferences is assumed in this paper. This assumption together with the weak convexity (**A2**) allows demand correspondences to be convex-valued, and therefore, the excess demand approach can be applied.

The plausible future work is surely to generalize the result of this paper by assuming nonordered preferences and weakening the assumptions **A8** to **A10**. It seems that the approaches of Debreu (1962) and McKenzie (2002) would be helpful to this line of research.

## References

Bergstrom, T. C. (1976), "How to Discard 'Free Disposability'-at No Cost," *Journal of Mathematical Economics*, Vol. 3, 131-134.

Cornet, B., M. Topuzu, and A. Yildiz (2003), "Equilibrium Theory with a Measure Space of Possibly Satiated Consumers," *Journal of Mathematical Economics*, Vol. 39, 175-196.

Debreu, G. (1962), "New Concepts and Techniques for Equilibrium Analysis," *International Economic Review*, Vol. 3, 257-273.

Debreu, G. (1982), "Existence of Competitive Equilibrium," in: K. Arrow and M. Intrilligator, eds., *Handbook of Mathematical Economics*, Vol. II. Amsterdam: North-Holland.

Gale, D. and A. Mas-Colell (1975), "An Equilibrium Existence Theorem for a General Model without Ordered Preferences," *Journal of Mathematical Economics*, Vol. 2, 9-15.

Gale, D. and A. Mas-Colell (1979), "Correction to an Equilibrium Existence Theorem for a General Model without Ordered Preferences," *Journal of Mathematical Economics*, Vol. 6, 297-298.

Kajii, A. (1996), "How to Discard Nonsatiation and Free Disposal with Paper Money," *Journal of Mathematical Economics*, Vol. 25, 75-84.

Loomes, G., C. Starmer, and R. Sugden (1991), "Observing Violations of Transitivity by Experimental Methods," *Econometrica*, Vol. 59, 425-439.

Martins-da-Rocha (2003), "Equilibria in Large Economies with a Separable Banach Commodity Space and Non-Ordered preferences," *Journal of Mathematical Economics*, Vol. 39, 863-889.

Mas-Colell, A. (1992), "Equilibrium Theory with Possibly Satiated Preferences," in *Equilibrium and Dynamics: Essays in Honor of David Gale*, edited by M. Majumdar, London: Macmillan, 201-213.

Mas-Colell, A. (1974), "An Equilibrium Existence Theorem without Complete orTransitive Preferences," *Journal of Mathematical Economics*, Vol. 1, 237-246.

May, K. O. (1954), "Intransitivity, Utility, and the Aggregation of Preference Patterns," *Econometrica*, Vol. 22, 1-13.

McKenzie, L.W. (2002), *Classical General Equilibrium Theory*, Cambridge: MIT University Press.

Michael, E. (1956), "Continuous Selections I," *Annals of Mathematics*, Vol. 63, 361-382.

Polemarchakis, H. M. and P. Siconolfi (1993), "Competitive Equilibria without

Free Disposal or Nonsatiation," *Journal of Mathematical Economics*, Vol. 22, 85-99.

Shafer, W. J. and H. Sonnenschein (1975), "Equilibrium in Abstract Economies without Ordered Preferences," *Journal of Mathematical Economics*, Vol. 2, 345-348.

Smale, S. (1982), "Global Analysis and Economics," in: K. Arrow and M. Intrilligator, eds., *Handbook of Mathematical Economics*, Vol. II. Amsterdam: North-Holland.

Sonnenschein, H. (1971), "Demand Theory without Transitivity Preference with Applications to the Theory of Competitive Equilibrium," in *Preferences, Utility, and Demand*, Chipman, J., L. Hurwicz, M. Richter, and H. Sonnenschein, eds., New York: Harcourt, Brace, Jovanovich.

Walker M. (1979), "A Generalization of the Maximum Theorem," *International Economic Review*, Vol. 20, 260-272.

Won, D. C. and N. C. Yannelis (2005), "Equilibrium Theory with Unbounded Consumption Sets and Non-Ordered Preferences: Part II, the Case of Satiation," Mimeo.

Yannelis, N. C. (1987), "Equilibria in Noncooperative Models of Competition," *Journal of Economic Theory*, Vol. 41, 96-111.