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NOTE ON THE DECOMPOSITION BY FACTORS IN DIRECT AND INDIRECT REQUIREMENTS*

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The decomposition by factors in direct and indirect requirements was introduced by Gim and Kim (2005). More specifically, the notion of direct and indirect input requirements of commodity i *to produce a unit of gross output of commodity* j *,* γ_{ij}^{g} *, can be decomposed into the direct, the technical indirect effects, and that to support a unit of final demand of commodity* j *,* γ_{ij}^{f} *, can be decomposed into the direct, the technical indirect, and the interrelated indirect effects in the open static input-output model. This note provides a complementation of the decomposition, which is based on more accurate and consecutively meaningful general relation given by* $\gamma_{ii}^{f} = \gamma_{ii}^{g} c_{ii}$ *(or* $\gamma_{ii}^{g} = 1 - 1/c_{ii}$ *) and* $\gamma_{ij}^{f} = \gamma_{ij}^{g} c_{jj}$ *(or* $\gamma_{ij}^{g} = c_{ij}/c_{jj}$ *) for* $i \neq j$*, where the element of the Leontief inverse* c_{ij} *represents the direct and indirect output requirements to support a unit of final demand.*

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* We wish to express our appreciation for valuable comments and suggestions given by three anonymous referees. This paper is the revision of Gim and Kim (2005) and has the following characteristics: (1) The paper is a note-type paper. (2) The approach in this note is based on the equation $(\mathbf{I}-\mathbf{A})\mathbf{C}=\mathbf{I}$, whereas the equation $\mathbf{C}(\mathbf{I}-\mathbf{A})=\mathbf{I}$ is used in Gim and Kim (1998, 2005). (3) The main reason for the change is that the product $(\mathbf{I}-\mathbf{A})\mathbf{C}$ is consecutively connected between the sectors in the input-output model, but the expression $\mathbf{C}(\mathbf{I}-\mathbf{A})$ is not consecutively connected in the input-output model although it has no problem mathematically. (4) This note is intended to follow the derivation and expressions given in Gim and Kim (2005) as much as possible to help the readers for comparisons between the two papers.

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I. INTRODUCTION

The decomposition by factors in direct and indirect requirements introduced by Gim and Kim (2005) was based on the general relation $\gamma_{ii}^f = c_{ii}\gamma_{ii}^g$ and $\gamma_{ij}^f = c_{ii}\gamma_{ij}^g$, $i \neq j$ proposed by Gim and Kim (1998). However, a close examination of the general relation reveals that there is no "consecutive connection" between the element of Leontief inverse c_{ii} and γ_{ij}^g when c_{ii} is post-multiplied by γ_{ij}^g, since the second subscript i in c_{ii} represents final demand of commodity i, whereas the first subscript i in γ_{ij}^g represents direct and indirect input requirements of commodity i. This note, therefore, provides a complementation of the decomposition, which is based on more accurate and consecutively meaningful general relation[1] given by $\gamma_{ii}^f = \gamma_{ii}^g c_{ii}$ (or $\gamma_{ii}^g = 1 - 1/c_{ii}$) and $\gamma_{ij}^f = \gamma_{ij}^g c_{jj}$ (or $\gamma_{ij}^g = c_{ij}/c_{jj}$) for $i \neq j$. The complemented result is also demonstrated by the same problems of estimating the pollution through the pollution-activity-augmented Leontief model and the waste generation in the case of Korea.

II. THE MODIFICATION: THE DECOMPOSITION OF THE DIRECT AND INDIRECT REQUIREMENTS

The Leontief inverse $\mathbf{C}$ represents $(\mathbf{I}-\mathbf{A})^{-1}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{A}$ the technical coefficient matrix. It follows then that

$$(\mathbf{I}-\mathbf{A})\mathbf{C} = \mathbf{I}. \tag{1}$$

Notice that in the case of $n = 2$, equation. (1) becomes

$$\begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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[1] The derivation of the complemented general relation is shown briefly in Appendix A.

and the first equation from it is given by $(1-a_{11})c_{11}-a_{12}c_{21}=1$. As can be observed, there is consecutive connection, for example, in the term $a_{12}c_{21}$ [2] since a_{12} represents the direct input requirements of sector 1 to produce one unit of output of sector 2, c_{21} means the total output requirements of sector 2 to support one unit of final demand of sector 1, and the subscript 2 in both a_{12} and c_{21} represents the output for sector 2. However, when one starts with $\mathbf{C}(\mathbf{I}-\mathbf{A})=\mathbf{I}$, which was used in Gim and Kim (1998, 2005), then for $n=2$,

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the first equation becomes $c_{11}(1-a_{11})-c_{12}a_{21}=1$. As can be seen, there is no consecutive connection, for example, in the term $c_{12}a_{21}$,[3] since c_{12} represents the total output requirements of sector 1 to support one unit of final demand of sector 2, a_{21} means the direct input requirements of sector 2 to produce one unit of output of sector 1, but the subscript 2 in c_{12} represents the final demand for sector 2, whereas the subscript 2 in a_{21} means the direct input requirements of sector 2. In consequence, both matrix equations, $(\mathbf{I}-\mathbf{A})\mathbf{C}=\mathbf{I}$ and $\mathbf{C}(\mathbf{I}-\mathbf{A})=\mathbf{I}$, hold mathematically, but the former is more accurate and consecutively meaningful in input-output economics. Therefore, the modification in this paper is wholly based on equation (1). This is the main reason why we inevitably write this note paper again.[4]

Furthermore, the Leontief inverse can be expressed as a power series (Waugh, 1950)

$$\mathbf{C}=\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{k+1}+\cdots \tag{2}$$

[2] The interpretation of the subscripts should be given in reverse order (from right to left). In the sense of economic interpretation, the cancellation occurs in the inner subscripts, so this expression is meaningful and consecutive.

[3] Likewise, the interpretation of the subscripts should be given in reverse order. However, no cancellation occurs in the inner subscripts, thus this term is meaningless and nonconsecutive.

[4] For the interrelation between the characteristics of the Leontief inverse and the consecutive connection, please refer to Appendix B.

and has been viewed as a term that consists of three different parts: the final demand($\mathbf{I}$), the direct effect($\mathbf{A}$), and the cumulative indirect effect $(\mathbf{A}^2 + \mathbf{A}^3 + \cdots)$.[5] The specific expression for each element c_{ij} can be obtained from (2) and is given by[6]

$$c_{ij} = \delta_{ij} + a_{ij} + (\sum_{r_1=1}^{n} a_{ir_1} a_{r_1 j} + \sum_{r_2=1}^{n}\sum_{r_1=1}^{n} a_{ir_1} a_{r_1 r_2} a_{r_2 j} + \cdots$$
$$+ \sum_{r_k=1}^{n}\sum_{r_{k-1}=1}^{n} \cdots \sum_{r_1=1}^{n} a_{ir_1} a_{r_1 r_2} \cdots a_{r_{k-1} r_k} a_{r_k j} + \cdots), \tag{3}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases} \tag{4}$$

for $i, j = 1, 2, \ldots, n$.

Now, the elements of the requirements matrix for a unit of final demand $\mathbf{\Gamma}^{\mathrm{f}} = (\gamma_{ij}^{f})$ are given by

$$\gamma_{ii}^{f} = c_{ii} - 1, \qquad \gamma_{ij}^{f} = c_{ij}, \quad i \neq j, \tag{5}$$

and the elements of the requirements matrix for a unit of gross output $\mathbf{\Gamma}^{\mathrm{g}} = (\gamma_{ij}^{g})$ are shown (Appendix A) to be[7]

$$\gamma_{ii}^{g} = 1 - \frac{1}{c_{ii}} = a_{ii} + \sum_{j=1,\, j\neq i}^{n} a_{ij} \frac{c_{ji}}{c_{ii}}, \tag{6}$$

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[5] For the interrelation between the characteristics of the technical (or input) coefficient matrix and the consecutive connection, please refer to Appendix C.

[6] The numerical value for c_{ij} is the same as that given in Gim and Kim (2005), but the order of expansion (i.e., the position of a_{ij}) is different since the modification begins with $(\mathbf{I}-\mathbf{A})\mathbf{C} = \mathbf{I}$, not with $\mathbf{C}(\mathbf{I}-\mathbf{A}) = \mathbf{I}$.

[7] One noticeable change in γ_{ij}^{g} is the position of a_{ik} and c_{kj}/c_{jj} when compared with the results given in Gim and Kim (2005), because of the same reason mentioned in note 2.

$$\gamma_{ij}^{g} = \frac{c_{ij}}{c_{jj}} = a_{ij} + \sum_{k=1, k\neq j}^{n} a_{ik}\frac{c_{kj}}{c_{jj}}, \quad i \neq j, \tag{7}$$

for $i, j = 1, 2, \ldots, n$. To obtain a specific expression and decomposition for each γ_{ij}^{g} in terms of a_{ij}, we use the same two approaches described in Gim and Kim (2005): one by expanding (6) and (7) and the other by considering the interrelated interdependence of the sectors.

Starting with $n = 2$ and considering only through the second-round indirect effect ($k = 2$ in (2)) for convenience of illustration, the elements of the Leontief inverse, c_{11} and c_{21}, for example, can be obtained from (3) as

$$\begin{aligned} c_{11} &= 1 + a_{11} + (a_{12}a_{21} + a_{12}a_{22}a_{21}) \\ &\quad + (a_{11}a_{11} + a_{11}a_{11}a_{11} + a_{11}a_{12}a_{21} + a_{12}a_{21}a_{11}), \\ c_{21} &= a_{21} + (a_{22}a_{21} + a_{22}a_{22}a_{21}) \\ &\quad + (a_{21}a_{11} + a_{21}a_{11}a_{11} + a_{21}a_{12}a_{21} + a_{22}a_{21}a_{11}). \end{aligned}$$

From (6) and (7) we obtain $\gamma_{11}^{g} = a_{11} + a_{12}(c_{21}/c_{11})$ and $\gamma_{21}^{g} = a_{21} + a_{22}(c_{21}/c_{11})$, and the expansion of (1) for $n = 2$ yields four equations, one of which is $-a_{21}c_{11} + (1 - a_{22})c_{21} = 0$. From this equation, we obtain $c_{21}/c_{11} = (1 - a_{22})^{-1}a_{21}$, and using the fact that

$$(1 - a_{ii})^{-1} = 1 + a_{ii} + a_{ii}^{2} + \cdots \quad \text{for} \quad (1 - a_{ii}) > 0,$$

we have

$$\begin{aligned} \gamma_{11}^{g} &= a_{11} + a_{12}(1 - a_{22})^{-1}a_{21} \\ &= a_{11} + a_{12}a_{21} + a_{12}a_{22}a_{21} + a_{12}a_{22}^{2}a_{21} + \cdots, \\ \gamma_{21}^{g} &= a_{21} + a_{22}(1 - a_{22})^{-1}a_{21} \\ &= a_{21} + a_{22}a_{21} + a_{22}^{2}a_{21} + a_{22}^{3}a_{21} + \cdots. \end{aligned}$$

For γ_{11}^{g}, the term a_{11} is the direct input requirement (direct effect), and the other terms $(a_{12}a_{21},\ a_{12}a_{22}a_{21}, \ldots)$ are the indirect input requirement (indirect effect), which indicate the relation between inputs and output; that is, these are indispensable direct and indirect effects of the production process. More specifically, a_{12} induces the indirect input requirements; it is connected technically with the element a_{21}, and it is also technically connected with a_{22}, which, in turn, is connected with a_{21}. In consequence, $a_{12}a_{21}+a_{12}a_{22}a_{21}+\cdots$ represent the total amount of the indirect effect, which we call the technical indirect effect. Comparing c_{11} with γ_{11}^{g}, we see that c_{11} contains an additional part $(a_{11}a_{11}+a_{11}a_{11}a_{11}+a_{11}a_{12}a_{21}+a_{12}a_{21}a_{11})$. We call this the interrelated indirect effect. Similar interpretation can be given for γ_{21}^{g}.

For $n=3$, we have, from (3) and after rearranging,[8]

$$\begin{aligned} c_{11} = 1 + a_{11} &+ (a_{12}a_{21} + a_{12}a_{22}a_{21} + a_{12}a_{23}a_{31} + a_{13}a_{31} \\ &+ a_{13}a_{32}a_{21} + a_{13}a_{33}a_{31}) + (a_{11}a_{11} + a_{11}a_{11}a_{11} \\ &+ a_{11}a_{12}a_{21} + a_{11}a_{13}a_{31} + a_{12}a_{21}a_{11} + a_{13}a_{31}a_{11}), \end{aligned} \tag{8}$$

$$\begin{aligned} c_{21} = a_{21} &+ (a_{22}a_{21} + a_{22}a_{22}a_{21} + a_{22}a_{23}a_{31} + a_{23}a_{31} \\ &+ a_{23}a_{32}a_{21} + a_{23}a_{33}a_{31}) + (a_{21}a_{11} + a_{21}a_{11}a_{11} \\ &+ a_{21}a_{12}a_{21} + a_{21}a_{13}a_{31} + a_{22}a_{21}a_{11} + a_{23}a_{31}a_{11}). \end{aligned} \tag{9}$$

The expressions for γ_{11}^{g} and γ_{21}^{g} can be obtained from (6) and (7) as

$$\gamma_{11}^{g} = a_{11} + a_{12}\frac{c_{21}}{c_{11}} + a_{13}\frac{c_{31}}{c_{11}}, \tag{10}$$

$$\gamma_{21}^{g} = a_{21} + a_{22}\frac{c_{21}}{c_{11}} + a_{23}\frac{c_{31}}{c_{11}}, \tag{11}$$

and the expansion of (1) for $n=3$ gives nine equations, two of which are

$$-a_{21}c_{11} + (1-a_{22})c_{21} - a_{23}c_{31} = 0,$$

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[8] The order of expansion (i.e., the position of a_{ij}) is different when comparing with the results given in Gim and Kim (2005), because of the same reason mentioned in the previous notes.

$$-a_{31}c_{11} - a_{32}c_{21} + (1-a_{33})c_{31} = 0.$$

Dividing the two equations above by c_{11} and rearranging, we obtain a linear system of equations

$$(1-a_{22})\frac{c_{21}}{c_{11}} - a_{23}\frac{c_{31}}{c_{11}} = a_{21},$$
$$-a_{32}\frac{c_{21}}{c_{11}} + (1-a_{33})\frac{c_{31}}{c_{11}} = a_{31}.$$

The second equation above gives

$$\frac{c_{31}}{c_{11}} = (1-a_{33})^{-1}(a_{31} + a_{32}\frac{c_{21}}{c_{11}}), \tag{12}$$

and substituting this into the first equation above and solving for c_{21}/c_{11} yields

$$\frac{c_{21}}{c_{11}} = [1-a_{22} - a_{23}(1-a_{33})^{-1}a_{32}]^{-1}[a_{21} + a_{23}(1-a_{33})^{-1}a_{31}]. \tag{13}$$

Notice that $1-a_{22} - a_{23}(1-a_{33})^{-1}a_{32}$ in (13) can not be a negative value or 0. It follows that $a_{22} + a_{23}(1-a_{33})^{-1}a_{32} < 1$, and this enables us to write

$$\begin{aligned}&[1-a_{22} - a_{23}(1-a_{33})^{-1}a_{32}]^{-1} \\ &\quad = 1 + [a_{22} + a_{23}(1-a_{33})^{-1}a_{32}] + [a_{22} + a_{23}(1-a_{33})^{-1}a_{32}]^2 + \cdots.\end{aligned} \tag{14}$$

Using $(1-a_{33})^{-1} = 1 + a_{33} + a_{33}^2 + \cdots$ and substituting (14) into (13) and simplifying, we obtain

$$\frac{c_{21}}{c_{11}} = a_{21} + a_{22}a_{21} + a_{23}a_{32}a_{21} + a_{23}a_{31} + a_{22}a_{23}a_{31} + \cdots. \tag{15}$$

Substituting (15) into (12) and simplifying, we have

$$\frac{c_{31}}{c_{11}} = a_{31} + a_{33}a_{31} + a_{33}a_{33}a_{31} + a_{32}a_{21} + a_{33}a_{32}a_{21} + \cdots \tag{16}$$

Finally, substituting (15) and (16) into (10) and (11), and simplifying and collecting terms only through the second-round indirect effect, we obtain

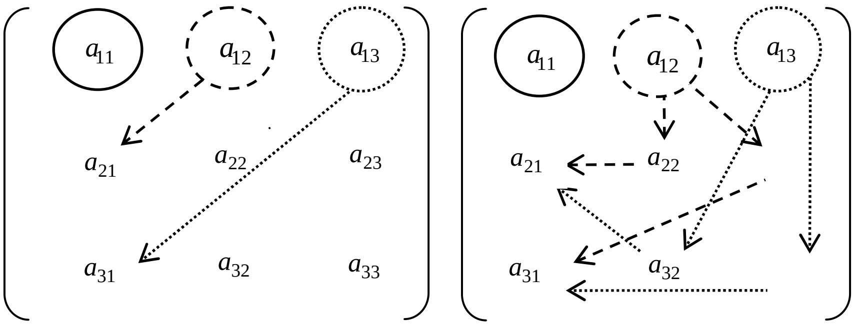
$$\begin{aligned}\gamma_{11}^{g} = a_{11} + (&a_{12}a_{21} + a_{12}a_{22}a_{21} + a_{12}a_{23}a_{31} + a_{13}a_{31} \\ &+ a_{13}a_{32}a_{21} + a_{13}a_{33}a_{31})\end{aligned} \tag{17}$$

and

$$\begin{aligned}\gamma_{21}^{g} = a_{21} + (&a_{22}a_{21} + a_{22}a_{22}a_{21} + a_{22}a_{23}a_{31} + a_{23}a_{31} \\ &+ a_{23}a_{32}a_{21} + a_{23}a_{33}a_{31}).\end{aligned} \tag{18}$$

Hence, as in the case with $n = 2$, both γ_{11}^{g} and γ_{21}^{g} consist of two parts: the direct effect and the second part of (17) and (18), respectively, which we call the technical indirect effect. Also, by comparing, we see that c_{11} and c_{21} contain an additional part: the last part in (8) and (9), respectively. We call this the interrelated indirect effect. In this manner, the expression for any other γ_{ij}^{g} can be obtained. In fact, it will turn out to be the expression given in (19). In γ_{11}^{g}, a_{11} is the direct effect, and the two elements a_{12} and a_{13} in the same row induce the first-round and second-round indirect effects, respectively. Figure 1 below shows the connections of up to second-round indirect effects induced by a_{12} and a_{13} (marked by dotted circles) for γ_{11}^{g}.

[Figure 1] Connections of the First-round(left) and the Second-round(right) Indirect Effects for γ_{11}^{g}



The second approach is by considering the interindustry interdependence without expanding (6) and (7), which reflects the complete technical relation of production. For γ_{ij}^{g} with n sectors, a_{ij} is the direct input requirement and the other elements in the i th row of $\mathbf{A}$, $a_{i1}, a_{i2}, \ldots, a_{i(j-1)}, a_{i(j+1)}, \ldots, a_{in}$, induce the technical indirect input requirements. Each one of these, a_{ir_1}, $r_1 = 1, 2, ..., j-1, j+1, ..., n$, is technically connected with the elements in the corresponding row r_1 (determined according to the second subscript of a_{ir_1}), and hence it should be post-multiplied by the elements in that corresponding row r_1 of $\mathbf{A}$. Consequently, the resultant terms have the form $a_{ir_1}a_{r_1r_2}$, $r_2 = 1, 2, ..., n$, and the terms with $r_2 = j$ will become the first-round indirect effect ($a_{ir_1}a_{r_1j}$, where $r_1 \neq j$). For the terms with $r_2 \neq j$, further post-multiplication is necessary, since each element $a_{r_1r_2}$ is technically connected with the elements in row r_2 of $\mathbf{A}$. Then, after the second post-multiplication, the resultant terms have the form $a_{ir_1}a_{r_1r_2}a_{r_2r_3}$, $r_3 = 1, 2, ..., n$, and the terms with $r_3 = j$ will become the second-round indirect effect ($a_{ir_1}a_{r_1r_2}a_{r_2j}$, where $r_1 \neq j, r_2 \neq j$). For the terms with $r_3 \neq j$, further post-multiplication is needed, since each element $a_{r_2r_3}$ is technically connected with the elements in row r_3 of $\mathbf{A}$. In this expansion, we must pay special attention to the term $a_{ir_1}a_{r_1r_2}a_{r_2r_3}$ for example. The expansion process took place from the left element to the right element, while the interpretation of the expansion was given from the right element to the left element. However, in the result given in Gim and Kim (2005), both the expansion and the interpretation were given from the right element to the left element in the same direction. This is one main difference between the results by the authors (2005 and present note paper).

In a like manner, the process can be continued to obtain the third-round indirect effect, the forth-round indirect effect, and on and on. Then, the sum of the direct input requirement a_{ij}, the first-round indirect effect, the second-round indirect effect, and so on, yield γ_{ij}^{g}. The modified general expression for each of the elements of $\mathbf{\Gamma}^{g}$ with n sectors can be formulated and is given by

$$\begin{aligned}\gamma_{ij}^{g} &= a_{ij} + \Big(\sum_{\substack{r_1=1 \\ r_1 \neq j}}^{n} a_{ir_1} a_{r_1 j} + \sum_{\substack{r_2=1 \\ r_2 \neq j}}^{n} \sum_{\substack{r_1=1 \\ r_1 \neq j}}^{n} a_{ir_1} a_{r_1 r_2} a_{r_2 j} \\ &\quad + \cdots + \sum_{\substack{r_k=1 \\ r_k \neq j}}^{n} \sum_{\substack{r_{k-1}=1 \\ r_{k-1} \neq j}}^{n} \cdots \sum_{\substack{r_1=1 \\ r_1 \neq j}}^{n} a_{ir_1} a_{r_1 r_2} \cdots a_{r_{k-1} r_k} a_{r_k j} + \cdots \Big),\end{aligned} \tag{19}$$

where k stands for k th-round and for $i, j = 1, 2, \ldots, n$.

Then, (19) can be written as

$$\gamma_{ij}^{g} = a_{ij} + t_{ij},$$

where t_{ij} is the second part in (19). We called t_{ij} the *technical* indirect effect since each term in t_{ij} indicates a "pure" (or complete) technical relation between inputs and output. It is named after the *technical* coefficient matrix **A** and the purely *technical* relation between inputs and output in the production system. As a consequence, each element of the Leontief inverse c_{ij} can be written in the following form:

$$c_{ij} = \delta_{ij} + a_{ij} + t_{ij} + r_{ij}, \tag{20}$$

where δ_{ij} is given in (4), a_{ij} is the element of the technical coefficient matrix $\mathbf{A}$, t_{ij} is the technical indirect effect defined above, and r_{ij} is given by

$$r_{ij} = a_{ij}a_{jj} + \Big(\sum_{r_1=1}^{n} a_{ir_1}a_{r_1 j}a_{jj} + \sum_{\substack{r_2=1 \\ r_2 \neq j}}^{n} a_{ij}a_{jr_2}a_{r_2 j} \Big)$$
$$+ \Big(\sum_{r_2=1}^{n}\sum_{r_1=1}^{n} a_{ir_1}a_{r_1 r_2}a_{r_2 j}a_{jj} + \sum_{\substack{r_3=1 \\ r_3 \neq j}}^{n}\sum_{r_1=1}^{n} a_{ir_1}a_{r_1 j}a_{jr_3}a_{r_3 j} \tag{21}$$
$$+ \sum_{\substack{r_3=1 \\ r_3 \neq j}}^{n}\sum_{\substack{r_2=1 \\ r_2 \neq j}}^{n} a_{ij}a_{jr_2}a_{r_2 r_3}a_{r_3 j} \Big) + \cdots$$

for $i, j = 1, 2, \ldots, n$. That is, r_{ij} is just the difference between the third part of the right-hand side of (3) and the second part of the right-hand side of (19) ($r_{ij} = (c_{ij} - \delta_{ij} - a_{ij}) - t_{ij}$). We called r_{ij} the *interrelated* indirect effect. It represents the interrelated interdependence indirect effect, which consists of terms that are not technically connected in the production process. In other words, every term in r_{ij} is "interrelated" with a term in γ_{ij}^{g}.

Since $\gamma_{ij}^{f} = c_{ij} - \delta_{ij}$, the expression for each element of $\mathbf{\Gamma}^{\mathrm{f}}$ can be obtained from (20) and can be written as

$$\gamma_{ij}^{f} = a_{ij} + t_{ij} + r_{ij}, \tag{22}$$

for $i, j = 1, 2, \ldots, n$.

III. SIMPLER FORMS AND MATRIX NOTATION FOR THE DECOMPOSITION[9]

Denoting $\mathbf{I}$, $\mathbf{A}$, $\mathbf{T}$, and $\mathbf{R}$ as the identity matrix, the direct effect, the technical indirect effect, and the interrelated indirect effect, and letting their elements be δ_{ij}, a_{ij}, t_{ij}, and r_{ij}, respectively, we have

[9] The decomposition of factors is applied in the newly defined model called "Output-Output model." For the more detailed results refer to Gim and Kim (2008).

$$\mathbf{\Gamma}^{\mathrm{g}} = \mathbf{A} + \mathbf{T}\,; \tag{23}$$
$$\mathbf{\Gamma}^{\mathrm{f}} = \mathbf{A} + \mathbf{T} + \mathbf{R}\,; \tag{24}$$
$$\mathbf{C} = \mathbf{I} + \mathbf{A} + \mathbf{T} + \mathbf{R}\,. \tag{25}$$

Note that combining (23) and (24) yields $\mathbf{R} = \mathbf{\Gamma}^{\mathrm{f}} - \mathbf{\Gamma}^{\mathrm{g}}$ and (23) gives $\mathbf{T} = \mathbf{\Gamma}^{\mathrm{g}} - \mathbf{A}$. If we denote $\mathbf{K}$ as the diagonal matrix that contains only the elements of the diagonal of $\mathbf{C}$, then from (5), (6), and (7), $\mathbf{\Gamma}^{\mathrm{g}} = \mathbf{\Gamma}^{\mathrm{f}}\mathbf{K}^{-1}$. Hence, we have $\mathbf{T} = \mathbf{\Gamma}^{\mathrm{f}}\mathbf{K}^{-1} - \mathbf{A}$ and $\mathbf{R} = \mathbf{\Gamma}^{\mathrm{f}}(\mathbf{I} - \mathbf{K}^{-1})$.

IV. APPLICATION TO ESTIMATING THE POLLUTION GENERATION

In this section we apply the modified results obtained in the previous sections to the environmental Leontief model (Leontief 1970), which is an input-output model augmented by pollution-generation and pollution-abatement sectors:[10]

$$\begin{pmatrix} 1-a_{11} & -a_{12} & -a_{1p} \\ -a_{21} & 1-a_{22} & -a_{2p} \\ -a_{p1} & -a_{p2} & 1-a_{pp} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_p \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ -d_p \end{pmatrix}. \tag{26}$$

When compared with the results given in Gim and Kim (2005), $\mathbf{C}_P$ and $\mathbf{\Gamma}_P^{\mathrm{f}}$ are the same, but $\mathbf{\Gamma}_P^{\mathrm{g}}$ becomes

$$\mathbf{\Gamma}_P^{\mathrm{g}} = \begin{pmatrix} 1-\dfrac{1}{c_{11}} & \dfrac{c_{12}}{c_{22}} & \dfrac{c_{1p}}{c_{pp}} \\ \dfrac{c_{21}}{c_{11}} & 1-\dfrac{1}{c_{22}} & \dfrac{c_{2p}}{c_{pp}} \\ \dfrac{c_{p1}}{c_{11}} & \dfrac{c_{p2}}{c_{22}} & 1-\dfrac{1}{c_{pp}} \end{pmatrix}.$$

[10] For the definitions of the terms a_{p1}, a_{1p}, a_{pp}, x_p and d_p, refer to Gim and Kim (2005).

When the example given in Miller and Blair (1985, p.247) is applied to the linear system of (26), the specific pollution-activity-augmented Leontief model becomes:

$$\begin{pmatrix} 0.85 & -0.25 & -0.10 \\ -0.20 & 0.95 & -0.20 \\ -0.05 & -0.04 & 1.00 \end{pmatrix} \begin{pmatrix} 100.0 \\ 100.0 \\ 6.0 \end{pmatrix} = \begin{pmatrix} 59.4 \\ 73.8 \\ -3.0 \end{pmatrix}.$$

The numerical values for $\mathbf{A}_P$, $\mathbf{C}_P$, and $\mathbf{\Gamma}_P^{\mathrm{f}}$ are the same when compared with the results given in Gim and Kim (2005), but the values of $\mathbf{\Gamma}_P^{\mathrm{g}}$, $\mathbf{T}_P$, and $\mathbf{R}_P$ changed as follows:

$$\mathbf{\Gamma}_P^{\mathrm{g}} = \begin{pmatrix} 0.2116 & 0.3006 & 0.1914 \\ 0.2229 & 0.1211 & 0.2508 \\ 0.0589 & 0.0550 & 0.0196 \end{pmatrix},$$

$$\mathbf{T}_P = \begin{pmatrix} 0.0616 & 0.0506 & 0.0914 \\ 0.0229 & 0.0711 & 0.0508 \\ 0.0089 & 0.0150 & 0.0196 \end{pmatrix} \quad \text{and}$$

$$\mathbf{R}_P = \begin{pmatrix} 0.0568 & 0.0414 & 0.0038 \\ 0.0598 & 0.0167 & 0.0050 \\ 0.0158 & 0.0076 & 0.0004 \end{pmatrix}.$$

It can be checked that

$$\mathbf{C}_P = \mathbf{I} + \mathbf{A}_P + \mathbf{T}_P + \mathbf{R}_P,$$
$$\mathbf{\Gamma}_P^{\mathrm{f}} = \mathbf{A}_P + \mathbf{T}_P + \mathbf{R}_P,$$

and

$$\mathbf{\Gamma}_P^{\mathrm{g}} = \mathbf{A}_P + \mathbf{T}_P$$

hold.

By the results of the modified decomposition, it can be viewed that the total amount of pollution generation of pollutant p (0.0747), $\mathbf{C}_{P(31)}$ = 0.0747 in $\mathbf{C}_P$, is generated by the direct effect (0.05), the technical indirect effect (0.0089), and the interrelated indirect effect (0.0158). Also, $\mathbf{\Gamma}^{g}_{P(31)} = 0.0589$, which represents the direct and indirect input units of pollutant p to produce one dollar's worth of gross output of commodity 1, is generated by the two effects: the direct (0.05) and the technical indirect effect (0.0089).

V. APPLICATION TO ESTIMATING THE WASTES GENERATION: THE CASE OF KOREA

In this section we again apply the modified results obtained in the previous sections to the environmental Leontief model, which is an input-output model augmented by waste generation. The waste-generation data and 15 rearranged sectors can be seen from the Table 1 (Gim and Kim 2005).

The augmented Leontief model for $n = 16$ can be written in compact matrix form as[11]

$$\left(\begin{array}{cccc:c} & & & & 0 \\ & (I-A) & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hdashline -a_{w1} & -a_{w2} & \cdots & -a_{w15} & 1 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{15} \\ \hdashline x_w \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{15} \\ \hdashline 0 \end{pmatrix},$$

of which the expanded coefficient matrix can be denoted as $(\mathbf{I}-\mathbf{A}_w)$. From the original values of $\mathbf{C}_w = (\mathbf{I}-\mathbf{A}_w)^{-1}$, we recomputed the corresponding matrices $\mathbf{\Gamma}^{g}_{w}$, $\mathbf{\Gamma}^{f}_{w}$, and $\mathbf{T}_w$, and showed only the bottom rows of these matrices in the following table:

[Table 1] Data Obtained from the Bottom Rows of $\mathbf{C}_w$, $\mathbf{\Gamma}^{g}_{w}$, $\mathbf{T}_w$, and $\mathbf{R}_w$

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[11] For the definitions of the elements a_{wj}, x_w, and d_i, refer to Gim and Kim (2005).

	j = 1	2	3	4	5	6	7	8
c_{wj}	0.7308	1.3642	1.6568	2.5885	4.1031	1.7024	47.7400	7.9449
γ^g_{wj}	0.6581	1.2980	1.3536	1.7050	2.4591	1.0460	39.2120	3.9269
t_{wj}	0.6411	0.8990	0.9796	0.7876	0.7811	0.8800	0.9710	0.8109
r_{wj}	0.0727	0.0662	0.3032	0.8835	1.6440	0.6564	8.5280	4.0180

	j = 9	10	11	12	13	14	15	16
c_{wj}	3.3693	5.5162	3.2693	6.5139	0.9789	1.1532	2.2951	1.0000
γ^g_{wj}	1.8256	3.9345	2.8369	6.4654	0.9164	0.9213	1.8168	0.0000
t_{wj}	1.6056	2.3225	0.9209	6.3824	0.8694	0.7983	0.5118	0.0000
r_{wj}	1.5437	1.5817	0.4324	0.0485	0.0625	0.2319	0.4783	0.0000

As we discussed in the previous example, the waste multipliers, c_{wj}, give an indication of the effects on waste generation. Moreover, due to the results obtained in this paper, $\mathbf{\Gamma}^g_w$ enables us to estimate the total waste generation associated with only the purely technical relation between inputs and output. Hence, $\gamma^g_{w1} = 0.6581$, for example, can be interpreted notionally as the direct and indirect input units of waste w to produce one million Korean Won worth of gross output of commodity 1. This amount is actually generated by the two effects: the direct (0.017) and the technical indirect effect (0.6411). The interrelated indirect effect for commodity 1(r_{w1}) turned out to be 0.0727. Consequently, an implication of the above modified results is that, by decomposing waste generation into factors, one can establish more effective methods for abatement of waste generation by approaching and examining factor by factor.

VI. CONCLUSIONS

The decomposition by factors in direct and indirect requirements introduced by Gim and Kim (2005) was based on the general relation $\gamma^f_{ii} = c_{ii}\gamma^g_{ii}$ and $\gamma^f_{ij} = c_{ii}\gamma^g_{ij}$, $i \neq j$ proposed by Gim and Kim (1998).

This general relation is based on the matrix equation $\mathbf{C}(\mathbf{I}-\mathbf{A})=\mathbf{I}$. However, it is observed that there is no consecutive connection between c_{ii} and γ_{ij}^{g} when c_{ii} is post-multiplied by γ_{ii}^{g} or γ_{ij}^{g}. This note paper then provides a complementation of the decomposition, which is based on more accurate and consecutively meaningful general relation given by $\gamma_{ii}^{f}=\gamma_{ii}^{g}c_{ii}$ (or $\gamma_{ii}^{g}=1-1/c_{ii}$) and $\gamma_{ij}^{f}=\gamma_{ij}^{g}c_{jj}$ (or $\gamma_{ij}^{g}=c_{ij}/c_{jj}$) for $i\neq j$. The complemented general relation is based on the matrix equation $(\mathbf{I}-\mathbf{A})\mathbf{C}=\mathbf{I}$. The complemented result is further applied to the same problems of estimating the pollution through the pollution-activity-augmented Leontief model and the waste generation in the case of Korea to observe changes in numerical values.

In preparing this note paper, we supplemented three significantly important appendices. They are as follows:

1. The derivation of new complemented general relation is included in Appendix A.
2. The interrelation between the characteristics of the Leontief inverse and the consecutive connection is described in detail in Appendix B.
3. The interrelation between the characteristics of the technical (or input) coefficient matrix and the consecutive connection is explained concisely in Appendix C.

In closing, we emphasize again one of the crucial and symbolical results: the matrix equation $(\mathbf{I}-\mathbf{A})\mathbf{C}=\mathbf{I}$ (or $(\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{I}$) is simultaneously satisfied both mathematically and economically, but the relation $\mathbf{C}(\mathbf{I}-\mathbf{A})=\mathbf{I}$ (or $(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})=\mathbf{I}$) holds only mathematically.

Appendix A: The Complemented General Relation

From the open static input-output model

$$\mathbf{Ax} + \mathbf{f} = \mathbf{x},$$

where $\mathbf{A}$ is the technical coefficient matrix, $\mathbf{x}$ the gross output vector, and $\mathbf{f}$ the final demand vector, the Leontief inverse $\mathbf{C}$ can be obtained as

$$\mathbf{C} = (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots. \tag{A1}$$

Here, $\mathbf{I}$ is the nth order identity matrix. Furthermore, the notion of γ_{ij}^f is already well-known, and its relation with c_{ij} can be written as

$$\gamma_{ii}^f = c_{ii} - 1, \quad i = 1, 2, \ldots, n$$

and

$$\gamma_{ij}^f = c_{ij}, \quad i, j = 1, 2, \ldots, n, \quad i \neq j.$$

Let $\mathbf{\Gamma}^{\mathrm{f}} = (\gamma_{ij}^f)$ be the matrix of which the ijth element is given by γ_{ij}^f. Then, it is clear from above that

$$\mathbf{\Gamma}^{\mathrm{f}} = \mathbf{C} - \mathbf{I} = \mathbf{A} + \mathbf{A}^2 + \cdots. \tag{A2}$$

In matrix notation, we have

$$\begin{pmatrix} \gamma_{11}^f & \gamma_{12}^f & \cdots & \gamma_{1n}^f \\ \gamma_{21}^f & \gamma_{22}^f & \cdots & \gamma_{2n}^f \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1}^f & \gamma_{n2}^f & \cdots & \gamma_{nn}^f \end{pmatrix} = \begin{pmatrix} c_{11} - 1 & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} - 1 & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} - 1 \end{pmatrix}.$$

Consequently, pre-multiplying $\mathbf{A}$ on $\mathbf{C}$ in (A1) and combining with (A2) yields the following familiar but compact relation

$$\mathbf{AC} = \mathbf{\Gamma}^{\mathrm{f}} \;,$$

that is,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} \gamma_{11}^{f} & \gamma_{12}^{f} & \cdots & \gamma_{1n}^{f} \\ \gamma_{21}^{f} & \gamma_{22}^{f} & \cdots & \gamma_{2n}^{f} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1}^{f} & \gamma_{n2}^{f} & \cdots & \gamma_{nn}^{f} \end{pmatrix}. \quad \text{(A3)}$$

Multiplying the i th row of **A** and j th column of **C** yields the element γ_{ij}^{f} , which means the direct and indirect input requirements of commodity i to support one unit of final demand of commodity j . Thus, equating the ij th element on the left side with that on the right side from (A3) gives

$$a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{i(j-1)}c_{(j-1)j} + a_{ij}c_{jj} + a_{i(j+1)}c_{(j+1)j} + \cdots + a_{in}c_{nj} = \gamma_{ij}^{f} \;,$$

and then dividing c_{jj} on both sides and rearranging by placing the term a_{ij} in front yields

$$\begin{aligned} & a_{ij} + a_{i1}\frac{c_{1j}}{c_{jj}} + a_{i2}\frac{c_{2j}}{c_{jj}} + \cdots + a_{i(j-1)}\frac{c_{(j-1)j}}{c_{jj}} \\ & \quad + a_{i(j+1)}\frac{c_{(j+1)j}}{c_{jj}} + \cdots + a_{in}\frac{c_{nj}}{c_{jj}} = \frac{\gamma_{ij}^{f}}{c_{jj}} \,. \end{aligned} \qquad \text{(A4)}$$

By using a similar approach to that described in Gim and Kim (2008, Sec. 3), it can be shown that the term on the left hand-side of (A4) coincides exactly with the concept of γ_{ij}^{g} . Therefore, in order to obtain the complemented general relation between the two different notions γ_{ij}^{g} and γ_{ij}^{f} , denote as follows:

$$\gamma_{ij}^{g} = a_{ij} + a_{i1}\frac{c_{1j}}{c_{jj}} + a_{i2}\frac{c_{2j}}{c_{jj}} + \cdots + a_{i(j-1)}\frac{c_{(j-1)j}}{c_{jj}}$$

$$+a_{i(j+1)}\frac{c_{(j+1)j}}{c_{jj}}+\cdots+a_{in}\frac{c_{nj}}{c_{jj}}\,. \tag{A5}$$

Substituting (A5) in (A4) finally yields

$$\gamma_{ij}^{g}=\frac{\gamma_{ij}^{f}}{c_{jj}}, \tag{A6}$$

for $i, j = 1, 2, \dots, n$.

Then, one can see clearly that the first term a_{ij} in γ_{ij}^{g} of (A5) represents the *direct* input requirement of commodity i to produce a unit of gross output of commodity j. The remaining terms represent the indirect input requirement of commodity i to produce a unit of gross output of commodity j, which occurred through all the intermediates of a_{ik} except $k = j$. Together, γ_{ij}^{g} represents the *direct and indirect* input requirements of commodity i to produce a unit of gross output of commodity j. Distinguishing between diagonal and nondiagonal elements in (A6), the modified general relation between the two different notions can be written as

$$\gamma_{ii}^{f}=\gamma_{ii}^{g}c_{ii} \quad\text{and}\quad \gamma_{ij}^{f}=\gamma_{ij}^{g}c_{jj},\;\; i\neq j, \tag{A7}$$

for $i, j = 1, 2, \dots, n$.

Furthermore, combining $\gamma_{ii}^{f}=c_{ii}-1$, $\gamma_{ij}^{f}=c_{ij}$ for $i \neq j$, and (A5) yields

$$\begin{aligned}\gamma_{ii}^{g}&=1-\frac{1}{c_{ii}}=a_{ii}+\sum_{j=1,\,j\neq i}^{n}a_{ij}\frac{c_{ji}}{c_{ii}},\\ \gamma_{ij}^{g}&=\frac{c_{ij}}{c_{jj}}=a_{ij}+\sum_{k=1,k\neq j}^{n}a_{ik}\frac{c_{kj}}{c_{jj}},\quad i\neq j,\end{aligned} \tag{A8}$$

for $i, j = 1, 2, \dots, n$.

In (A7), the second subscript j in γ_{ij}^{g} and the first subscript j in c_{jj} both represent the gross output of commodity j. Thus, there is *consecutive connection* between γ_{ij}^{g} and c_{jj}, and the product $\gamma_{ij}^{g}c_{jj}$ makes sense. Moreover, the economic interpretation of $\gamma_{ij}^{g}c_{jj}$ exactly coincides with the definition of γ_{ij}^{f}, namely the direct and indirect input requirements of commodity i to support a unit of final demand of commodity j. A similar interpretation can be given to the product $\gamma_{ii}^{g}c_{ii}$.

Appendix B: The Characteristics of the Leontief Inverse and the Consecutive Connection

The open static input-output model $\mathbf{Ax}+\mathbf{f}=\mathbf{x}$ can be equivalently expressed as

$$\mathbf{x}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}=\mathbf{Cf},$$

where $\mathbf{C}=(c_{ij})$ with c_{ij}, the direct and indirect output requirements of commodity i to support a unit of final demand of commodity j. By the definition of c_{ij}, it is self-evident that the product $c_{ij}f_j$, where f_j is the jth element of **f**, makes sense, since both the subscript js in c_{ij} and f_j represent the final demand of commodity j. Similarly, the product x_ic_{ij}, where x_i is the ith element of **x**, makes sense because both the subscript is in x_i and c_{ij} represent the gross output of commodity i. However, $c_{ij}x_j$ makes no sense, since the subscript j in c_{ij} represents the final demand of commodity j, but the subscript j in x_j means the gross output for commodity j (i.e., neither of the subscript js can cancell each other out in the units of measurement, implying that there is no consecutive connection between **C** and **x** when **C** is post-multiplied by **x**).

Accordingly, to have a proper economic meaning of interindustry interdependence among sectors, the term that should be post-multiplied by **C** is limited to only final demand **f** (or $\Delta\mathbf{f}$), and the term that must be pre-multiplied by **C** is restricted to only the gross output **x** (or $\Delta\mathbf{x}$), by the fundamental characteristic of the multi-sector multiplier (or Leontief inverse) **C**.

The misconception of post-multiplying **x** to **C** often occurs, directly or indirectly, in the literature. For example, Oosterhaven and Stelder (2002) in their paper mentioned that practitioners often multiply the gross output

x by the value-added multipliers $\mathbf{v}_c{}'\mathbf{L}$ (namely, $\mathbf{v}_c{}'\mathbf{Lx}$) to measure the importance of an activity, where $\mathbf{v}_c{}'$ represents a row with value-added coefficients and **L** represents the Leontief inverse (**C** in our note paper). De Mesnard (2002) proposed $\mathbf{i}'\mathbf{AL} = \mathbf{i}'(\mathbf{L} - \mathbf{I})$ as alternative net multipliers that accept outputs (**x**) as entries. His derivation process, however, is based on a misconception, because he implicitly assumed that $\mathbf{L} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots$ can be multiplied by $\Delta\mathbf{x}^e$ (exogenous effect of output **x**). As can be perceived now, the expression $\mathbf{L}\Delta\mathbf{x}^e$ does not make sense, since there is no consecutive connection between **L** and $\Delta\mathbf{x}^e$.

Dietzenbacher (2005) showed that $\mathbf{L}\hat{\mathbf{x}}$ yields output multipliers, where $\hat{\mathbf{x}}$ denotes the diagonal matrix whose *j*th diagonal element is given by $\mathbf{x}_j$, the *j*th component of **x**. In the same vein, $\mathbf{L}\hat{\mathbf{x}}$ also makes no sense, since there is no consecutive connection between **L** and $\hat{\mathbf{x}}$.

Further more, we discussed in Section II that there is a consecutive connection, for instance, in the term $a_{12}c_{21}$, whereas the term $c_{12}a_{21}$ exhibits no economically meaningful results. Similar arguments also apply to the products $\gamma_{ij}^g c_{jj}$ (consecutive connection) and $c_{ii}\gamma_{ij}^g$ (no consecutive connection), as mentioned in Section I.

Therefore, the characteristic of the Leontief inverse **C** with regard to the consecutive connection is summarized in the following table by means of listing a number of examples of multipliable and unmultipliable cases. The essence is that post-multiplication by **C** is limited to only the terms that can be cancelled by the final demand as the unit of measurement (e.g., **f**, $\Delta\mathbf{f}$), whereas pre-multiplication by **C** is restricted to only the terms that can be offset by the measurement unit of gross output (e.g., **xC**, **AC**, $\gamma_{ij}^g c_{jj}$).

[Table B1] Multipliable and Unmultipliable Forms in the Consecutive Connections

Multipliable Form	Unmultipliable Form
$\mathbf{Cf}$	$\mathbf{Cx}$
$\mathbf{AC}$	$\mathbf{CA}$
$(\mathbf{I}-\mathbf{A})\mathbf{C} = \mathbf{I}$	$\mathbf{C}(\mathbf{I}-\mathbf{A}) = \mathbf{I}$ (satisfied only mathematically)
$\gamma_{ij}^g c_{jj} = \gamma_{ij}^f$	$c_{ii}\gamma_{ij}^g$
$\gamma_{ii}^g c_{ii} = \gamma_{ii}^f$	$c_{ii}\gamma_{ii}^g$

Appendix C: The Characteristics of the Technical (or Input) Coefficient Matrix and the Consecutive Connection

In the open static input-output model

$$\mathbf{Ax} + \mathbf{f} = \mathbf{x}, \tag{C1}$$

the input coefficient matrix **A** is denoted by $\mathbf{A} = (a_{ij})$, of which the element $a_{ij} = X_{ij} / x_j$, where X_{ij} is defined as the flow of input from i to j. With the tantamount expression $X_{ij} = a_{ij} x_j$, and since a_{ij} is defined as the input of commodity i per unit of gross output of commodity j, the product $\mathbf{Ax}$ forms naturally as intermediate demand and does hold the consecutive connection.

A debatable issue occurs when the input coefficient matrix **A** is post-multiplied by final demand **f** to form $\mathbf{Af}$, $\mathbf{A}^2\mathbf{f}$, etc. This situation often arises, for example, in the following expansion:

$$\begin{aligned} \mathbf{x} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{Cf} \\ &= (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots)\mathbf{f} \\ &= \mathbf{f} + \mathbf{Af} + \mathbf{A}^2\mathbf{f} + \cdots. \end{aligned}$$

One might be ambiguous or confused about whether the product $\mathbf{Af}$ is multipliable or not, since **f** is always known as or understood to be *final demand*, and in the literal sense of the letters **A** and **f**, there might seem to be no consecutive connection between **A** and **f**.

However, as can be seen from equation (C1), the final demand **f** is a portion of the gross output **x**, and also, although **f** is called “final demand” only as the name, **f** actually has the real character of gross output; the product of gross output **x** is only distributed into the terms intermediate demand (**Ax**) and final demand (**f**). Thus, post-multiplication of **A** by **f** to form $\mathbf{Af}$ is also multipliable and meaningful. This result (**A** can be multiplied by **x** and **f**) might also be caused partly by the fact that the characters of **C**, as the multi-sector multiplier, and **A**, as the coefficient matrix which represents the direct effect, are disparate (recall that **Cf** is multipliable, but **Cx** is not). Keeping this in mind, the Leontief inverse **C** possesses the concept of input-output multipliers that compute the direct and indirect effects (output requirements), while the matrix **A** is not an inverse matrix nor has it the concept of input-output multipliers. It is merely an input coefficient matrix that describes the technical ratio of input to gross output in the production system.

Therefore, $\mathbf{Ax}$ and $\mathbf{Af}$ both are multipliable and meaningful. As $\mathbf{Af}$ is multipliable, $\mathbf{A}^2\mathbf{f} = \mathbf{A}(\mathbf{Af})$ becomes the first-round indirect input requirement, $\mathbf{A}^3\mathbf{f}$ the second-round indirect input requirement, and so on.

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